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Research Article

On A_α–eigenvalues of graphs and topological indices

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Abstract

Let $A(G)$ and $D(G)$ be the adjacency matrix and the degree diagonal matrix of a graph G, respectively. For any real number $\alpha \in [0,1]$, the A_α-matrix of G is defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$. In this paper, new lower bounds on the A_α-spectral radius and A_α-spread are obtained in terms of the maximum degree, general Randić index, first general Zagreb index, eccentric connectivity index, and general Randić eccentricity index. The obtained bounds provide new lower bounds on the A-spectral radius, Q-spectral radius, A-spread, and Q-spread.

Keywords: A_α-matrix; A_α-spectral radius; A_α-spread; topological index.

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1. Introduction

Let G be a simple undirected graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For $v_i \in V(G)$, $d_i = d_G(v_i)$ denotes the degree of v_i and $N(v_i)$ denotes the set of all neighbors of the vertex v_i in G. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, or simply by δ and Δ, respectively. For the vertices $v_i, v_j \in V(G)$, the distance $d(v_i, v_j)$ is defined as the length of a shortest path between v_i and v_j in G. The eccentricity $\varepsilon_i = \varepsilon_G(v_i)$ of a vertex v_i is the maximum distance from v_i to any other vertex. The radius of G, $\rho = \rho(G)$, is the minimum eccentricity while the diameter, $\phi = \phi(G)$, is the maximum eccentricity. Let K_n and $K_{1,\,n-1}$ be the complete graph and star with n vertices, respectively.

The study of topological indices of various graphs has been of interest to chemists, mathematicians, and scientists from related fields due to the fact that the topological indices play a significant role in mathematical chemistry especially in the QSPR/QSAR (quantitative structure-property/activity relationships) modeling. The general Randić index of a graph G, introduced by Bollobás and Erdős [1] in 1998, is defined as

$$R^{(t)} = \sum_{v_iv_j \in E(G)} (d_id_j)^t, \quad t \in \mathbb{R}.$$

Clearly, $R^{(0)}$ is the number of edges, $R^{(-\frac{1}{2})}$ is the Randić index R [22], $R^{(\frac{1}{2})}$ is the reciprocal Randić index [7], $R^{(1)}$ is the second Zagreb index M_2 [8], etc. The first general Zagreb index of a graph G, introduced by Li and Zheng [12] in 2005, is defined as

$$Z^{(t)} = \sum_{v_i \in V(G)} d_i^t, \quad t \in \mathbb{R}.$$

It is easy to see that $Z^{(0)}$ is the number of vertices, $Z^{(1)}$ is twice the number of edges, $Z^{(2)}$ is the first Zagreb index M_1 [8], $Z^{(3)}$ is the forgotten topological index F [5], etc.

The total eccentricity of a connected graph G, introduced by Smith et al. [25], is the sum of the eccentricities of its vertices, that is,

$$\mathcal{E} = \sum_{v_i \in V(G)} \varepsilon_i.$$

The eccentric connectivity index of a connected graph G, introduced by Sharma et al. [24], is defined as

$$\xi^c = \sum_{v_i \in V(G)} d_i\varepsilon_i.$$

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Similar to the general Randić index, the general Randić eccentricity index of a connected graph G is defined as

$$\zeta^{(t)} = \sum_{v_iv_j \in E(G)} (\varepsilon_i \varepsilon_j)^t, \quad t \in \mathbb{R}.$$

It is not difficult to see that $\zeta^{(0)}$ is the number of edges and $\zeta^{(1)}$ is the second Zagreb eccentricity index ξ_2 [27].

