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Research Article

The clique number and some Hamiltonian properties of graphs

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Abstract

A graph is said to be Hamiltonian (respectively, traceable) if it has a Hamiltonian cycle (respectively, Hamiltonian path), where a Hamiltonian cycle (respectively, Hamiltonian path) is a cycle (respectively, path) containing all the vertices of the graph. In this short note, sufficient conditions involving the clique number for the Hamiltonian and traceable graphs are presented.

Keywords: clique number; Hamiltonian graph; traceable graph.

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1. Introduction and statements of the main results

Throughout this note, only finite undirected graphs without loops or multiple edges are considered. The notation and terminology used in this note, but not defined here, can be found in the book [1]. For a graph $G = (V, E)$, its order is denoted by n, that is $n = |V|$. Denote by $\delta(G)$, $\omega(G)$, and $\alpha(G)$ the minimum degree, the clique number, and the independence number of a graph G, respectively. For each positive integer $r \leq \alpha(G)$, define

$$\sigma_r(G) := \min\{d(v_1) + d(v_2) + \cdots + d(v_r) : \text{ where } \{v_1, v_2, \ldots, v_r\} \text{ is an independent set in } G\}.$$

A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. Recall the following well-known results obtained by Chvátal and Erdős in [2].

Theorem 1.1. *Let G be a k-connected graph of order $n \geq 3$. If $\alpha \leq k$, then G is Hamiltonian.*

Theorem 1.2. *Let G be a k-connected graph of order n. If $\alpha \leq k + 1$, then G is traceable.*

From Theorem 1.1, one can see that it is reasonable to find sufficient conditions for the Hamiltonicity of k-connected graphs when $\alpha \geq k + 1$ and $k \geq 2$. Also, from Theorem 1.2, one can see that it is reasonable to find sufficient conditions for the traceability of k-connected graphs when $\alpha \geq k + 2$ and $k \geq 1$. In this short note, sufficient conditions involving the clique number, for the Hamiltonicity of k-connected ($k \geq 2$) graphs with the constraint $\alpha \geq k + 1$ are presented. Sufficient conditions involving the clique number, for the traceability of k-connected ($k \geq 1$) graphs with the constraint $\alpha \geq k + 2$ are also presented. The main results of this note are as follows.

Theorem 1.3. *Let G be a k-connected graph of order $n \geq 3$ with $\alpha \geq k + 1 \geq 3$. If*

$$\sigma_\alpha \geq 2\alpha(n - \alpha)\left(1 - \frac{1}{\omega}\right),$$

then G is Hamiltonian or $K_{\alpha,\, n-\alpha}$ with $k \leq n - \alpha \leq \alpha - 1$.

Theorem 1.4. *Let G be a k-connected graph of order n with $\alpha \geq k + 2 \geq 3$. If*

$$\sigma_\alpha \geq 2\alpha(n - \alpha)\left(1 - \frac{1}{\omega}\right),$$

then G is traceable or $K_{\alpha,\, n-\alpha}$ with $k \leq n - \alpha \leq \alpha - 2$.

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Notice that
$$\delta = \frac{\sigma_1}{1} \leq \frac{\sigma_2}{2} \leq \cdots \leq \frac{\sigma_{\alpha-1}}{\alpha-1} \leq \frac{\sigma_\alpha}{\alpha}.$$
Thus, Theorem 1.3 and Theorem 1.4 have the following Corollary 1.1 and Corollary 1.2, respectively.

Corollary 1.1. *Let G be a k-connected graph of order $n \geq 3$ with $\alpha \geq k+1 \geq 3$. If*
$$\frac{\sigma_{\alpha-1}}{\alpha-1} \geq \cdots \geq \frac{\sigma_2}{2} \geq \frac{\sigma_1}{1} = \delta \geq 2(n-\alpha)\left(1-\frac{1}{\omega}\right),$$
then G is Hamiltonian or $K_{\alpha,\,n-\alpha}$ with $k \leq n-\alpha \leq \alpha-1$.

Corollary 1.2. *Let G be a k-connected graph of order n with $\alpha \geq k+2 \geq 3$. If*
$$\frac{\sigma_{\alpha-1}}{\alpha-1} \geq \cdots \geq \frac{\sigma_2}{2} \geq \frac{\sigma_1}{1} = \delta \geq 2(n-\alpha)\left(1-\frac{1}{\omega}\right),$$
then G is traceable or $K_{\alpha,\,n-\alpha}$ with $k \leq n-\alpha \leq \alpha-2$.

