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Research Article

Weakly irreducible filter in strong quasi-ordered residuated systems

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Abstract

In this article, the notion of weakly irreducible filters in strong quasi-ordered residuated systems is introduced and analyzed. It is shown that any weakly irreducible filter is a prime (and therefore, irreducible) filter. It is also proved that if the lattice $\mathfrak{F}(A)$ of all filters in a strong quasi-ordered residuated system $\mathfrak{A}$ is distributive, then any irreducible filter in $\mathfrak{A}$ is weakly irreducible in $\mathfrak{A}$.

Keywords: quasi-ordered residuated system; strong quasi-ordered residuated system; prime and irreducible filters; weakly irreducible filter.

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1. Introduction

Algebraic systems derived from logical systems are the focus of many studies (for example, see [5, 10]). These algebraic systems and their substructures have their own importance in mathematics. The concept of filters is mostly studied by the researchers working in logical algebras and in systems related to them [4, 10].

The idea of residuated relational systems ordered under a quasi-order relation, or quasi-ordered residuated systems (briefly, QRSs (and QRS for quasi-ordered residuated system)), was proposed by Bonzio and Chajda [2] in 2018. Previously, this concept was discussed in [1]. A similar system is considered also in [7]. The present author introduced and developed the concepts of ideals [18] and filters [11] (and their several types such as implicative filters [13], weak implicative filters [16], comparative filters [14] and associated filters [12]) in this algebraic structure.

The concept of strong QRSs was introduced and discussed in [15]. In such systems, comparative and implicative filters coincide. The specificity of strong QRSs is that these systems allow us to determine the least upper bound $x \sqcup y$ for each pair of elements x and y belonging to a strong quasi-ordered residuated system $\mathfrak{A}$. In [15], it was shown that $(\mathfrak{A}, \sqcup)$ is an upper semi-lattice. In [17], the idea of prime filter (of the first kind) and the concept of irreducible filters in strong QRSs were introduced and analyzed. It was shown in [17] that every prime filter in such a system is an irreducible filter.

In this article, as a continuation of the studies [15, 17], the concept of weakly irreducible filters in strong QRSs is introduced (Definition 3.1) and analyzed (Theorem 3.1). It is shown that any weakly irreducible filter is a prime (Theorem 3.2) (and, therefore, an irreducible (Theorem 3.3)) filter. It is also proved that if the lattice $\mathfrak{F}(A)$ of all filters in a strong quasi-ordered residuated system $\mathfrak{A}$ is distributive, then any irreducible filter in $\mathfrak{A}$ is weakly irreducible in $\mathfrak{A}$ (Theorem 3.4).

2. Preliminaries

In this section, terminology and notation and some processes with them (i.e., their interrelationships) taken from [2, 11–17] and used in this report, are given. The notation and terminology that are used in this paper, but not given here, can be found in [3, 5, 6, 10].

Concept of quasi-ordered residuated systems

In [2], Bonzio and Chajda introduced and analyzed the idea of residual relational systems.

Definition 2.1 ([2], Definition 2.1)**.** *A* residuated relational system *is a structure* $\mathfrak{A} = \langle A, \cdot, \to, 1, R\rangle$*, where* $\langle A, \cdot, \to, 1\rangle$ *is an algebra of type* $\langle 2, 2, 0\rangle$ *and* R *is a binary relation on* A *and satisfying the following properties:*

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(1). $(A, \cdot, 1)$ *is a commutative monoid;*

(2). $(\forall x \in A)\,((x, 1) \in R)$*;*

(3). $(\forall x, y, z \in A)\,((x \cdot y, z) \in R \Longleftrightarrow (x, y \to z) \in R)$.

In Definition 2.1, we refer to the operation "$\cdot$" as multiplication, to "$\to$" as its residuum and to condition (3) as residuation. The basic properties for residuated relational systems are subsumed in the following (see Proposition 2.1 in [2]): If $\mathfrak{A} = \langle A, \cdot, \to, 1, R\rangle$ is a residuated relational system then

(4). $(\forall x, y \in A)\,(x \to y = 1 \Longrightarrow (x, y) \in R)$;

(5). $(\forall x \in A)\,((x, 1 \to 1) \in R)$;

(6). $(\forall x \in A)\,((1, x \to 1) \in R)$;

(7). $(\forall x, y, z \in A)\,(x \to y = 1 \Longrightarrow (z \cdot x, y) \in R)$;

(8). $(\forall x, y \in A)\,((x, y \to 1) \in R)$.