For any real number $\alpha \in [0,1]$, Nikiforov [17] defined the A_α-matrix of a graph G as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G),$$

where $D(G)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the adjacency matrix. Let

$$\lambda_1(A_\alpha(G)) \geq \lambda_2(A_\alpha(G)) \geq \cdots \geq \lambda_n(A_\alpha(G))$$

be A_α-eigenvalues of a graph G with n vertices. Then $\lambda_1(A_\alpha(G))$ and $S_\alpha(G) = S_{A_\alpha}(G) = \lambda_1(A_\alpha(G)) - \lambda_n(A_\alpha(G))$ are called A_α-spectral radius and A_α-spread of the graph G, respectively. Since $A_0(G) = A(G)$ and $2A_{1/2}(G) = Q(G)$, we have $\lambda_1(A_0(G)) = \lambda_1(A(G))$, $2\lambda_1(A_{1/2}(G)) = \lambda_1(Q(G))$, $S_0(G) = S_A(G)$ and $2S_{1/2}(G) = S_Q(G)$, where $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G. Therefore, the A_α-eigenvalues can be regard as a common generalization of the A-eigenvalues and Q-eigenvalues. The investigation of A_α-eigenvalues is a popular topic in the theory of graph spectra at present.

In this paper, some lower bounds on the A_α-spectral radius and A_α-spread are given. The relations between A_α-eigenvalues and topological indices of graphs are also established. Due to the fact that most of the popular topological indices of graphs have a large number of results on the estimation of bounds, the newly established relations give a lot of lower bounds on the A_α-spectral radius and A_α-spread. Moreover, lower bounds on the A_α-spectral radius in terms of the maximum degree and minimum degree are also obtained. Detail about the A_α-spectral radius can be found in [3, 4, 6, 9, 10, 13, 18, 19, 21, 23, 26, 28, 29, 31, 32, 34]. For detail about the A_α-spread, one may refer to [14, 15]. Detail concerning the topological indices considered in this paper can be found in [2, 11, 20, 33] and in the references therein.

2. Lower bounds on the A_α-spectral radius

Let $X = (x_1, x_2, \ldots, x_n)^T$ be a real vector. The quadratic form $X^T A_\alpha(G) X$ can be represented in the following form:

$$X^T A_\alpha(G) X = \alpha \sum_{v_i \in V(G)} d_i x_i^2 + 2(1-\alpha) \sum_{v_iv_j \in E(G)} x_i x_j.$$

If X is an eigenvector of $A_\alpha(G)$ with respect to $\lambda_1(A_\alpha(G))$, then by the Perron-Frobenius theorem, X is positive and unique if G is connected. The eigenequations for the matrix $A_\alpha(G)$ can be written as

$$\lambda_1(A_\alpha(G)) x_i = \alpha d_i x_i + (1-\alpha) \sum_{v_j \in N(v_i)} x_j.$$

Theorem 2.1. *Let G be a connected graph with n vertices. If $0 \leq \alpha < 1$, then*

$$\lambda_1(A_\alpha(G)) \geq \frac{1}{Z^{(t)}} \left[\alpha Z^{(t+1)} + 2(1-\alpha) R^{(\frac{t}{2})} \right] \tag{1}$$

with equality if and only if

$$\frac{\Gamma\left(d_i^{\frac{t}{2}}\right) - \Gamma\left(d_j^{\frac{t}{2}}\right)}{d_j - d_i} = \frac{\alpha}{1-\alpha}$$

for $1 \leq i < j \leq n$, where

$$\Gamma(d_i) = \frac{\sum\limits_{v_j \in N(v_i)} d_j}{d_i}.$$

Proof. Let $\lambda_1(A_\alpha(G)) = \lambda_1$ and $X = \frac{1}{\sqrt{Z^{(t)}}} \left(d_1^{\frac{t}{2}}, d_2^{\frac{t}{2}}, \ldots, d_n^{\frac{t}{2}} \right)^T$. By the Rayleigh-Ritz theorem, we have

$$\begin{aligned} \lambda_1 &\geq X^T A_\alpha(G) X \\ &= \alpha \sum_{v_i \in V(G)} d_i x_i^2 + 2(1-\alpha) \sum_{v_iv_j \in E(G)} x_i x_j \\ &= \frac{1}{Z^{(t)}} \left[\alpha Z^{(t+1)} + 2(1-\alpha) R^{(\frac{t}{2})} \right]. \end{aligned}$$