2. Proofs of Theorems 1.3 and 1.4

In order to prove Theorem 1.3 and Theorem 1.4, we need the following result obtained in [3].

Lemma 2.1. *Let $G = (V,E)$ be a graph of order n such that $V = \{v_1, v_2, \ldots, v_n\}$. Let $(x_1, x_2, \ldots, x_n)$ be any n-vector with $x_1 + x_2 + \cdots + x_n = 1$ and $x_i \geq 0$ for each i, $1 \leq i \leq n$. Then*
$$\sum_{v_iv_j \in E} x_ix_j \leq \frac{1}{2}\left(1-\frac{1}{\omega}\right).$$

Proof of Theorem 1.3. Let G be a graph satisfying the conditions of Theorem 1.3. Suppose that G is not Hamiltonian. Let $I := \{v_1, v_2, \ldots, v_\alpha\}$ be a maximum independent set in G. Take $V - I := \{v_{\alpha+1}, v_{\alpha+2}, \ldots, v_n\}$. Define an n-vector $(x_1, x_2, \ldots, x_n)$ as follows. For each i with $1 \leq i \leq \alpha$, set
$$x_i := \frac{1}{2\alpha}$$
and for each i with $\alpha+1 \leq i \leq n$, set
$$x_i := \frac{1}{2(n-\alpha)}.$$
Then $x_1 + x_2 + \cdots + x_n = 1$ and $x_i \geq 0$ for each i, $1 \leq i \leq n$.

By applying Lemma 2.1 on the graph G with the n-vector $(x_1, x_2, \ldots, x_n)$ defined above, one has
$$\begin{aligned}
\frac{1}{2}\left(1-\frac{1}{\omega}\right) &\leq \frac{\sigma_\alpha}{4\alpha(n-\alpha)} \\
&\leq \frac{d(v_1)+d(v_2)+\cdots+d(v_\alpha)}{4\alpha(n-\alpha)} + 0 \\
&\leq \sum_{v_iv_j \in E,\, v_i \in I,\, v_j \in V-I} x_ix_j + \sum_{v_iv_j \in E,\, v_i \in V-I,\, v_j \in V-I} x_ix_j \\
&= \sum_{v_iv_j \in E} x_ix_j \\
&\leq \frac{1}{2}\left(1-\frac{1}{\omega}\right).
\end{aligned}$$
Therefore, all the above inequalities become equalities. This implies that
$$\sum_{v_iv_j \in E,\, v_i \in V-I,\, v_j \in V-I} x_ix_j = 0$$
and
$$d(v_1)+d(v_2)+\cdots+d(v_\alpha) = \sigma_\alpha = 2\alpha(n-\alpha)\left(1-\frac{1}{\omega}\right).$$

From

$$\sum_{v_iv_j\in E,\, v_i\in V-I,\, v_j\in V-I} x_ix_j = 0,$$

it follows that $v_iv_j \notin E$ for each pair of distinct vertices v_i and v_j of $V-I$. Thus, $\omega = 2$. Therefore,

$$d(v_1) + d(v_2) + \cdots + d(v_\alpha) = \sigma_\alpha = 2\alpha(n-\alpha)\left(1 - \frac{1}{\omega}\right) = \alpha(n-\alpha).$$

This implies that $v_iv_j \in E$ for each $v_i \in I$ and each $v_j \in V-I$. Notice that $V-I$ is independent and I is a maximum independent set in G. One has $\alpha \geq n-\alpha$. If $\alpha = n-\alpha$, then G is Hamiltonian, which is a contradiction. Thus, $\alpha \geq n-\alpha+1$. Since G is k-connected, one has $n-\alpha \geq k$. Therefore, $k \leq n-\alpha \leq \alpha-1$. This completes the proof. □

Since the proof of Theorem 1.4 is similar to the proof of Theorem 1.3 (we just need to note that $K_{\alpha,\, n-\alpha}$ is traceable when $\alpha = n-\alpha+1$), the details of the proof of Theorem 1.4 are omitted.

References

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