Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **4** (2021) 14–19
DOI: 10.47443/cm.2021.0040

Research Article

Fuzzy implications based on strong negations

Anestis G. Hatzimichailidis[1], Georgios Souliotis[2], Basil Papadopoulos[2,*]

[1]*Department of Forestry and Natural Environment, International Hellenic University, Drama 66100, Greece*
[2]*Department of Civil Engineering, Section of Mathematics and Informatics, Democritus University of Thrace, Kimeria 67100, Greece*

(Received: 19 July 2021. Received in revised form: 11 August 2021. Accepted: 11 August 2021. Published online: 13 August 2021.)

Abstract

In this paper we introduce fuzzy implications stemming from a class of strong negations, which are generated via conical sections. The strong negations form a structural element in the production of fuzzy implications.

Keywords: fuzzy logic; fuzzy implications; fuzzy negations; conical sections.

2020 Mathematics Subject Classification: 03B50, 03B52, 03B70.

1. Introduction

Fuzzy implications offer a new possibility to describe in a more adequate way the truth value of proposition: "if P then Q", where P and Q are (fuzzy) propositions. Various classes of fuzzy implications have been studied in the last years. Also, several techniques have been used to develop new classes of fuzzy implications and used in various applications [1,4,5,8,13].

The purpose of this paper is to propose an algorithm for the production of fuzzy implications based on strong negations. In [12], an algorithm for producing negations via conical sections was found. Fuzzy implications that stemming from a class of strong negations represent a generalization of some known fuzzy implications. For particular values of parameters, various fuzzy implications can be obtained: some of them are known but others are new, as it is shown in Sections 3 and 4. The layout of this paper is as follows. In Section 2, we recall some basic concepts and definitions on fuzzy implications. In Section 3, we study a class of fuzzy implications which arise from a class of fuzzy negations via conical sections. Section 4 discusses the complete sets of connectives in fuzzy logic.

2. Mathematical background: Basic connectives in fuzzy logic

The following definitions and notations can be found in [1–5, 7, 11]. A binary operation i on the unit interval, i.e. the mapping

$$i : [0,1] \times [0,1] \to [0,1]$$

is called a *fuzzy intersection* if it is an extension of the classical Boolean intersection:

$$i(a,b) \in [0,1]$$

for all $a, b \in [0,1]$ and

$$i(0,0) = i(0,1) = i(1,0) = 0, i(1,1) = 1.$$

A canonical model of fuzzy intersections is the family of triangular norms (briefly t-norms). A *t-norm T* is a function of the form

$$T : [0,1] \times [0,1] \to [0,1]$$

which is commutative, associative, non-decreasing, and $T(a,1) = a$ for every $a \in [0,1]$.
A *t-norm T* is called *Archimedean* if it is continuous and for $a \in (0,1)$, it holds that

$$T(a,a) < a.$$

A t-norm T is *nilpotent* if it is continuous and if, for all $a \in [0,1)$, there is a $v \in N$, such that

$$T(\overbrace{a, ..., a}^{v}) = 0.$$

*Corresponding author (papadob@civil.duth.gr).

Archimedean norms have two forms: nilpotent ones and those which are not nilpotent. Those, which are not nilpotent, are called *strict*.
A function $n:[0,1]\to[0,1]$ is called a *negation* if it is non-increasing $(n(a)\le n(b)$ when $a\ge\ b)$ and also $n(0)=1,\ n(1)=0$.

A negation n is called *strict* if and only if n is continuous and strictly decreasing $(n(a)<n(b)$, if $a>b$ for all $a,b\in[0,1])$. A strict negation n is called *strong* if and only if it is self-inverse, i.e., $n(n(a))=a$ for all $a\in[0,1]$. The dual negation based on a fuzzy negation n is given by $n^d=1-n(1-a),a\in[0,1]$, see [2]. The most important and most widely used strong negation is the standard negation $n_S:[0,1]\to[0,1]$ given by $n_S(a)=1-a$.

