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Research Article

Some new integral inequalities via general forms of proportional fractional integral operators

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Abstract

In this paper, some new integral inequalities for integrable geometrically convex mappings via the general forms of proportional fractional integral operators are proved. Basic definitions, various classical inequalities and generalized proportional fractional integral operators are used to prove the main findings.

Keywords: AG-convexity; generalized proportional fractional operators; Hölder inequality; general Cauchy inequality.

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1. Introduction and preliminaries

Fractional analysis has brought a new dimension to many fields in mathematics and has become one of the most popular topics in recent years with its applications in several disciplines such as engineering, physics, modeling and control theory. Researchers have started to work intensively on fractional integral and derivative operators, and many new concepts and new applications have been included in the literature. The new features added by each new operator tries to prove their effectiveness in the real world problems solutions and the adventure continues in the search for the most effective operators. Many studies were conducted with the help of these operators to explain physical phenomena and demonstrate wide usage area in inequality theory (see the papers [3, 7, 12, 19, 20, 24, 27, 30, 34–36]). Now, we take a look at fractional integrals from a historical perspective by recalling these operators.

Definition 1.1. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(\epsilon) = \frac{1}{\Gamma(\alpha)} \int_a^{\epsilon} (\epsilon - \kappa)^{\alpha-1} f(\kappa) d\kappa, \qquad \epsilon > a$$

and

$$J^{\alpha}_{b-}f(\epsilon) = \frac{1}{\Gamma(\alpha)} \int_{\epsilon}^{b} (\kappa - \epsilon)^{\alpha-1} f(\kappa) d\kappa, \qquad \epsilon < b$$

respectively. Here $\Gamma(\epsilon)$ is the Gamma function and its definition is

$$\Gamma(\epsilon) = \int_0^{\infty} e^{-\epsilon} \epsilon^{\kappa-1} d\kappa.$$

It is to be noted that $J^{0}_{a+}f(\epsilon) = J^{0}_{b-}f(\epsilon) = f(\epsilon)$ in the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Riemann-Liouville integral operators are presented as a generalization of classical integrals. Then a more general version of this useful operator is given as follows.

Definition 1.2. [22] *Let (a, b) with $-\infty < a < b < \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $Re(\alpha) > 0$. Also let h be a strictly increasing function on (a, b), having a continuous derivative h' on (a, b). The generalized left and right sided Riemann-Liouville fractional integrals of a function f with respect to another function h on $[a, b]$ defined as*

$${}^{h}_{a+}I^{\alpha} f(\epsilon) = \frac{1}{\Gamma(\alpha)} \int_a^{\epsilon} (h(\epsilon) - h(\kappa))^{\alpha-1} f(\kappa) h'(\kappa) d\kappa, \qquad \epsilon > a$$

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and

$$ {}^{h}_{b-}I^{\alpha}f(\epsilon)=\frac{1}{\Gamma(\alpha)}\int_{\epsilon}^{b}(h(\kappa)-h(\epsilon))^{\alpha-1}f(\kappa)h'(\kappa)d\kappa, \qquad \epsilon<b. $$

In [15], Jarad et al. investigated the generalized proportional fractional integrals with a different kernel structure that satify several important properties as follows:

Definition 1.3. *The left and right generalized proportional fractional integral operators are respectively defined by*

$$ {}_{a+}\mathcal{J}^{\alpha,\zeta}f(\epsilon)=\frac{1}{\zeta^{\alpha}\Gamma(\alpha)}\int_{a}^{\epsilon}e^{\left[\frac{\zeta-1}{\zeta}(\epsilon-\kappa)\right]}(\epsilon-\kappa)^{\alpha-1}f(\kappa)d\kappa, \qquad \epsilon>a $$

and

$$ {}_{b-}\mathcal{J}^{\alpha,\zeta}f(\epsilon)=\frac{1}{\zeta^{\alpha}\Gamma(\alpha)}\int_{\epsilon}^{b}e^{\left[\frac{\zeta-1}{\zeta}(\kappa-\epsilon)\right]}(\kappa-\epsilon)^{\alpha-1}f(\kappa)d\kappa, \qquad \epsilon<b $$

where $\zeta\in(0,1]$ *and* $\alpha\in\mathbb{C}$ *and* $\mathbb{R}(\alpha)>0$.