Recall that a *quasi-order relation* "$\preccurlyeq$" on a set A is a binary relation which is reflexive and transitive. A *quasi-ordered residuated system* (QRS) is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \to, 1, \preccurlyeq\rangle$, where $\preccurlyeq$ is a quasi-order relation in the monoid $(A, \cdot)$ (see Definition 3.1 in [2]).

Example 2.1. *Let* $A = \{1, a, b, c, d\}$ *and operations "$\cdot$" and "$\to$" defined on* A *as follows:*

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

and

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	a	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ *is a QRS where the relation "$\preccurlyeq$" is defined as follows*

$$\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (d,1), (b,b), (a,a), (c,c), (d,d), (c,a), (d,a), (d,b), (d,c)\}.$$

Example 2.2. *For a commutative monoid* A*, let* $\mathfrak{P}(A)$ *denote the powerset of* A *ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of* A*. Then* $\langle \mathfrak{P}(A), \cdot, \to, A, \subseteq\rangle$ *is a QRS in which the residuum are given by*

$$(\forall X, Y \in \mathfrak{P}(A))\,(Y \to X := \{z \in A : Yz \subseteq X\}).$$

Example 2.3. *Let* $\mathbb{R}$ *be the field of real numbers. Define a binary operations "$\cdot$" and "$\to$" on* $A = [0, 1] \subset \mathbb{R}$ *by*

$$(\forall\; x, y \in [0, 1])\,(x \cdot y := \max\{0, x + y - 1\}) \text{ and } x \to y := \min\{1, 1 - x + y\}).$$

Then, A *is a commutative monoid with the identity* 1 *and* $\langle A, \cdot, \to, \leqslant, 1\rangle$ *is a QRS.*

Example 2.4. *Any commutative residuated lattice* $\langle A, \cdot, \to, 0, 1, \sqcap, \sqcup, R\rangle$*, where* R *is a lattice quasi-order, is a QRS.*

Example 2.5. *Let* $A = \{1, a, b, c, d, e\}$ *and the operations "$\cdot$" and "$\to$" be defined on* A *as follows:*

$\cdot$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	b	a	c	d	e
b	b	b	a	c	d	e
c	c	b	c	c	d	e
d	d	d	d	d	e	e
e	e	e	e	e	e	d

and

$\to$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	c	1	1	1
b	1	c	1	1	1	1
c	1	c	c	1	1	1
d	1	d	d	d	1	e
e	1	e	e	e	d	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ *is a QRS, where the relation "$\preccurlyeq$" is defined as follows*

$$\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (d,1), (e,1), (a,a), (a,c), (a,d), (a,e), (b,b), (b,c), (b,d), (b,e), (c,c), (c,d), (c,e), (d,d), (e,e)\}.$$

It should be noted that the elements a *and* b *are not comparable.*

Example 2.6 ([9], Example 2.2)**.** *Every commutative residuated po-monoid (briefly, a CRPM)* $(A, \cdot, \rightarrow, 1, \leqslant)$ *is a quasi-ordered system.*

The following proposition shows the basic properties of quasi-ordered residuated systems (QRSs).

Proposition 2.1 ([2], Proposition 3.1)**.** *Let* A *be a QRS. Then*

(9). $(\forall\, x, y, z \in A)\,(x \preccurlyeq y \implies (x \cdot z \preccurlyeq y \cdot z \,\wedge\, z \cdot x \preccurlyeq z \cdot y))$;

(10). $(\forall\, x, y, z \in A)\,(x \preccurlyeq y \implies (y \rightarrow z \preccurlyeq x \rightarrow z \,\wedge\, z \rightarrow x \preccurlyeq z \rightarrow y))$;

(11). $(\forall\, x, y \in A)\,(x \cdot y \preccurlyeq x \,\wedge\, x \cdot y \preccurlyeq y)$.

Remark 2.1. *A quasi-ordered residuated system, generally speaking, differs from the commutative residuated lattice* $\langle A, \cdot, \rightarrow, 0, 1, \sqcap, \sqcup, R\rangle$ *where* R *is a lattice quasi-order. First, our observed system does not have to be limited from below. Second, the observed system does not have to be a lattice. However, the difference between a quasi-ordered relational system and a CRPM (see Example 2.6) is only in order relations since a quasi-order relation does not have to be antisymmetric. Additional information about the last-mentioned algebraic structure can be found in* [8].

A non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ is said to be a *filter* of $\mathfrak{A}$ (see Definition 3.1 in [11]) if it satisfies the following two conditions:

(F2). $(\forall\, u, v \in A)\,((u \in F \,\wedge\, u \preccurlyeq v) \implies v \in F)$, and

(F3). $(\forall\, u, v \in A)\,((u \in F \,\wedge\, u \rightarrow v \in F) \implies v \in F)$.