A function $S:[0,1]\times[0,1]\to[0,1]$ is called a *triangular conorm* (briefly *t-conorm*) if it satisfies the following properties for all $a,b,c,d\in[0,1]$:

i. $S(a,0)=a$, *boundary condition*.

ii. if $a\le c$ and $b\le d$ then $S(a,b)\le S(c,d)$, *monotonicity*.

iii. $S(a,b)=S(b,a)$, *commutativity*.

iv. $S(S(a,b),c)=S(a,S(b,c))$, *associativity*.

A *fuzzy implication* is a function $I:[0,1]\times[0,1]\to[0,1]$, which for any truth values $a,b\in[0,1]$ of (fuzzy) propositions ***P*** and ***Q***, respectively, gives the truth value $I(a,b)$ of conditional proposition: "if ***P*** then ***Q***".

Function $I(.,.)$ should be an extension of the *classical implication* from the domain $\{0,1\}$ to the domain $[0,1]$.

Recall that the *implication operator* of classical logic is a mapping

$$\mapsto:\{0,1\}\times\{0,1\}\to\{0,1\},$$

that satisfies the conditions: $\mapsto(0,0)=\mapsto(0,1)=\mapsto(1,1)=1$ and $\mapsto(1,0)=0$. The latter conditions are typically the minimum requirements for a fuzzy implication operator. In other words, fuzzy implications are required to reduce to the classical implication when truth-values are restricted to 0 and 1; i.e.,

$$I(0,0)=I(0,1)=I(1,1)=1\ \text{ and }\ I(1,0)=0.$$

One way of defining an implication operator $\mapsto$ in classical logic is using the formula

$$a\mapsto b\equiv\neg a\vee b,\qquad a,b\in\{0,1\}.$$

This formula can also be rewritten, for all $a,b\in\{0,1\}$ based on the law of *absorption of negation* in classical logic, as either:

$$a\mapsto b\equiv max\{x\in\{0,1\}:a\wedge x\le b\}$$

or

$$a\mapsto b\equiv\neg a\vee(a\wedge b)$$

or

$$a\mapsto b\equiv(\neg a\wedge\neg b)\vee b$$

Fuzzy logic extensions of the previous four formulas respectively, for all $a,b\in[0,1]$, are:

$$I(a,b)=S(n(a),b) \tag{1}$$

$$I(a,b)=sup\{x\in[0,1]:T(a,x)\le b\} \tag{2}$$

$$I(a,b)=S(n(a),T(a,b)) \tag{3}$$

$$I(a,b)=S(T(n(a),n(b)),b), \tag{4}$$

where S, T and n denote a *t-conorm*, a *t-norm* and a *fuzzy negation* on $[0,1]$, respectively, and the triple $\langle T,S,n\rangle$ is required to satisfy the *De Morgan laws*:

$$n(T(a,\ b))=S(n(a),n(b))$$

and

$$n(S(a,b))=T(n(a),n(b)),$$

for all $a,b\in[0,1]$. Note that fuzzy implications obtained from (1) are usually referred to as *S-implications* (the symbol S is often used for denoting *t-conorms*) whereas fuzzy implications obtained from (2) are called *R-implications* as they are closely connected with the so-called residuated semi group. Fuzzy implications obtained from (3) are called *QL-implications* since they were originally employed in quantum logic and fuzzy implications obtained from (4) are called *D-implications*. Identifying various properties of the classical implication and generalizing them appropriately leads to the following properties, which may be viewed as reasonable axioms of fuzzy implications.