We will continue with the Hadamard integral operators and the Katugampola integral operators, which are a general variant of the Riemann-Liouville integral operators as follows:

Definition 1.4. [21] *Let* a,b *be two non-negative real numbers satisfying the inequality* $a<b$, *and* α,ρ *be two positive real numbers and* $f:[a,b]\to\mathbb{R}$ *be an integrable function. The left and right Katugampola fractional integrals defined as*

$$ {}_{a+}\mathcal{I}^{\alpha}f(\epsilon)=\frac{1}{\Gamma(\alpha)}\int_{a}^{\epsilon}\left(\frac{\epsilon^{\rho}-\kappa^{\rho}}{\rho}\right)^{\alpha-1}\frac{f(\kappa)d\kappa}{\kappa^{1-\rho}}, \qquad \epsilon>a $$

and

$$ {}_{b-}\mathcal{I}^{\alpha}f(\epsilon)=\frac{1}{\Gamma(\alpha)}\int_{\epsilon}^{b}\left(\frac{\kappa^{\rho}-\epsilon^{\rho}}{\rho}\right)^{\alpha-1}\frac{f(\kappa)d\kappa}{\kappa^{1-\rho}}, \qquad \epsilon<b. $$

The associated integral operator based on the Hadamard derivative operator is given as follows.

Definition 1.5. [22, 29] *Let* a,b *be two reals with* $0<a<b$ *and* $f(\kappa):[a,b]\to\mathbb{R}$ *be an integrable function. The left and right Hadamard fractional integrals of order* $\alpha>0$ *are defined as*

$$ {}_{a+}\mathfrak{F}^{\alpha}f(\epsilon)=\frac{1}{\Gamma(\alpha)}\int_{a}^{\epsilon}\frac{f(\kappa)}{\kappa\left(ln\frac{\epsilon}{\kappa}\right)^{1-\alpha}}d\kappa, \qquad \epsilon>a $$

and

$$ {}_{b-}\mathfrak{F}^{\alpha}f(\epsilon)=\frac{1}{\Gamma(\alpha)}\int_{\epsilon}^{b}\frac{f(\kappa)}{\kappa\left(ln\frac{\kappa}{\epsilon}\right)^{1-\alpha}}d\kappa, \qquad \epsilon<b. $$

Within the scope of fractional analysis studies, the search for the operator with the most effective and general kernels led the researchers to define the operator named generalized proportional Hadamard fractional integrals. This operator, which has a different structure, is given as follows.

Definition 1.6. [26] *The left and right generalized proportional Hadamard fractional integrals of order* $\alpha>0$ *and proportionality index* $\zeta\in(0,1]$ *is defined by*

$$ {}_{a+}\mathcal{F}^{\alpha,\zeta}f(\epsilon)=\frac{1}{\zeta^{\alpha}\Gamma(\alpha)}\int_{a}^{\epsilon}\frac{e^{\left[\frac{\zeta-1}{\zeta}(ln\frac{\epsilon}{\kappa})\right]}}{\left(ln\frac{\epsilon}{\kappa}\right)^{1-\alpha}}\frac{f(\kappa)}{\kappa}d\kappa, \qquad \epsilon>a $$

and

$$ {}_{b-}\mathcal{F}^{\alpha,\zeta}f(\epsilon)=\frac{1}{\zeta^{\alpha}\Gamma(\alpha)}\int_{a}^{\epsilon}\frac{e^{\left[\frac{\zeta-1}{\zeta}(ln\frac{\kappa}{\epsilon})\right]}}{\left(ln\frac{\kappa}{\epsilon}\right)^{1-\alpha}}\frac{f(\kappa)}{\kappa}d\kappa, \qquad \epsilon<b. $$

On all of these, the generalized proportional fractional (GPF) integral operator in the sense of another function h has been defined as follows.

Definition 1.7. [18, 28] *Let* $f\in X_h^q(0,\infty)$, *there is an increasing, positive monotone function* h *defined on* $[0,\infty)$ *having continuous derivative* h' *on with* $h(0)=0$. *Then the left-sided and right-sided GPF-integral operator of a function* f *in the sense of another function* h *of order* $\alpha>0$ *are stated as:*

$$ {}^{h}_{a+}\mathfrak{I}^{\alpha,\zeta}f(\epsilon)=\frac{1}{\zeta^{\alpha}\Gamma(\alpha)}\int_{a}^{\epsilon}e^{\left[\frac{\zeta-1}{\zeta}(h(\epsilon)-h(\kappa))\right]}(h(\epsilon)-h(\kappa))^{\alpha-1}f(\kappa)h'(\kappa)d\kappa, \qquad \epsilon>a $$