It was shown in [11] that if a non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the following two conditions:

(F0). $1 \in F$ and

(F1). $(\forall\, u, v \in A)\,((u \cdot v \in F \implies (u \in F \,\wedge\, v \in F))$.

If $\{F_k\}_k$ is a family of filters in $\mathfrak{A}$, then $\bigcap_k F_k$ is also a filter in $\mathfrak{A}$. If H is a nonempty subset of A, then

$$F(H) := \bigcap\{F : F \text{ is a filter in } \mathfrak{A} \text{ and } H \subseteq F\}$$

is a minimal filter in A containing H. If $\mathfrak{F}(A)$ is the family of all filters in a quasi-ordered residuated system $\mathfrak{A}$, then $(\mathfrak{F}(A).\sqcap, \sqcup)$ is a complete lattice (see Theorem 3.1 in [11]) where $S \sqcap T = S \cap T$ and $S \sqcup T = F(S \cup T)$ for any two filters S and T in $\mathfrak{A}$.

It needs to be mentioned here that the concept of filters in QRSs is somewhat different from the concept of filters in implicative algebras (for example, see [5, 10]). It is clear that a filter in the considered sense (a nonempty subset F that satisfies the conditions (F2) and (F3)) is a filter in the sense of its descriptions (a nonempty subset F that satisfies the conditions (F0) and (F3)) in the books [5, 10].

Strong QRSs

In this subsection, we analyze the concept of strong QRSs, which was introduced and analyzed in [15]. Considering the fact that the quasi-order relation "$\preccurlyeq$", which appears in the determination of this algebraic system, does not have to be antisymmetric, the following definition gets a clearer meaning. It is generally known that a quasi-order relation "$\preccurlyeq$" on a set A generates an equivalence relation "$\equiv_{\preccurlyeq} := \preccurlyeq \,\cap\, \preccurlyeq^{-1}$" on A. Due to properties (9) and (10), this equivalence is compatible with the operations in $\mathfrak{A}$. Thus, the relation "$\equiv_{\preccurlyeq}$" is a congruence on $\mathfrak{A}$.

Definition 2.2 ([15], Definition 3.1)**.** *A quasi-ordered residuated system* $\mathfrak{A}$ *is said to be a* strong QRS *if the following condition holds:*

(12). $(\forall\, u, v \in A)\,((u \rightarrow v) \rightarrow v \equiv_{\preccurlyeq} (v \rightarrow u) \rightarrow u)$.

Now, we give an example of a strong QRS.

Example 2.7. *Let* $A = \{1, a, b, c\}$ *and the operations "*$\cdot$*" and "*$\rightarrow$*" be defined on* A *as follows:*

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

and

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	c
c	1	b	b	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ *is a QRS where the relation "*$\preccurlyeq$*" is defined as follows*

$$\preccurlyeq := \{(1,1),(a,1),(b,1),(c,1),(a,a),(b,b),(c,c),(a,b),(a,c)\}.$$

From direct verification, it is proved that $\mathfrak{A}$ *is a strong QRS.*

The following theorem shows that in a strong QRS one can construct the least upper bound for each pair of its elements.

Theorem 2.1 ([15], Theorem 3.3)**.** *Let* $\mathfrak{A}$ *be a strong QRS. For any* $u, v \in A$*, the element*

$$u \sqcup v := (v \rightarrow u) \rightarrow u \equiv_{\preccurlyeq} (u \rightarrow v) \rightarrow v$$

is the least upper bound of u *and* v*.*

Theorem 2.2 ([15], Theorem 3.4)**.** *If* $\mathfrak{A}$ *is a strong QRS, then* $(\mathfrak{A}, \sqcup)$ *is a distributive upper semi-lattice in the following sense*

$$(\forall\, x, y, z \in A)\,((x \sqcup y) \sqcup z \equiv_{\preccurlyeq} (x \sqcup z) \sqcup (y \sqcup z)).$$

It should be noted here that the notion of distributive semi-lattice, used here, is somewhat different from the concept of distributive semi-lattice used in the book [6].

Definition 2.3 ([17], Definition 9)**.** *Let* F *be a filter of a strong quasi-ordered residuated system* $\mathfrak{A}$*. Then* F *is said to be a* prime filter *in* $\mathfrak{A}$ *if the following condition holds:*

(PF). $(\forall\, u, v \in A)\,(u \sqcup v \in F \implies (u \in F \vee v \in F))$.