A1. $a \leq b$ implies $I(a, x) \geq I(b, x)$ — *Monotonicity in first argument*

A2. $a \leq b$ implies $I(x, a) \leq I(x, b)$ — *Monotonicity in second argument*

A3. $I(a, I(b, x)) = I(b, I(a, x))$ — *Exchange property*

A4. $I(a, b) = I(n(b), n(a))$ — *Contraposition*

A5. $I(1, b) = b$ — *Neutrality of truth*

A6. $I(0, a) = 1$ — *Dominance of falsity*

A7. $I(a, a) = 1$ — *Identity*

A8. $I(a, b) = 1$ if and only if $a \leq b$ — *Boundary Condition*

A9. I is a continuous function — *Continuity*

3. A novel class of fuzzy implications

3.1. Fuzzy negations based on conical sections

In [9, 10, 12], a generation of a new class of fuzzy negations was discussed. This class of fuzzy negations is based on the conical sections. We consider the following special form of the conical sections:

$$ax^2 + by^2 + 2cxy + dx + ey + f = 0, \quad x, y \in [0, 1]. \tag{5}$$

If (5) satisfies the basic property of the negation: $n(0) = 1$ and $n(1) = 0$, the conical section of Equation (5) should pass from the points $A(1, 0)$ and $B(0, 1)$. Thus, the following relations result

$$b + e + f = 0 \Rightarrow e = -b - f$$

$$a + d + f = 0 \Rightarrow d = -a - f.$$

Thus, Equation (5) takes the form:

$$ax^2 + by^2 + 2cxy + (-a - f)x + (-b - f)y + f = 0, \quad x, y \in [0, 1], \ f \neq 0.$$

Furthermore, $f \neq 0$ since the point $O(0, 0)$ does not verify (5).
Moreover, the equation

$$ax^2 + ay^2 + 2cxy + (-a - f)x + (-a - f)y + f = 0, \quad x, y \in [0, 1], \ f \neq 0 \tag{6}$$

is a conical section which has as an axis of symmetry with the straight line $y = x$ passing through the points $(1, 0)$ and $(0, 1)$. Equation (6) transforms to an equivalent one, given in the following form:

$$kx^2 + ky^2 + 2mxy - (k + 1)x - (k + 1)y + 1 = 0, \quad x, y \in [0, 1], \tag{7}$$

where

$$k = \frac{a}{f} \quad \text{and} \quad m = \frac{c}{f}.$$

Equation (7) expresses conical sections, where $k = 0$ produces Sugeno negations

$$N(x) = \frac{1 - x}{1 + mx}, \ m > -1,$$

while for $k = -1$ it expresses conical sections, which produce strong fuzzy negations with the formula:

$$N(x) = \sqrt{(m^2 - 1)x^2 + 1} + mx, \quad x \in [0, 1], \ m \leq 0.$$

Remark 3.1. Due to the symmetry of the conical section to the straight line $y = x$, if Equation (7), for $k = -1$, is solved for the variable x, it generates the same formula of function, namely

$$N^{-1}(y) = \sqrt{(m^2-1)\,y^2+1} + my, \quad y \in [0,1], \quad m \leq 0.$$

As it is well known that Archimedean t-norms have been represented by maps $f : [0,1] \to [0,+\infty]$, where f is a continuous and strictly decreasing function satisfying $0 < f(0) \leq +\infty$ and $f(1) = 0$. In this case, the operation T satisfies

$$f(T(x,y)) = \min(f(x) + f(y), f(0))$$

and since this minimum is in the range of f, one has

$$T(x,y) = f^{-1}(\min(f(x) + f(y), f(0))), \quad x,y \in [0,1]. \tag{8}$$

Such functions are called *additive generators* of the *t-norm T*.
The choice of the function

$$N(x) = \sqrt{(m^2-1)\,x^2+1} + mx, \quad x \in [0,1], \quad m \leq 0$$

as an additive generator of the *t-norm* T in formula (8) gives

$$T_N(x,y) = N(\min(N(x) + N(y), N(0))), \quad x,y \in [0,1].$$

This formula can also be rewritten as

$$T_N(x,y) = N(\min(N(x) + N(y), 1)), \quad x,y \in [0,1]. \tag{9}$$

Note that t-norms T_N obtained from (9) are nilpotent because for the generator N we have $N(0) = 1 > 0$.

Example 3.1. *In the t-norm* $N(x) = \sqrt{(m^2-1)\,x^2+1} + mx$*, which is based on the conical sections, the choice of* $m = -1$ *results in the standard negation* $N_S(x) = 1 - x$*. Therefore, in formula* (8)*, if we choose as generator the standard negation* N_S*, we result in the t-norm:*

$$T_{N_S}(x,y) = 1 - \min((1-x) + (1-y), 1) = \max(x + y - 1, 0),$$

which usually referred to as bounded difference.