and

$$ {}^{h}_{b-}\mathfrak{I}^{\alpha,\zeta}f(\epsilon)=\frac{1}{\zeta^{\alpha}\Gamma(\alpha)}\int_{\epsilon}^{b}e^{\left[\frac{\zeta-1}{\zeta}(h(\kappa)-h(\epsilon))\right]}(h(\kappa)-h(\epsilon))^{\alpha-1}f(\kappa)h'(\kappa)d\kappa, \qquad \epsilon<b $$

where $\zeta\in(0,1]$ *and* $\alpha\in\mathbb{C}$ *and* $\mathbb{R}(\alpha)>0$.

If we set the parameters with different choices in Definition 1.7, one can obtain Riemann-Liouville integrals, generalized Riemann-Liouville fractional integrals, generalized proportional fractional integrals, Katugampola fractional integrals, Hadamard fractional integrals and generalized proportional Hadamard fractional integrals.
In [25], Pečarić et al. mentioned about some different classes of convex functions as followings:

A function $f : I \to [0,\infty)$ is said to be $\log -$convex or multiplicatively convex ($AG-$convex) if $\log f$ is convex, or, equivalently, for all $x, y \in I$ and $t \in [0,1]$ one has the inequality:

$$f(tx + (1-t)y) \leq f(x)^t f(y)^{1-t}.$$

A function $f : I \to [0,\infty)$ is said to be $GA-$convex if for all $x, y \in I$ and $t \in [0,1]$, one has the inequality:

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y).$$

Example 1.1. *The function* $f(x) = \frac{1}{x}$ *,* $x \in (0,1)$ *is* $log-$*convex on* $(0,1)$.

The researchers have performed numerous research articles on various integral inequalities by using different kinds of fractional integral operators with applications, see [1, 2, 4–6, 8–11, 13, 14, 16–18, 23, 28, 31–33].

The main aim of this paper is to establish some new integral inequalities for product of two geometrically convex functions via the general forms of proportional fractional integral operators.

2. Main results

Theorem 2.1. *Assume that* $f, g : (0,\infty) \to R$ *be differentiable function and* ψ *be a positive monotone increasing function that defined on* $[0,\infty)$. *Let* $\psi'(\tau)$ *be continuous and* $\psi(0) = 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $|\psi'(\tau)| \leq M$. *Then, if* f *and* g *are* $AG-$*convex functions, we have the following inequality;*

$$\begin{aligned}\left({}^{\Psi}\mathcal{T}^{\eta,\xi}_{0^+,\tau} fg\right)(\tau) \leq\ & \frac{fg(b) - fg(a)}{b-a} \frac{M^{\frac{p-1}{p}}}{\xi^\eta \Gamma(\eta)} \left(\frac{e^{\frac{\xi-1}{\xi} p\psi(\tau)} - 1}{\frac{\xi-1}{\xi} p}\right)^{\frac{1}{p}} \left(\int_0^\tau \left|(\psi(\tau) - \psi(x))^{\eta-1}(x-b)\right|^q dx\right)^{\frac{1}{q}} \\ & + \frac{fg(b)}{\xi^\eta \Gamma(\eta)} \frac{\psi^\eta(\tau)}{\left(\frac{1-\xi}{\xi}\psi(\tau)\right)^\eta} \left(\Gamma(\eta) - \Gamma\left(\eta, \frac{1-\xi}{\xi}\psi(\tau)\right)\right)\end{aligned}$$

for $\xi \in (0,1)$*,* $\eta \in \mathbb{C}$*,* $Re(\eta) > 0, \tau > 0$.

Proof. By using the definition of AG-convex functions, we can write

$$fg(ta + (1-t)b) \leq\ [fg(a)]^t [fg(b)]^{1-t}.$$

By changing of the variable such that $x = ta + (1-t)b$, we have

$$fg(x) \leq\ [fg(a)]^{\frac{b-x}{b-a}}\ [fg(b)]^{\frac{x-a}{b-a}}.$$

By applying the General Cauchy inequality to above inequality, we get

$$fg(x) \leq \left(\frac{b-x}{b-a}\right) fg(a) + \left(\frac{x-a}{b-a}\right) fg(b).$$