Example 2.8. *Let* $A = \{1, a, b, c\}$ *and the operations "*$\cdot$*" and "*$\rightarrow$*" be defined on* A *as follows:*

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	a	a
c	c	b	a	a

and

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	1
c	1	b	c	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ *is a QRS where the relation "*$\preccurlyeq$*" is defined as follows*

$$\preccurlyeq := \{(1,1),(a,1),(b,1),(c,1),(a,a),(b,b),(c,c),(a,b),(a,c),(b,c)\}.$$

By direct verification, it is proved that $\mathfrak{A}$ *is a strong QRS. The only proper filter in this system is the subset* $F := \{1\}$*. It is concluded by direct checking that* F *is a prime filter.*

Example 2.9. *Let* $\mathfrak{A}$ *be as in Example 2.7. The subsets* $\{1\}$*,* $\{1, b\}$ *and* $\{1, c\}$ *are filters in* $\mathfrak{A}$*. It is checked that* $F_1 := \{1, b\}$ *and* $F_2 := \{1, c\}$ *are prime filters in* $\mathfrak{A}$ *while the filter* $\{1\}$ *is not prime because one has* $b \sqcup c = 1 \in \{1\}$ *but* $b \notin \{1\}$ *and* $c \notin \{1\}$*.*

Definition 2.4 ([17], Definition 10)**.** *A filter* F *of a quasi-ordered residuated system* $\mathfrak{A}$ *is said to be an* irreducible filter *in* $\mathfrak{A}$ *if for every two filters* S *and* T *of* $\mathfrak{A}$*, the following implication holds:*

(IRF). $(F = S \cap T \implies (F = S \vee F = T))$.

Theorem 2.3 ([17], Theorem 9)**.** *Any prime filter of a strong QRS is an irreducible filter.*

Example 2.10. *Let* A *be as in Example 2.5. It is determined that* $(A, \cdot, \rightarrow)$ *is a strong quasi-ordered system. For example,* $a \sqcup b = (a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a = c$ *is valid. The semi-lattice* $(A, \sqcup)$ *is not a lattice because there is no lower bound for the elements* a *and* b*.*

3. Main results: Weakly irreducible filters in strong QRSs

This section is the central part of this paper. The notion of *weakly irreducible filters* in a strong QRS is introduced in the following definition by weakening the condition that determine the concept of irreducible filters.

Definition 3.1. *Let F be a proper filter of a strong quasi-ordered residuated system $\mathfrak{A}$. The filter F is a* weakly irreducible filter *in $\mathfrak{A}$ if and only if for every pair of filters S and T of A satisfying $S \cap T \subseteq F$, either $S \subseteq F$ or $T \subseteq F$.*

The next theorem gives an important characterization of weakly irreducible filters in strong QRSs.

Theorem 3.1. *Let F be a proper filter in a quasi-ordered residuated system $\mathfrak{A}$. Then F is weakly irreducible if and only if the following condition holds:*

(WIrF). $(\forall\, x, y \in A)\,((x \notin F \wedge y \notin F) \implies (\exists\, z \notin F)\,(x \leqslant z \wedge y \leqslant z))$.

Proof. Assume that F is a weakly irreducible filter in $\mathfrak{A}$ and take the elements $x, y \in A$ such that $x \notin F$ and $y \notin F$. Let us design filters $F_x = F(F \cup \{x\})$ and $F_y = F(F \cup \{y\})$. It is obvious that $F \subseteq F_x \cap F_y$. If $F = F_x \cap F_y$ then we have $x \in F_x \subseteq F$ or $y \in F_y \subseteq F$ which is impossible because of the hypothesis. So, $F \subset F_x \cap F_y$. Thus, there exists an element $z \in F_x \cap F_y$ such that $z \notin F$. In addition to the above, it is obvious that $x \leqslant z$ and $y \leqslant z$ are valid.

Conversely, suppose that the filter F in a strong quasi-ordered residuated system $\mathfrak{A}$ satisfies the condition (WIrF) and let S and T be UP filters in $\mathfrak{A}$ such that $S \cap T \subseteq F$. Suppose now that $\neg(S \subseteq F)$ and $\neg(T \subseteq F)$ are valid. Then there exists an element $x \in S$ such that $x \notin F$ and there exists an element $y \in T$ such that $y \notin F$. By assumption, there exists a $z \notin F$ such that $x \leqslant z$ and $y \leqslant z$. On the other hand, we have $z \in S$ and $z \in T$ and consequently, $z \in S \cap T \subseteq F$ which is a contradiction. Thus, $S \subseteq F$ or $T \subseteq F$, proving that F is a weakly irreducible filter in $\mathfrak{A}$. □