3.2. Fuzzy implications stemming from strong negations

In most study-applications of fuzzy sets, the standard negation $n_S = 1 - x$ is implicitly used. The replacement of this negation with the negations produced via conical sections offers a new approach in the application of fuzzy implications. As it is well known, a function $I : [0,1]^2 \to [0,1]$ is called *R-implication* if there exists a t-norm T such that

$$I(x,y) = \sup\{t \in [0,1] \mid T(x,t) \leq y\}, \quad x,y \in [0,1]$$

and it is denoted as I_T.

Proposition 3.1. [2] *For a t-norm* T*, the following statements are equivalent.*

i. T *is left-continuous.*

ii. T *and* I_T *form an adjoint pair, i.e., they satisfy the following residual principle:*

$$T(x,z) < y \Leftrightarrow I_T(x,y) \geq z, \quad x,y,z \in [0,1].$$

iii. The supremum is the maximum, i.e.,

$$I_T(x,y) = \max\{z \in [0,1] \mid T(x,z) \leq y\}.$$

Therefore, if we choose t-norm T in Proposition 3.1 as the t-norm N, which is based on the conical sections, we have the following class of fuzzy implications:

$$I(x,y) = \max\{t \in [0,1] \mid N(\min(N(x) + N(t), 1)) \leq y\}, \quad x,y \in [0,1].$$

This formula can also be rewritten as

$$I(x,y) = \max\{t \in [0,1] \mid \min(N(x) + N(t), 1) \geq N(y)\}, \quad x,y \in [0,1].$$

4. Compete sets of connectives in fuzzy logic

In classical logic, a set C of connectives is called complete if each propositional type is equivalent to a propositional type, containing only connectives belonging to C, see [6]. By using only certain connectives we can have the functionality of others. For example, some complete sets of connectives are the following

$$\{\neg, \wedge\}, \{\neg, \vee\}, \{\neg, \mapsto\}, \{\neg, \wedge, \vee\}, \text{ etc.}$$

In classical logic, the completeness of most sets of connectives has no practical but theoretical interest.
Also, as is well known, the set $\{\neg\}$ is not complete. Complete sets with a single element are only $\{\downarrow\}$ and $\{|\}$. The binary connectives $\downarrow$ and $|$ are defined, for any two logical propositions p and q, as:

$$p \downarrow q \equiv \neg (p \vee q)$$

and

$$p|q \equiv \neg (p \wedge q).$$

Example 4.1. *The set $\{\downarrow\}$ is a complete set of connectives.*

Proof. For any logical propositions p and q, we have

i. $\neg p \equiv \neg (p \vee p) \equiv p \downarrow p$

ii. $p \wedge q \equiv \neg (\neg p \vee \neg q)$

$\equiv \neg [\neg (p \vee p) \vee \neg (q \vee q)]$

$\equiv \neg [(p \downarrow p) \vee (q \downarrow q)]$

$\equiv (p \downarrow p) \downarrow (q \downarrow q).$

The set $\{\neg, \wedge\}$ is a complete set of connectives, therefore $\{\downarrow\}$ is also complete. □

The developments in the theory of fuzzy implications (if...X...then...Y...) indicate that fuzzy negation is enough to generate an algorithmic process of production for fuzzy implications, see [2, 12].
For example, suppose that in the context of an application, the *Yager implication* is selected, which is generated in the following way:

$$I(x,y) = f^{-1}(x \cdot f(y)),$$

where f is a decreasing function; if f is replaced by a fuzzy negation, then we have an algorithm for producing fuzzy implications:

$$\begin{aligned} I(x,y) = N(x \cdot N(y)) &= \sqrt{(m^2-1)\,x^2 N^2(y) + 1} + mxN(y) \\ &= \sqrt{(m^2-1)\,x^2 \left(\sqrt{(m^2-1)\,y^2+1} + my\right)^2 + 1} \\ &\quad + mx\left(\sqrt{(m^2-1)\,y^2+1} + my\right), \quad x, y \in [0,1]. \end{aligned}$$