Namely

$$fg(x) \leq \frac{fg(b) - fg(a)}{b-a}(x-b) + fg(b).$$

If we denote $A = \frac{fg(b)-fg(a)}{b-a}$ and $B = fg(b)$, we get

$$fg(x) \leq A(x-b) + B.$$

By multiplying both sides of the resulting inequality with

$$\frac{1}{\xi^\eta \Gamma(\eta)} \int_0^\tau \frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau) - \psi(x))^{1-\eta}} \psi'(x)\, dx,$$

we get

$$\left({}^{\Psi}\mathcal{T}^{\eta,\xi}_{0^+,\tau} fg\right)(\tau) \leq \frac{A}{\xi^\eta \Gamma(\eta)} \int_0^\tau \frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau) - \psi(x))^{1-\eta}} (x-b)\psi'(x)\, dx + \frac{B}{\xi^\eta \Gamma(\eta)} \int_0^\tau \frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau) - \psi(x))^{1-\eta}} \psi'(x)\, dx.$$

By applying the well known Hölder inequality, we get

$$\begin{aligned}\left({}^{\Psi}\mathcal{T}^{\eta,\xi}_{0^+,\tau}fg\right)(\tau) &\leq \frac{A}{\xi^{\eta}\Gamma(\eta)}\left(\int_0^{\tau}\left|e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}\psi'(x)\right|^{p}dx\right)^{\frac{1}{p}}\left(\int_0^{\tau}\left|(\psi(\tau)-\psi(x))^{\eta-1}(x-b)\right|^{q}dx\right)^{\frac{1}{q}}\\ &+\frac{B}{\xi^{\eta}\Gamma(\eta)}\int_0^{\tau}\frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau)-\psi(x))^{1-\eta}}\psi'(x)\,dx.\end{aligned}$$

By making use of some necessary operation and by taking into account $|\psi'(x)|\leq M$, we obtain

$$\begin{aligned}\left({}^{\Psi}\mathcal{T}^{\eta,\xi}_{0^+,\tau}fg\right)(\tau) &\leq \frac{fg(b)-fg(a)}{b-a}\frac{M^{\frac{p-1}{p}}}{\xi^{\eta}\Gamma(\eta)}\left(\frac{e^{\frac{\xi-1}{\xi}p\psi(\tau)}-1}{\frac{\xi-1}{\xi}p}\right)^{\frac{1}{p}}\left(\int_0^{\tau}\left|(\psi(\tau)-\psi(x))^{\eta-1}(x-b)\right|^{q}dx\right)^{\frac{1}{q}}\\ &+\frac{fg(b)}{\xi^{\eta}\Gamma(\eta)}\frac{\psi^{\eta}(\tau)}{\left(\frac{1-\xi}{\xi}\psi(\tau)\right)^{\eta}}\left(\Gamma(\eta)-\Gamma\left(\eta,\frac{1-\xi}{\xi}\psi(\tau)\right)\right).\end{aligned}$$

The proof is completed. □

Theorem 2.2. *Assume that* $f,g:(0,\infty)\to R$ *be differentiable function and* ψ *be a positive monoton increasing function that defined on [0,∞). Let* $\psi'(\tau)$ *be continuous and* $\psi(0)=0,\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, |\psi'(\tau)|\leq M$. *Then, if* f *and* g *are* $AG-$*convex functions, we have the following inequality;*

$$\begin{aligned}\left({}^{\Psi}\mathcal{T}^{\eta,\xi}_{0^+,\tau}fg\right)(\tau) &\leq \frac{fg(b)-fg(a)}{b-a}\frac{M^{\frac{p-1}{p}}}{\xi^{\eta}\Gamma(\eta)}\left(\frac{\psi^{p(\eta-1)+1}(\tau)}{\left(\frac{1-\xi}{\xi}p\psi(\tau)\right)^{p(\eta-1)+1}}\left(\Gamma(p(\eta-1)+1)-\Gamma\left(p(\eta-1)+1,\frac{1-\xi}{\xi}p\psi(\tau)\right)\right)\right)^{\frac{1}{p}}\\ &\times\left(\frac{|\tau-b|^{q+1}-|-b|^{q+1}}{q+1}\right)^{\frac{1}{q}}+\frac{fg(b)}{\xi^{\eta}\Gamma(\eta)}\frac{\psi^{\eta}(\tau)}{\left(\frac{1-\xi}{\xi}\psi(\tau)\right)^{\eta}}\left(\Gamma(\eta)-\Gamma\left(\eta,\frac{1-\xi}{\xi}\psi(\tau)\right)\right)\end{aligned}$$

for $\xi\in(0,1)$, $\eta\in\mathbb{C}$, $Re(\eta)>0, Re(p(\eta-1))>0, \tau>0$.