The equality holds in (1) if and only if X is an eigenvector corresponding to the eigenvalue λ_1. Then, we have

$$\begin{aligned}\lambda_1 d_1^{\frac{t}{2}} &= \alpha d_1^{1+\frac{t}{2}} + (1-\alpha)\sum_{v_j\in N(v_1)} d_j^{\frac{t}{2}},\\ \lambda_1 d_2^{\frac{t}{2}} &= \alpha d_2^{1+\frac{t}{2}} + (1-\alpha)\sum_{v_j\in N(v_2)} d_j^{\frac{t}{2}},\\ &\vdots\\ \lambda_1 d_n^{\frac{t}{2}} &= \alpha d_n^{1+\frac{t}{2}} + (1-\alpha)\sum_{v_j\in N(v_n)} d_j^{\frac{t}{2}}.\end{aligned}$$

Thus,

$$\frac{\Gamma\left(d_i^{\frac{t}{2}}\right)-\Gamma\left(d_j^{\frac{t}{2}}\right)}{d_j-d_i}=\frac{\alpha}{1-\alpha}$$

for $1\le i<j\le n$, where

$$\Gamma(d_i)=\frac{\sum_{v_j\in N(v_i)} d_j}{d_i}.$$

□

Remark 2.1. *In particular, one has*

$$\lambda_1(A(G))\ge\frac{2R^{(\frac{t}{2})}}{Z^{(t)}}\quad and \quad \lambda_1(Q(G))\ge\frac{1}{Z^{(t)}}\left[Z^{(t+1)}+2R^{(\frac{t}{2})}\right].$$

By taking $t=0,1,2,$ in Theorem 2.1, one obtains the following corollaries.

Corollary 2.1. *Let G be a connected graph with n vertices and m edges. If $0\le\alpha<1$, then*

$$\lambda_1(A_\alpha(G))\ge\frac{2m}{n}.$$

Corollary 2.2. *Let G be a connected graph with n vertices and m edges. If $0\le\alpha<1$, then*

$$\lambda_1(A_\alpha(G))\ge\frac{1}{2m}[\alpha M_1+2(1-\alpha)R_{-1}].$$

Corollary 2.3. *Let G be a connected graph with n vertices. If $0\le\alpha<1$, then*

$$\lambda_1(A_\alpha(G))\ge\frac{1}{M_1}[\alpha F+2(1-\alpha)M_2].$$

Theorem 2.2. *Let G be a connected graph with n vertices. If $0\le\alpha<1$, then*

$$\lambda_1(A_\alpha(G))\ge\frac{1}{\mathcal{E}}\left[\alpha\xi^c+2(1-\alpha)\zeta^{(\frac{1}{2})}\right]. \tag{2}$$

with equality if and only if

$$\frac{\Psi(\sqrt{\varepsilon_i})-\Psi(\sqrt{\varepsilon_j})}{d_j-d_i}=\frac{\alpha}{1-\alpha}$$

for $1\le i<j\le n$, where

$$\Psi(\varepsilon_i)=\frac{\sum_{v_j\in N(v_i)}\varepsilon_j}{\varepsilon_i}.$$

Proof. Let $\lambda_1(A_\alpha(G))=\lambda_1$ and $X=\frac{1}{\sqrt{\mathcal{E}}}(\sqrt{\varepsilon_1},\sqrt{\varepsilon_2},\dots,\sqrt{\varepsilon_n})^T$. By the Rayleigh-Ritz theorem, we have

$$\begin{aligned}\lambda_1 &\ge X^TA_\alpha(G)X\\ &= \alpha\sum_{v_i\in V(G)} d_ix_i^2+2(1-\alpha)\sum_{v_iv_j\in E(G)}x_ix_j\\ &= \frac{1}{\mathcal{E}}\left[\alpha\xi^c+2(1-\alpha)\zeta^{(\frac{1}{2})}\right].\end{aligned}$$

The equality holds in (2) if and only if X is an eigenvector corresponding to the eigenvalue λ_1. Thus, we have

$$\lambda_1\sqrt{\varepsilon_1} = \alpha d_1\sqrt{\varepsilon_1}+(1-\alpha)\sum_{v_j\in N(v_1)}\sqrt{\varepsilon_j},$$

$$\begin{aligned}\lambda_1\sqrt{\varepsilon_2} &= \alpha d_2\sqrt{\varepsilon_2}+(1-\alpha)\sum_{v_j\in N(v_2)}\sqrt{\varepsilon_j},\\ &\vdots\\ \lambda_1\sqrt{\varepsilon_n} &= \alpha d_n\sqrt{\varepsilon_n}+(1-\alpha)\sum_{v_j\in N(v_n)}\sqrt{\varepsilon_j}.\end{aligned}$$

