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Research Article

Estimates for the ratio of the first two eigenvalues of the Dirichlet-Laplace operator with a drift

Şerban Bărbuleanu[1], Mihai Mihăilescu[1,*], Denisa Stancu-Dumitru[2]

[1]*Department of Mathematics, University of Craiova, 200585 Craiova, Romania*
[2]*Department of Mathematics and Computer Sciences, University Politehnica of Bucharest, 060042 Bucharest, Romania*

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Abstract

Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Consider the eigenvalue problem of the Laplace operator with a drift term $-\Delta u - x \cdot \nabla u = \lambda u$ in Ω subject to the homogeneous Dirichlet boundary condition ($u = 0$ on $\partial\Omega$). Denote by $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$ the first two eigenvalues of the problem. We show that $\lambda_2(\Omega)\lambda_1(\Omega)^{-1} \le 1 + 4N^{-1}$. In particular, we complement a similar result obtained by Thompson [*Stud. Appl. Math.* **48** (1969) 281–283] for the classical eigenvalue problem of the Laplace operator.

Keywords: Laplace operator; drift; eigenvalue; eigenfunction.

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1. Motivation and main result

For each positive integer N denote by $\mathbb{R}^N$ the N-dimensional Euclidean space and by $|\cdot|_N$ the Euclidean norm in $\mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set.

1.1. The eigenvalue problem of the Dirichlet-Laplace operator

The eigenvalue problem for the Dirichlet-Laplace operator on Ω reads as follows

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{1}$$

It is well-known that the spectrum of problem (1) consists of an increasing and unbounded sequence of positive real numbers (see, e.g. [10, Theorem 8.2.1]). Denote by $\mu_1(\Omega)$ and $\mu_2(\Omega)$ the first two eigenvalues of problem (1). Let us also denote by $\Omega^\star$ a ball from $\mathbb{R}^N$ which has the same volume as Ω (i.e., $|\Omega^\star| = |\Omega|$). In 1955-1956, Payne, Pólya and Weinberger [7] showed that

$$\frac{\mu_2(\Omega)}{\mu_1(\Omega)} \le 3\,,$$

when $N = 2$ and conjectured that the right-hand side could be replaced by

$$\frac{\mu_2(\Omega^\star)}{\mu_1(\Omega^\star)} \approx 2.539\,.$$

This result was extended to all dimensions in 1969 by Thompson [8], who showed that

$$\frac{\mu_2(\Omega)}{\mu_1(\Omega)} \le 1 + \frac{4}{N}\,,$$

and, again, it was conjectured that the right-hand side could be replaced by

$$\frac{\mu_2(\Omega^\star)}{\mu_1(\Omega^\star)}\,.$$

In the case $N = 2$ important advances on this problem were obtained by Brands [3], de Vries [5], Chiti [4], and the conjecture was finally settled positively by Asbaugh and Benguria [1] in 1992. The above historical pieces of information are mainly taken from the book by Kesavan [6, Section 4.4, pp. 98-99].

*Corresponding author (mmihailes@yahoo.com).

1.2. The eigenvalue problem of the Dirichlet-Laplace operator with a drift

Consider the eigenvalue problem of the Dirichlet-Laplace operator with a drift term

$$\begin{cases} -\Delta u - x\cdot\nabla u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2}$$

where λ is a real parameter. We say that λ is an *eigenvalue* of problem (2) if there exists $u_\lambda \in X_0 \setminus \{0\}$ such that

$$\int_\Omega \nabla u_\lambda \cdot \nabla\varphi \; dm_N = \lambda \int_\Omega u_\lambda \varphi \; dm_N\,, \tag{3}$$

where $dm_N := e^{|x|_N^2/2}dx$ and $X_0 := H_0^1(\Omega; dm_N)$. Function u_λ from the above definition is called an *eigenfunction* corresponding to the eigenvalue λ.