Theorem 3.2. *Every weakly irreducible filter in a strong QRS is a prime filter.*

Proof. Let F be a weakly irreducible filter in a strong quasi-ordered residuated system $\mathfrak{A}$ and take $x, y \in A$ such that $x \sqcup y \in F$. If we assume that $x \notin F$ and $y \notin F$, then there exists an element $z \notin F$ such that $x \leqslant z$ and $y \leqslant z$. Since z is the upper bound for x and y, one has $x \sqcup y \leqslant z$. Then, $z \in F$. We arrive at a contradiction. Therefore, either $x \in F$ or $y \in F$, which means that F is a prime filter. □

Theorem 3.3. *Every weakly irreducible filter in a strong QRS is an irreducible filter.*

Proof. The proof of this theorem follows directly from Theorem 2.3 and Theorem 3.2. □

Remark 3.1. *The proof of Theorem 3.3 can be derived without reference to Theorem 3.2. Indeed, let F be a weakly irreducible filter in $\mathfrak{A}$. Let S and T be two filters in $\mathfrak{A}$ such that $F = S \cap T$. Then, $F \subseteq S$ and $F \subseteq T$. If we assume that $F \neq S$ and $F \neq T$, then there are elements $x \in S$ and $y \in T$ such that $x \notin F$ and $y \notin F$. Since F is a weakly irreducible filter in $\mathfrak{A}$, by Theorem 3.1 there is an element $z \notin F$ with $x \leqslant z$ and $y \leqslant z$. On the other hand, one has $z \in S \cap T = F$, which is a contradiction. So, F is irreducible.*

As shown in [17], any prime filter in a strong QRS is an irreducible filter. In this paper, it is shown that any weakly irreducible filter is a prime filter and, therefore, an irreducible filter. It is quite natural to ask the following question: When will the converse be valid? One of the possible answers to this question can be recognized immediately from the next theorem.

Theorem 3.4. *If the lattice $\mathfrak{F}(A)$ of a strong quasi-ordered residuated system $\mathfrak{A}$ is distributive, then any irreducible filter in $\mathfrak{A}$ is a weakly irreducible filter in $\mathfrak{A}$.*

Proof. Let F, S and T be fillers in a strong quasi-ordered residuated system $\mathfrak{A}$ such that $S \cap T \subseteq F$ and F is irreducible. Then

$$F = (S \cap T) \sqcup F = (S \sqcap T) \sqcup F = (S \sqcup F) \sqcap (T \sqcup F) = (S \sqcup F) \cap (T \sqcup F)$$

since $\mathfrak{F}(A)$ is a distributive lattice. As F is an irreducible filter in $\mathfrak{A}$, either $S \sqcup F = F$ or $T \sqcup F = F$. This implies that $S \subseteq F$ or $T \subseteq F$. Therefore, F is a weakly irreducible filter in $\mathfrak{A}$. □

This section is ended with the following question:

Question: Is the condition specified in Theorem 3.4 necessary?

4. Conclusion and future work

The concept of residuated relational systems ordered under a quasi-order relation, or quasi-ordered residuated systems (briefly, QRSs), was introduced by Bonzio and Chajda [2] in 2018. The present author proposed the notion of filters [11] and several types of filters in these algebraic systems such as implicative, comparative, associated and shift filters in the articles that followed [11]. It was shown that every comparative filter is an in implicative filter in QRSs (see Theorem 5 in [14]). In [15], the idea of strong QRSs was proposed in which implicative and comparative filters coincide. In [17], prime and irreducible filters in strong QRSs were introduced and analyzed. It was shown that any prime filter is an irreducible filter.

In this paper, the notion of weakly irreducible filters in strong QRSs is proposed and analyzed. It is shown that any weakly irreducible filter is a prime (ant therefore, irreducible) filter. It is also proved that if the lattice $\mathfrak{F}(A)$ of all filters in a strong quasi-ordered residuated system $\mathfrak{A}$ is distributive, then any irreducible filter in $\mathfrak{A}$ is weakly irreducible in $\mathfrak{A}$. The statement of Theorem 3.4 suggests some natural questions:

- Since $(\mathfrak{A}, \sqcup)$ is an upper semi-lattice, if $\mathfrak{A}$ is a strong QRS, can general results on semi-lattices be applied in this case?
- Is it possible (if yes then how) to design a modification of Stone's Theorem [19] (see also [6]: Part II, Chapter 1, Theorem 15, Page 63, and Chapter 5, Lemma 2, Page 100)?
- Does every proper filter is the intersection of some irreducible filters in a strong QRS?

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