Therefore,

$$\frac{\Psi(\sqrt{\varepsilon_i})-\Psi(\sqrt{\varepsilon_j})}{d_j-d_i}=\frac{\alpha}{1-\alpha}$$

for $1\le i<j\le n$, where

$$\Psi(\varepsilon_i)=\frac{\sum_{v_j\in N(v_i)}\varepsilon_j}{\varepsilon_i}.$$

□

Remark 2.2. *In particular, we have*

$$\lambda_1(A(G))\ge\frac{2\zeta^{(\frac{t}{2})}}{\mathcal{E}}\quad and\quad \lambda_1(Q(G))\ge\frac{1}{\mathcal{E}}\left[\xi^c+2\zeta^{(\frac{1}{2})}\right].$$

Question 2.1. *Characterize all graphs for which equality holds in* (1) *and* (2).

Theorem 2.3. *Let G be a connected graph with n vertices and m edges. If $0\le\alpha<1$ and $y>0$, then*

$$\lambda_1(A_\alpha(G))\ge\frac{1}{y^2+n-1}[\alpha\Delta y^2+\alpha(2m-\Delta)+2(1-\alpha)(\Delta y+m-\Delta)]. \tag{3}$$

The equality in (3) *holds if and only if the following two cases are respectively satisfied:*

(i) $\Delta=n-1$. *Then $d_2=\cdots=d_n$, and y is the root of*

$$(1-\alpha)x^2+(d_2+2\alpha-\alpha n-1)x-(1-\alpha)(n-1)=0.$$

(ii) $\Delta<n-1$. *Then $d_2=\cdots=d_{\Delta+1}$, $d_{\Delta+2}=\ldots=d_n$ and*

$$y=\frac{(1-\alpha)\Delta}{d_n-\alpha\Delta}=\frac{d_n-d_2}{1-\alpha}+1.$$

Proof. Let $\lambda_1(A_\alpha(G))=\lambda_1$, $d_1=\Delta$ and $N(v_1)=\{v_2,\ldots,v_{\Delta+1}\}$. We take

$$X=\frac{1}{\sqrt{y^2+n-1}}(y,1,\ldots,1)^T,$$

where $y>0$. By the Rayleigh-Ritz theorem, we have

$$\begin{aligned}\lambda_1 &\ge X^TA_\alpha(G)X\\ &= \alpha\sum_{v_i\in V(G)}d_ix_i^2+2(1-\alpha)\sum_{v_iv_j\in E(G)}x_ix_j\\ &= \frac{1}{y^2+n-1}\left[\alpha\Delta y^2+\alpha(2m-\Delta)+2(1-\alpha)(\Delta y+m-\Delta)\right].\end{aligned}$$

The equality holds in (3) if and only if X is an eigenvector corresponding to an eigenvalue λ_1. Then we have

$$\begin{aligned}\lambda_1 y &= \alpha\Delta y+(1-\alpha)\Delta,\\ \lambda_1 &= \alpha d_2+(1-\alpha)(y+d_2-1),\\ &\vdots\\ \lambda_1 &= \alpha d_{\Delta+1}+(1-\alpha)(y+d_{\Delta+1}-1),\\ \lambda_1 &= \alpha d_{\Delta+2}+(1-\alpha)d_{\Delta+2},\\ &\vdots\\ \lambda_1 &= \alpha d_n+(1-\alpha)d_n.\end{aligned}$$

If $\Delta = n-1$, then $d_2 = \cdots = d_n$, and y is the root of

$$(1-\alpha)x^2 + (d_2 + 2\alpha - \alpha n - 1)x - (1-\alpha)(n-1) = 0.$$

If $\Delta < n-1$, then $d_2 = \cdots = d_{\Delta+1}$, $d_{\Delta+2} = \ldots = d_n$ and

$$y = \frac{(1-\alpha)\Delta}{d_n - \alpha\Delta} = \frac{d_n - d_2}{1-\alpha} + 1.$$