Using [2, relation (2.10) on page 715] (see also [9, Théorème 8.7] with $H = X_0$ applied for the particular case induced by problem (2)), we deduce that the first eigenvalue of problem (2) has the following variational characterization

$$\lambda_1(\Omega) := \inf_{u\in X_0\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^2 \; dm_N}{\displaystyle\int_\Omega u^2 \; dm_N}\,,$$

and there exists a corresponding eigenfunction corresponding to $\lambda_1(\Omega)$, $e_1 \in X_0 \setminus \{0\}$ such that

$$\int_\Omega |\nabla e_1|_N^2 \; dm_N = \lambda_1(\Omega) \quad \text{and} \quad \int_\Omega e_1^2 \; dm_N = 1\,. \tag{4}$$

Moreover, the second eigenvalue of problem (2) has the following variational characterization

$$\lambda_2(\Omega) := \inf_{u\in X_0\setminus\{0\};\; \int_\Omega ue_1 \; dm_N = 0} \frac{\displaystyle\int_\Omega |\nabla u|_N^2 \; dm_N}{\displaystyle\int_\Omega u^2 \; dm_N}\,.$$

The goal of this paper is to prove the following theorem:

Theorem 1.1. *The following estimate holds true*

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \le 1 + \frac{4}{N}\,.$$

Consequently, we show that the result by Thompson [8] established in the case of the Laplace operator continues to hold true in the case of the Laplace operator with the drift $x\cdot\nabla\cdot$. Note that even if the two cases seem to be very similar we can point out differences between them. For example, it is easy to check that the ratio $\frac{\mu_2(\Omega)}{\mu_1(\Omega)}$ is invariant on rescaled domains. More precisely, if for some $t > 0$ we denote $\Omega_t := t\Omega = \{tx : \; x \in \Omega\}$ then

$$\frac{\mu_2(\Omega)}{\mu_1(\Omega)} = \frac{\mu_2(\Omega_t)}{\mu_1(\Omega_t)}, \quad \forall\, t > 0\,.$$

This equality holds since a simple change of variable shows that $\mu_i(\Omega_t) = t^{-2}\mu_i(\Omega)$ for $i \in \{1, 2\}$. Such a relation fails to hold true in the case of $\lambda_i(\Omega_t)$ and consequently in general

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \ne \frac{\lambda_2(\Omega_t)}{\lambda_1(\Omega_t)}\,.$$

In other words, we want to point out the fact that in the case of the Laplace operator with the drift $x\cdot\nabla\cdot$ the dependence of the ratio $\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)}$ on the domain Ω is more involved than in the case of the ratio $\frac{\mu_2(\Omega)}{\mu_1(\Omega)}$. Despite this fact, we can establish the same bound from above $(1 + 4N^{-1})$ which depends only on the dimension of the Euclidean space and not on the domain Ω on which the eigenvalue problem is analysed.

2. Proof of Theorem 1.1

For each $i \in \{1, ..., N\}$, define

$$A_i := \int_\Omega \left(\frac{\partial e_1}{\partial x_i}\right)^2 \; dm_N\,.$$

Assume that $A_1 = \min_{i\in\{1,...,N\}} A_i$. Then we have

$$A_1 \le \frac{1}{N}\sum_{i=1}^{N} A_i\,. \tag{5}$$

Next, note that taking $t_i := \int_\Omega x_i e_1^2\, dm_N$ we have

$$\int_\Omega (x_i - t_i)e_1^2\, dm_N = 0, \quad \forall\, i \in \{1,...,N\} \tag{6}$$

since function e_1 satisfies (4).

By the definition of $\lambda_2(\Omega)$ and the fact that $u \in H_0^1(\Omega; dm_N)$ if and only if $ue^{|x|_N^2/4} \in H_0^1(\Omega; dx)$ and $|x|_N ue^{|x|_N^2/4} \in L^2(\Omega)$ (see [2, Proposition 2.3]) we deduce

$$\lambda_2(\Omega) \le \frac{\displaystyle\int_\Omega |\nabla[(x_i - t_i)e_1]|_N^2\, dm_N}{\displaystyle\int_\Omega (x_i - t_i)^2 e_1^2\, dm_N}, \quad \forall\, i \in \{1,...,N\}\,. \tag{7}$$