Proof. By using the definition of $AG-$convexity, we can write

$$fg(ta+(1-t)b)\leq [fg(a)]^{t}[fg(b)]^{1-t}.$$

By changing of the variable, we get

$$fg(x)\leq [fg(a)]^{\frac{b-x}{b-a}}[fg(b)]^{\frac{x-a}{b-a}}.$$

By applying General Cauchy inequality and by making some operations, we have

$$fg(x)\leq\frac{fg(b)-fg(a)}{b-a}(x-b)+fg(b).$$

If we denote $A=\frac{fg(b)-fg(a)}{b-a}$, $B=fg(b)$, then

$$fg(x)\leq A(x-b)+B.$$

By multiplying both sides of the above inequality

$$\frac{1}{\xi^{\eta}\Gamma(\eta)}\int_0^{\tau}\frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau)-\psi(x))^{1-\eta}}\psi'(x)\,dx,$$

we obtain

$$\left({}^{\Psi}\mathcal{T}^{\eta,\xi}_{0^+,\tau}fg\right)(\tau) \leq \frac{A}{\xi^{\eta}\Gamma(\eta)}\int_0^{\tau}\frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau)-\psi(x))^{1-\eta}}(x-b)\psi'(x)\,dx+\frac{B}{\xi^{\eta}\Gamma(\eta)}\int_0^{\tau}\frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau)-\psi(x))^{1-\eta}}\psi'(x)\,dx.$$

By applying Hölder inequality to the resulting inequality, we get

$$\left({}^{\Psi}\mathcal{T}^{\eta,\xi}_{0^+,\tau}fg\right)(\tau) \leq \frac{A}{\xi^{\eta}\Gamma(\eta)}\left(\int_0^{\tau}\left|\frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau)-\psi(x))^{1-\eta}}\psi'(x)\right|^{p}dx\right)^{\frac{1}{p}}\left(\int_0^{\tau}|(x-b)|^{q}dx\right)^{\frac{1}{q}}$$

$$+\frac{B}{\xi^{\eta}\Gamma(\eta)}\int_0^{\tau}\frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau)-\psi(x))^{1-\eta}}\psi'(x)\,dx.$$

By using the boundedness of $|\psi'(x)|\leq M$ and some further calculation, we have

$$\begin{aligned}\left({}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\xi}fg\right)(\tau) &\leq \frac{fg(b)-fg(a)}{b-a}\frac{M^{\frac{p-1}{p}}}{\xi^{\eta}\Gamma(\eta)}\left(\frac{\psi^{p(\eta-1)+1}(\tau)}{\left(\frac{1-\xi}{\xi}p\psi(\tau)\right)^{p(\eta-1)+1}}\left(\Gamma(p(\eta-1)+1)-\Gamma\left(p(\eta-1)+1,\frac{1-\xi}{\xi}p\psi(\tau)\right)\right)\right)^{\frac{1}{p}}\\ &\times\left(\frac{|\tau-b|^{q+1}-|-b|^{q+1}}{q+1}\right)^{\frac{1}{q}}+\frac{fg(b)}{\xi^{\eta}\Gamma(\eta)}\frac{\psi^{\eta}(\tau)}{\left(\frac{1-\xi}{\xi}\psi(\tau)\right)^{\eta}}\left(\Gamma(\eta)-\Gamma\left(\eta,\frac{1-\xi}{\xi}\psi(\tau)\right)\right).\end{aligned}$$

This is the desired result. □

Theorem 2.3. *Assume that $f,g:(0,\infty)\to R$ be differentiable function and ψ be a positive monoton increasing function that defined on [0,∞). Let $\psi'(\tau)$ be continuous and $\psi(0)=0,\ \ p,q>1$ ve $\frac{1}{p}+\frac{1}{q}=1$, $|\psi'(\tau)|\leq M$. Then, if f and g are $GA-$convex functions, we have the following inequality;*