This completes the proof. □

Remark 2.3. *If $\alpha = 0$ and there exist chromatic number χ such that $\Delta^2 - (n-1)(\chi-1)^2 + 2(\chi-1)(m-\Delta) > 0$, then the lower bound in (3) is better than the lower bound $\lambda_1(A(G)) \geq \chi - 1$ obtained in [30]. In particular, if G is a bipartite graph, then the lower bound in (3) is better than that the one established in [30].*

It is remarked here that the bound given in Theorem 2.3 is better than the known results of similar kind when y takes different values. By taking y as the A_α-spectral radius $\lambda_1(A_\alpha(G))$, the maximum degree Δ and the average degree $\frac{2m}{n}$ in Theorem 2.3, respectively, one gets the following corollaries.

Corollary 2.4. *Let G be a connected graph with n vertices and m edges. If $0 \leq \alpha < 1$, then*

$$\lambda_1(A_\alpha(G)) \geq \lambda_0, \tag{4}$$

where λ_0 is the largest root of $x^3 - \alpha\Delta x^2 + [n-1-2(1-\alpha)\Delta]x - 2m + (2-\alpha)\Delta = 0$.

Remark 2.4. *If $\alpha = 0$, then the equality in (4) holds if and only if $G \cong K_{1,n-1}$.*

Remark 2.5. *Let $f(x) = x^3 - \alpha\Delta x^2 + [n-1-2(1-\alpha)\Delta]x - 2m + (2-\alpha)\Delta$. If $0 \leq \alpha \leq \frac{1}{2}$ and $m > \frac{n^2}{8} + n + \frac{1}{2}$, then*

$$f(\alpha\Delta + 1) = (3\alpha^2 - 2\alpha)\Delta^2 + (\alpha n + 2\alpha)\Delta - 2m + n < 0.$$

This implies that

$$\lambda_1(A_\alpha(G)) > \alpha\Delta + 1 \tag{5}$$

for $0 \leq \alpha \leq \frac{1}{2}$ and $m > \frac{n^2}{8} + n + \frac{1}{2}$. It is clear that the lower bound in (5) is better than the one given in Corollary 13 of [17] for $0 \leq \alpha \leq \frac{1}{2}$ and $m > \frac{n^2}{8} + n + \frac{1}{2}$.

Corollary 2.5. *Let G be a connected graph with n vertices and m edges. If $0 \leq \alpha < 1$, then*

$$\lambda_1(A_\alpha(G)) \geq \frac{1}{\Delta^2 + n - 1}[\alpha\Delta^3 + 2(1-\alpha)\Delta^2 - (2-\alpha)\Delta + 2m]. \tag{6}$$

Remark 2.6. *If $0 \leq \alpha \leq \frac{1}{2}$ and $m > \frac{n^2}{16} + \frac{n}{2}$, then the lower bound in (6) is better than the bound given in Corollary 13 of [17]. If $\frac{1}{2} < \alpha \leq \frac{5}{6}$ and $m \geq \frac{3n^2}{10} + \frac{n}{4}$, then the lower bound in (6) is better than that in Corollary 13 in [17].*

Corollary 2.6. *Let G be a connected graph with n vertices and m edges. If $0 \leq \alpha < 1$, then*

$$\lambda_1(A_\alpha(G)) \geq \frac{4\alpha\Delta m^2 + 4(1-\alpha)\Delta mn + 2mn^2 - (2-\alpha)\Delta n^2}{4m^2 + n^3 - n^2}. \tag{7}$$

Remark 2.7. *If $0 \leq \alpha < 1$ and $\Delta > \frac{2m^2}{n^2} + \frac{m}{n}$, then the lower bound in (7) is better than the lower bound $\lambda_1(A_\alpha(G)) \geq \frac{2m}{n}$ given in Corollary 19 of [17].*

The following theorem is a generalization of Theorem 2.3.