The above equation expresses the algorithm of a family of fuzzy implications for the different values of $m < 0$. For example, for $m = -1$ it results the fuzzy implication $I(x,y) = 1 - x + xy$, which usually referred to as *Reichenbach implication*. This is an *S-implication*, which satisfies the reasonable axioms of fuzzy implications: A1, A2, A3, A4, A6, A8 and A9 (given in Section 2). Thus, based on a negation we can create other basic connectives (e.g., implications), as we have shown above. Therefore, the set with only one element, the fuzzy negation, is complete. This could be one of the reasons that makes fuzzy logic a useful tool for many applications, e.g., in technology, decision making, pattern recognition problems, etc.

5. Conclusions

In this paper we introduced fuzzy implications stemming from a class of strong negations, which are generated via conical sections. The strong negations form a structural element in the production of fuzzy implications. Thus, we have a machine for producing fuzzy implications, which can be useful in many areas, as in artificial intelligence, neural networks, etc. The present work constitutes a study of such type of fuzzy implications based on strong negations and future research on several real problems is needed to establish the proposed algorithm, e.g., decision-making problems, pattern recognition problems, medical diagnostic reasoning, assignment problems, sale analysis, financial services, etc.

References

[1] V. Balopoulos, A. G. Hatzimichailidis, B. K. Papadopoulos, Distance and similarity measures for fuzzy operators, *Inform. Sci.* **177** (2007) 2336–2348.
[2] M. Baczynski, G. Beliakov, S. H. Bustince, A. Pradera, *Advances in Fuzzy Implication Functions*, Springer, Berlin, 2013.
[3] J. Fodor, On fuzzy implication operators, *Fuzzy Sets and Systems* **42** (1991) 293–300.
[4] A. G. Hatzimichailidis, G. A. Papakostas, V. G. Kaburlasos, On constructing distance and similarity measures based on fuzzy implications, In: G. A. Papakostas, A. G. Hatzimichailidis, V. G. Kaburlasos (Eds.), *Handbook of Fuzzy Sets Comparison – Theory, Algorithms and Applications*, Sience Gate Publishing, Thrace, 2016, pp.1–21.
[5] A. G. Hatzimichailidis, G. A. Papakostas, V. G. Kaburlasos, A distance measure based on fuzzy d-implications: application in pattern recognition, *British J. Math. Comput. Sci.* **14** (2016) 1–14.
[6] D. J. Hunter, *Essentials of Discrete Mathematics*, Third Edition, Jones & Bartlett Learning, Burlington, 2017.
[7] G. Klir, B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, Upper Saddle River, 1995.
[8] A. Kolesárová, S. Massanet, R. Mesiar, J. V. Riera, J. Torrens, Polynomial constructions of fuzzy implication functions: the quadratic case, *Inform. Sci.* **494** (2019) 60–79.
[9] S. Makariadis, G. Souliotis, B. Papadopoulos, *Application of Algorithmic Fuzzy Implications on Climatic Data*,İ Proceedings of the 21st EANN (Engineering Applications of Neural Networks), 2020.
[10] S. Makariadis, G. Souliotis, B. Papadopoulos, Parametric fuzzy implications produced via fuzzy negations with a case study in environmental variables, *Symmetry* **13** (2021) 509–529.
[11] H. T. Nguyen, E. A. Walker, *A First Course in Fuzzy Logic*, CRC Press, Boca Raton, 1997.
[12] G. Souliotis, B. Papadopoulos, An algorithm for producing fuzzy negations via conical sections, *Algorithms* **12** (2019) Art# 89.
[13] C. Y. Wang, L. Wan, New results on granular variable precision fuzzy rough sets based on fuzzy (co)implications, *Fuzzy Sets and Systems*, DOI: 10.1016/j.fss.2020.08.011, In press.