Simple computations imply that for each $i \in \{1,...,N\}$ we have

$$\nabla[(x_i - t_i)e_1] = \left((x_i - t_i)\frac{\partial e_1}{\partial x_1}, ..., (x_i - t_i)\frac{\partial e_1}{\partial x_i} + e_1, ..., (x_i - t_i)\frac{\partial e_1}{\partial x_N}\right)\,,$$

and, thus, we find

$$|\nabla[(x_i - t_i)e_1]|_N^2 = (x_i - t_i)^2|\nabla e_1|_N^2 + 2(x_i - t_i)e_1\frac{\partial e_1}{\partial x_i} + e_1^2\,.$$

Multiplying the last equality by $e^{|x|_N^2/2}$ and then integrating on Ω we get

$$\int_\Omega |\nabla[(x_i - t_i)e_1]|_N^2\, dm_N = \int_\Omega (x_i - t_i)^2|\nabla e_1|_N^2 dm_N + 2\int_\Omega (x_i - t_i)e_1\frac{\partial e_1}{\partial x_i}\, dm_N + \int_\Omega e_1^2\, dm_N\,. \tag{8}$$

On the other hand, since $\lambda_1(\Omega)$ is the first eigenvalue of problem (2) with the corresponding eigenfunction e_1 satisfying (4), by (3) with $\lambda = \lambda_1(\Omega)$, $u_\lambda = e_1$ and $\varphi = (x_i - t_i)^2 e_1$ we know that

$$\int_\Omega \nabla e_1 \cdot \nabla[(x_i - t_i)^2 e_1]\, dm_N = \lambda_1(\Omega)\int_\Omega (x_i - t_i)^2 e_1^2\, dm_N, \quad \forall\, i \in \{1,...,N\}\,. \tag{9}$$

Simple computations imply that for each $i \in \{1,...,N\}$ we have

$$\nabla[(x_i - t_i)^2 e_1] = \left((x_i - t_i)^2\frac{\partial e_1}{\partial x_1}, ..., (x_i - t_i)^2\frac{\partial e_1}{\partial x_i} + 2(x_i - t_i)e_1, ..., (x_i - t_i)^2\frac{\partial e_1}{\partial x_N}\right)\,,$$

and, thus, we find

$$\nabla e_1 \cdot \nabla[(x_i - t_i)^2 e_1] = (x_i - t_i)^2|\nabla e_1|_N^2 + 2(x_i - t_i)e_1\frac{\partial e_1}{\partial x_i}\,.$$

Multiplying the last equality by $e^{|x|_N^2/2}$ and then integrating on Ω and using (9) we find

$$\int_\Omega (x_i - t_i)^2|\nabla e_1|_N^2\, dm_N + 2\int_\Omega (x_i - t_i)e_1\frac{\partial e_1}{\partial x_i}\, dm_N = \lambda_1(\Omega)\int_\Omega (x_i - t_i)^2 e_1^2\, dm_N, \quad \forall\, i \in \{1,...,N\}\,. \tag{10}$$

By (7), (8) and (10) we infer

$$\begin{aligned}
\lambda_2(\Omega) &\le \frac{\displaystyle\int_\Omega |\nabla[(x_i - t_i)e_1]|_N^2\, dm_N}{\displaystyle\int_\Omega (x_i - t_i)^2 e_1^2\, dm_N} \\
&= \frac{\displaystyle\int_\Omega (x_i - t_i)^2|\nabla e_1|_N^2 dm_N + 2\int_\Omega (x_i - t_i)e_1\frac{\partial e_1}{\partial x_i}\, dm_N + \int_\Omega e_1^2\, dm_N}{\displaystyle\int_\Omega (x_i - t_i)^2 e_1^2\, dm_N} \\
&= \lambda_1(\Omega) + \frac{\displaystyle\int_\Omega e_1^2\, dm_N}{\displaystyle\int_\Omega (x_i - t_i)^2 e_1^2\, dm_N}, \quad \forall\, i \in \{1,...,N\}\,.
\end{aligned} \tag{11}$$