Theorem 2.4. *Let $0 \leq \alpha < 1$ and G be a connected graph with n vertices and m edges.*

(i) *If $y_i, y_j > 0$ and $v_iv_j \notin E(G)$ for $i \neq j$ and $i, j = 1, 2, \ldots, n$, then*

$$\lambda_1(A_\alpha(G)) \geq \frac{\alpha(d_iy_i^2 + d_jy_j^2) + 2(1-\alpha)(d_iy_i + d_jy_j) + 2m - (2-\alpha)(d_i + d_j)}{y_i^2 + y_j^2 + n - 2}.$$

(ii) *If $y_i, y_j > 0$ and $v_iv_j \in E(G)$ for $i \neq j$ and $i, j = 1, 2, \ldots, n$, then*

$$\lambda_1(A_\alpha(G)) \geq \frac{\alpha(d_iy_i^2 + d_jy_j^2) + 2(1-\alpha)[y_iy_j + (d_i-1)y_i + (d_j-1)y_j] + \Upsilon}{y_i^2 + y_j^2 + n - 2},$$

where $\Upsilon = 2m - (2-\alpha)(d_i + d_j) + 2(1-\alpha)$.

Proof. Let $X = \frac{1}{\sqrt{y_i^2+y_j^2+n-2}}(y_i, y_j, 1, \ldots, 1)^T$, where $y_i, y_j > 0$. If $v_iv_j \notin E(G)$, then by the Rayleigh-Ritz theorem we have

$$\begin{aligned}\lambda_1 &\geq X^TA_\alpha(G)X \\ &= \alpha\sum_{u\in V(G)} d_ux_u^2 + 2(1-\alpha)\sum_{uv\in E(G)} x_ux_v \\ &= \frac{\alpha(d_iy_i^2+d_jy_j^2)+2(1-\alpha)(d_iy_i+d_jy_j)+2m-(2-\alpha)(d_i+d_j)}{y_i^2+y_j^2+n-2}.\end{aligned}$$

If $v_iv_j \in E(G)$, then again by the Rayleigh-Ritz theorem we have

$$\begin{aligned}\lambda_1 &\geq X^TA_\alpha(G)X \\ &= \alpha\sum_{u\in V(G)} d_ux_u^2 + 2(1-\alpha)\sum_{uv\in E(G)} x_ux_v \\ &= \frac{\alpha(d_iy_i^2+d_jy_j^2)+2(1-\alpha)[y_iy_j+(d_i-1)y_i+(d_j-1)y_j]+\Upsilon}{y_i^2+y_j^2+n-2},\end{aligned}$$

where $\Upsilon = 2m-(2-\alpha)(d_i+d_j)+2(1-\alpha)$. This completes the proof. □

3. Lower bounds on the A_α-spread

Lemma 3.1 (see [16]). *If M is an $n\times n$ real symmetric matrix, then*

$$S_M = \max\left\{\left|X^TMX - Y^TMY\right| : ||X|| = ||Y|| = 1\right\}.$$

Theorem 3.1. *Let G be a connected graph with n vertices. If $0 \leq \alpha < 1$, then*

$$S_\alpha(G) \geq \begin{cases}\frac{1}{Z^{(t)}}[\alpha Z^{(t+1)} + 2(1-\alpha)R^{(\frac{t}{2})}] - \frac{\alpha d_1d_2^t + \alpha d_2d_1^t - 2(1-\alpha)d_1^{\frac{t}{2}}d_2^{\frac{t}{2}}}{d_1^t+d_2^t}, & v_1v_2 \in E(G), \\ \frac{1}{Z^{(t)}}[\alpha Z^{(t+1)} + 2(1-\alpha)R^{(\frac{t}{2})}] - \frac{\alpha d_1d_2^t + \alpha d_2d_1^t}{d_1^t+d_2^t}, & v_1v_2 \notin E(G).\end{cases}$$