$$\begin{aligned}\left({}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\xi}fg\right)(\tau) &\leq \frac{A}{\xi^{\eta}\Gamma(\eta)}\left(\frac{M^{\frac{p-1}{p}}}{p}\frac{\psi^{p(\eta-1)+1}(\tau)}{\left(\frac{1-\xi}{\xi}p\psi(\tau)\right)^{p(\eta-1)+1}}\right.\\ &\left.\left(\Gamma(p(\eta-1)+1)-\Gamma\left(p(\eta-1)+1,\frac{1-\xi}{\xi}p\psi(\tau)\right)\right)+\frac{1}{q}\int_0^{\tau}\left|\log_{\frac{a}{b}}\frac{x}{b}\right|^q dx\right)\\ &+\frac{B}{\xi^{\eta}\Gamma(\eta)}\frac{\psi^{\eta}(\tau)}{\left(\frac{1-\xi}{\xi}\psi(\tau)\right)^{\eta}}\left(\Gamma(\eta)-\Gamma\left(\eta,\frac{1-\xi}{\xi}\psi(\tau)\right)\right)\end{aligned}$$

for $\xi\in(0,1)$, $\eta\in\mathbb{C}$, $Re(\eta)>0, Re(p(\eta-1))>0,\tau>0$.

Proof. By using the definition of $GA-$convexity, we can write

$$fg\left(a^tb^{1-t}\right)\leq tfg(a)+(1-t)fg(b).$$

By making use of some arrangements, we have

$$fg(x)\leq\ \log_{\frac{a}{b}}\frac{x}{b}\ fg(a)+\left(1-\log_{\frac{a}{b}}\frac{x}{b}\ \right)fg(b).$$

Namely,

$$fg(x)\leq\log_{\frac{a}{b}}\frac{x}{b}\ \ (fg(a)-fg(b))+fg(b).$$

By denoting $A=fg(a)-fg(b)$, $B=fg(b)$, we get

$$fg(x)\leq A\log_{\frac{a}{b}}\frac{x}{b}\ \ +B.$$

By multiplying the both sides of the above inequality by

$$\frac{1}{\xi^{\eta}\Gamma(\eta)}\int_0^{\tau}\frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau)-\psi(x))^{1-\eta}}\psi'(x)\,dx,$$

it yields,

$$\left({}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\xi}fg\right)(\tau)\ \leq\ \frac{A}{\xi^{\eta}\Gamma(\eta)}\int_0^{\tau}\frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau)-\psi(x))^{1-\eta}}\log_{\frac{a}{b}}\frac{x}{b}\ \psi'(x)\,dx+\frac{B}{\xi^{\eta}\Gamma(\eta)}\int_0^{\tau}\frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau)-\psi(x))^{1-\eta}}\psi'(x)\,dx.$$

By applying the Young inequality, we obtain

$$\begin{aligned}\left({}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\xi}fg\right)(\tau) &\leq \frac{A}{\xi^{\eta}\Gamma(\eta)}\left(\frac{1}{p}\int_0^{\tau}\left|\frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau)-\psi(x))^{1-\eta}}\psi'(x)\right|^p dx+\frac{1}{q}\int_0^{\tau}\left|\log_{\frac{a}{b}}\frac{x}{b}\right|^q dx\right)\\ &+\frac{B}{\xi^{\eta}\Gamma(\eta)}\int_0^{\tau}\frac{e^{\frac{\xi-1}{\xi}(\psi(\tau)-\psi(x))}}{(\psi(\tau)-\psi(x))^{1-\eta}}\psi'(x)\,dx.\end{aligned}$$

By making use of necessary operation and changing of the vairables, we get

$$\begin{aligned}\left({}^{\Psi}\mathcal{T}^{\eta,\xi}_{0^+,\tau}fg\right)(\tau) \quad &\leq \quad \frac{A}{\xi^{\eta}\Gamma(\eta)}\left(\frac{M^{\frac{p-1}{P}}}{p}\frac{\psi^{p(\eta-1)+1}(\tau)}{\left(\frac{1-\xi}{\xi}p\psi(\tau)\right)^{p(\eta-1)+1}}\right.\\
&\left.\left(\Gamma(p(\eta-1)+1)-\Gamma\left(p(\eta-1)+1,\frac{1-\xi}{\xi}p\psi(\tau)\right)\right)+\frac{1}{q}\int_0^{\tau}\left|\log_{\frac{a}{b}}\frac{x}{b}\right|^{q}dx\right)\\
&+\frac{B}{\xi^{\eta}\Gamma(\eta)}\frac{\psi^{\eta}(\tau)}{\left(\frac{1-\xi}{\xi}\psi(\tau)\right)^{\eta}}\left(\Gamma(\eta)-\Gamma\left(\eta,\frac{1-\xi}{\xi}\psi(\tau)\right)\right),\end{aligned}$$

which completes the proof. □

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