Further, note that, integrating by parts, we get

$$\begin{aligned}
2\int_\Omega (x_i-t_i)e_1\frac{\partial e_1}{\partial x_i}\,dm_N &= 2\int_\Omega (x_i-t_i)e_1\frac{\partial e_1}{\partial x_i}e^{|x|_N^2/2}\,dx \\
&= \int_\Omega (x_i-t_i)\frac{\partial e_1^2}{\partial x_i}e^{|x|_N^2/2}\,dx \\
&= \int_{\partial\Omega}(x_i-t_i)\frac{\partial e_1^2}{\partial x_i}e^{|x|_N^2/2}\sigma_i d\sigma(x) - \int_\Omega \frac{\partial}{\partial x_i}\left((x_i-t_i)e^{|x|_N^2/2}\right)e_1^2\,dx \\
&= -\int_\Omega [e^{|x|_N^2/2}+(x_i-t_i)x_ie^{|x|_N^2/2}]e_1^2\,dx \\
&= -\int_\Omega e_1^2\,dm_N - \int_\Omega (x_i-t_i)x_ie_1^2dm_N, \quad \forall\, i\in\{1,...,N\}\,.
\end{aligned}$$

Since relation (6) holds true, we have

$$\int_\Omega (x_i-t_i)x_ie_1^2dm_N = \int_\Omega (x_i-t_i)^2e_1^2dm_N + t_i\int_\Omega (x_i-t_i)e_1^2dm_N = \int_\Omega (x_i-t_i)^2e_1^2dm_N, \quad \forall\, i\in\{1,...,N\}$$

and then we deduce

$$\int_\Omega e_1^2dm_N = -\int_\Omega (x_i-t_i)^2e_1^2dm_N - 2\int_\Omega (x_i-t_i)e_1\frac{\partial e_1}{\partial x_i}\,dm_N, \quad \forall\, i\in\{1,...,N\}\,.$$

The above equality implies

$$\int_\Omega e_1^2e^{|x|_N^2/2}\,dx \le -2\int_\Omega (x_i-t_i)e_1\frac{\partial e_1}{\partial x_i}e^{|x|_N^2/2}\,dx, \quad \forall\, i\in\{1,...,N\}\,,$$

or, using Hölder's inequality, we get

$$\begin{aligned}
\left(\int_\Omega e_1^2e^{|x|_N^2/2}\,dx\right)^2 &\le 4\left(\int_\Omega (x_i-t_i)e_1\frac{\partial e_1}{\partial x_i}e^{|x|_N^2/2}\,dx\right)^2 \\
&\le 4\left(\int_\Omega (x_i-t_i)^2e_1^2e^{|x|_N^2/2}\,dx\right)\left(\int_\Omega\left(\frac{\partial e_1}{\partial x_i}\right)^2e^{|x|_N^2/2}\,dx\right), \quad \forall\, i\in\{1,...,N\}\,.
\end{aligned}$$

Equivalently, we have

$$\frac{\displaystyle\int_\Omega e_1^2\,dm_N}{\displaystyle\int_\Omega (x_i-t_i)^2e_1^2\,dm_N} \le 4\,\frac{\displaystyle\int_\Omega\left(\frac{\partial e_1}{\partial x_i}\right)^2dm_N}{\displaystyle\int_\Omega e_1^2\,dm_N}, \quad \forall\, i\in\{1,...,N\}\,. \tag{12}$$

Relations (11) and (12) yield

$$\lambda_2(\Omega)\le\lambda_1(\Omega)+4\,\frac{\displaystyle\int_\Omega\left(\frac{\partial e_1}{\partial x_i}\right)^2dm_N}{\displaystyle\int_\Omega e_1^2\,dm_N} = \lambda_1(\Omega)+4A_i, \quad \forall\, i\in\{1,...,N\}\,.$$

Letting $i=1$ above and using (5) we conclude that

$$\lambda_2(\Omega)\le\lambda_1(\Omega)+\frac{4}{N}\int_\Omega |\nabla e_1|_N^2\,dm_N = \lambda_1(\Omega)+\frac{4}{N}\lambda_1(\Omega)\,.$$

The proof of Theorem 1.1 is now complete.

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