Proof. Let $X = \frac{1}{\sqrt{Z^{(t)}}}\left(d_1^{\frac{t}{2}}, d_2^{\frac{t}{2}}, \ldots, d_n^{\frac{t}{2}}\right)^T$ and $Y = \frac{1}{\sqrt{d_1^t+d_2^t}}\left(d_2^{\frac{t}{2}}, -d_1^{\frac{t}{2}}, 0, 0, \ldots, 0\right)^T$. By Lemma 3.1, we have

$$\begin{aligned}S_\alpha(G) &= \lambda_1(A_\alpha(G)) - \lambda_n(A_\alpha(G)) \\ &= \max\left\{\left|X^TA_\alpha(G)X - Y^TA_\alpha(G)Y\right| : ||X|| = ||Y|| = 1\right\} \\ &\geq X^TA_\alpha(G)X - Y^TA_\alpha(G)Y \\ &= \alpha\sum_{v_i\in V(G)} d_ix_i^2 + 2(1-\alpha)\sum_{v_iv_j\in E(G)} x_ix_j - \alpha\sum_{v_i\in V(G)} d_iy_i^2 - 2(1-\alpha)\sum_{v_iv_j\in E(G)} y_iy_j \\ &= \begin{cases}\frac{1}{Z^{(t)}}[\alpha Z^{(t+1)} + 2(1-\alpha)R^{(\frac{t}{2})}] - \frac{\alpha d_1d_2^t + \alpha d_2d_1^t - 2(1-\alpha)d_1^{\frac{t}{2}}d_2^{\frac{t}{2}}}{d_1^t+d_2^t}, & v_1v_2 \in E(G), \\ \frac{1}{Z^{(t)}}[\alpha Z^{(t+1)} + 2(1-\alpha)R^{(\frac{t}{2})}] - \frac{\alpha d_1d_2^t + \alpha d_2d_1^t}{d_1^t+d_2^t}, & v_1v_2 \notin E(G).\end{cases}\end{aligned}$$

□

Corollary 3.1. *Let G be a connected k-regular graph with n vertices. If $0 \leq \alpha < 1$, then*

$$S_\alpha(G) \geq (1-\alpha)(k+1).$$

The equality holds for $G \cong K_n$.

By taking $d_1 = \Delta$ and $d_2 = \delta$ in Theorem 3.1, one gets the next result.

Corollary 3.2. *Let G be a connected graph with n vertices. If $0 \le \alpha < 1$, then*

$$S_\alpha(G) \ge \begin{cases} \frac{1}{Z^{(t)}}[\alpha Z^{(t+1)} + 2(1-\alpha)R^{(\frac{t}{2})}] - \frac{\alpha\Delta\delta^t + \alpha\delta\Delta^t - 2(1-\alpha)\Delta^{\frac{t}{2}}\delta^{\frac{t}{2}}}{\Delta^t + \delta^t}, & v_1v_2 \in E(G), \\ \frac{1}{Z^{(t)}}[\alpha Z^{(t+1)} + 2(1-\alpha)R^{(\frac{t}{2})}] - \frac{\alpha\Delta\delta^t + \alpha\delta\Delta^t}{\Delta^t + \delta^t}, & v_1v_2 \notin E(G). \end{cases}$$

In particular, by taking $t = 0, 1, 2$ in Corollary 3.2, one obtain the following corollaries.

Corollary 3.3. *Let G be a connected graph with n vertices and m edges. If $0 \le \alpha < 1$, then*

$$S_\alpha(G) \ge \begin{cases} \frac{2m}{n} - \frac{\alpha(\Delta+\delta)}{2} + 1 - \alpha, & v_1v_2 \in E(G), \\ \frac{2m}{n} - \frac{\alpha(\Delta+\delta)}{2}, & v_1v_2 \notin E(G). \end{cases}$$

Corollary 3.4. *Let G be a connected graph with n vertices and m edges. If $0 \le \alpha < 1$, then*

$$S_\alpha(G) \ge \begin{cases} \frac{1}{2m}[\alpha M_1 + 2(1-\alpha)R_{-1}] - \frac{2\alpha\Delta\delta - 2(1-\alpha)\sqrt{\Delta\delta}}{\Delta+\delta}, & v_1v_2 \in E(G), \\ \frac{1}{2m}[\alpha M_1 + 2(1-\alpha)R_{-1}] - \frac{2\alpha\Delta\delta}{\Delta+\delta}, & v_1v_2 \notin E(G). \end{cases}$$

Corollary 3.5. *Let G be a connected graph with n vertices. If $0 \le \alpha < 1$, then*

$$S_\alpha(G) \ge \begin{cases} \frac{1}{M_1}[\alpha F + 2(1-\alpha)M_2] - \frac{\Delta\delta(\alpha\Delta + \alpha\delta + 2\alpha - 2)}{\Delta^2+\delta^2}, & v_1v_2 \in E(G), \\ \frac{1}{M_1}[\alpha F + 2(1-\alpha)M_2] - \frac{\alpha\Delta\delta(\Delta+\delta)}{\Delta^2+\delta^2}, & v_1v_2 \notin E(G). \end{cases}$$

Theorem 3.2. *Let G be a connected graph with n vertices. If $0 \le \alpha < 1$, then*

$$S_\alpha(G) \ge \begin{cases} \frac{1}{\mathcal{E}}[\alpha\xi^c + 2(1-\alpha)\zeta^{(\frac{1}{2})}] - \frac{\alpha d_1\varepsilon_2 + \alpha d_2\varepsilon_1 - 2(1-\alpha)\sqrt{\varepsilon_1\varepsilon_2}}{\varepsilon_1+\varepsilon_2}, & v_1v_2 \in E(G), \\ \frac{1}{\mathcal{E}}[\alpha\xi^c + 2(1-\alpha)\zeta^{(\frac{1}{2})}] - \frac{\alpha d_1\varepsilon_2 + \alpha d_2\varepsilon_1}{\varepsilon_1+\varepsilon_2}, & v_1v_2 \notin E(G). \end{cases}$$

Proof. Let $X = \frac{1}{\sqrt{\mathcal{E}}}(\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}, \ldots, \sqrt{\varepsilon_n})^T$ and $Y = \frac{1}{\sqrt{\varepsilon_1+\varepsilon_2}}(\sqrt{\varepsilon_2}, -\sqrt{\varepsilon_1}, 0, 0, \cdots, 0)^T$. By Lemma 3.1, we have

$$\begin{aligned} S_\alpha(G) &= \lambda_1(A_\alpha(G)) - \lambda_n(A_\alpha(G)) \\ &= \max\left\{ \left|X^T A_\alpha(G) X - Y^T A_\alpha(G) Y\right| : \|X\| = \|Y\| = 1 \right\} \\ &\ge X^T A_\alpha(G) X - Y^T A_\alpha(G) Y \\ &= \alpha \sum_{v_i \in V(G)} d_i x_i^2 + 2(1-\alpha) \sum_{v_iv_j \in E(G)} x_i x_j - \alpha \sum_{v_i \in V(G)} d_i y_i^2 - 2(1-\alpha) \sum_{v_iv_j \in E(G)} y_i y_j \\ &= \begin{cases} \frac{1}{\mathcal{E}}[\alpha\xi^c + 2(1-\alpha)\zeta^{(\frac{1}{2})}] - \frac{\alpha d_1\varepsilon_2 + \alpha d_2\varepsilon_1 - 2(1-\alpha)\sqrt{\varepsilon_1\varepsilon_2}}{\varepsilon_1+\varepsilon_2}, & v_1v_2 \in E(G), \\ \frac{1}{\mathcal{E}}[\alpha\xi^c + 2(1-\alpha)\zeta^{(\frac{1}{2})}] - \frac{\alpha d_1\varepsilon_2 + \alpha d_2\varepsilon_1}{\varepsilon_1+\varepsilon_2}, & v_1v_2 \notin E(G). \end{cases} \end{aligned}$$

□

If one takes $\varepsilon_1 = \phi(G)$ and $\varepsilon_2 = \rho(G)$, then the following corollary is obtained.

Corollary 3.6. *Let G be a connected graph with n vertices. If $0 \le \alpha < 1$, then*

$$S_\alpha(G) \ge \begin{cases} \frac{1}{\mathcal{E}}[\alpha\xi^c + 2(1-\alpha)\zeta^{(\frac{1}{2})}] - \frac{\alpha d_1\rho + \alpha d_2\phi - 2(1-\alpha)\sqrt{\phi\rho}}{\phi+\rho}, & v_1v_2 \in E(G), \\ \frac{1}{\mathcal{E}}[\alpha\xi^c + 2(1-\alpha)\zeta^{(\frac{1}{2})}] - \frac{\alpha d_1\rho + \alpha d_2\phi}{\phi+\rho}, & v_1v_2 \notin E(G). \end{cases}$$

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