**Contributions to Mathematics**
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*Contrib. Math.* **4** (2021) 12–13
DOI: 10.47443/cm.2021.0028

*Research Article*

# An alternative proof of a harmonic mean inequality for Nielsen's beta function

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(Received: 8 May 2021. Received in revised form: 2 August 2021. Accepted: 4 August 2021. Published online: 9 August 2021.)



**Abstract**

In this short note, an alternative proof of a harmonic mean inequality involving Nielsen's beta function is provided. This inequality was first posed as a conjecture by Nantomah [*Bull. Int. Math. Virtual Inst.* **9** (2019) 263–269] and subsequently proved by Matejíčka [*Probl. Anal. Issues Anal.* **8(26)** (2019) 105–111]. The present proof is more compact and relatively simple.

**Keywords:** harmonic mean inequality; Nielsen's beta function.

**2020 Mathematics Subject Classification:** 33B15, 33Bxx, 26D07, 26A48.

## 1. Introduction

The Nielsen's beta function, which was introduced in [10, p. 16], may be defined by any of the following equivalent forms:

$$\begin{aligned}\beta(s) &= \int_0^1 \frac{r^{s-1}}{1+r}\,dr,\\ &= \int_0^\infty \frac{e^{-sr}}{1+e^{-r}}\,dr,\\ &= \sum_{v=0}^{\infty} \frac{(-1)^v}{v+s},\\ &= \frac{1}{2}\left\{\psi\left(\frac{s+1}{2}\right) - \psi\left(\frac{s}{2}\right)\right\},\\ &= \psi(s) - \psi\left(\frac{s}{2}\right) - \ln 2,\end{aligned}$$

for $s > 0$, where

$$\psi(s) = \frac{d}{ds}\ln\Gamma(s)$$

is the digamma function and $\Gamma(s)$ is the gamma function. This special function provides a powerful tool for evaluating certain integrals and it is directly connected to several famous mathematical constants. For some properties, generalizations and applications of this function, one may refer to [2–12].

In [6], the author posed the following conjecture among other results.

**Conjecture 1.1.** *For $s \in (0,\infty)$, the inequality*

$$\frac{2\beta(s)\beta(1/s)}{\beta(s)+\beta(1/s)} \le \ln 2, \tag{1}$$

*is satisfied, with equality if $s = 1$. In other words, for $s \in (0,\infty)$, the harmonic mean of $\beta(s)$ and $\beta(1/s)$ is at most $\ln 2$.*

Subsequently, Matejíčka [1] came out with a solid proof of Conjecture 1.1. In this work, our objective is to provide a more compact and relatively simple proof of the inequality (1).

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## 2. A proof of Conjecture 1.1

**Lemma 2.1.** [1] *The inequality*

$$2(\beta'(s))^2 - \beta''(s)\beta(s) > 0$$

*holds for $x > 0$.*

**Lemma 2.2.** *The function*

$$h(s) = \frac{s\beta'(s)}{\beta^2(s)}$$

*is decreasing on $(0, \infty)$.*

*Proof.* By direct computations and using Lemma 2.1, we obtain

$$\begin{aligned} \beta^3(s)h'(s) &= \beta(s)\beta'(s) + s\left[\beta(s)\beta''(s) - 2(\beta'(s))^2\right] \\ &< 0. \end{aligned}$$

Thus, $h'(s) < 0$ which completes the proof. □

*Proof of Conjecture 1.1.* The case for equality is trivial. Let

$$A(s) = \frac{2\beta(s)\beta(1/s)}{\beta(s) + \beta(1/s)}$$

and

$$Q(s) = \ln A(s)$$

for $s \in (0, 1) \cup (1, \infty)$. Then direct computations yield

$$Q'(s) = \frac{\beta'(s)}{\beta(s)} - \frac{1}{s^2}\frac{\beta'(1/s)}{\beta(1/s)} - \frac{s\beta'(s) - \frac{1}{s^2}\beta(1/s)}{\beta(s) + \beta(1/s)}$$

which implies that

$$\begin{aligned} s\left[\frac{1}{\beta(s)} + \frac{1}{\beta(1/s)}\right] Q'(s) &= s\frac{\beta'(s)}{\beta^2(s)} - \frac{1}{s}\frac{\beta'(1/s)}{\beta^2(1/s)} \\ &= K(s). \end{aligned}$$

As a result of Lemma 2.2, we have $K(s) > 0$ if $s \in (0, 1)$ and $K(s) < 0$ if $s \in (1, \infty)$. Hence $Q(s)$ is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. Consequently, $A(s)$ is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. For both cases, we have

$$A(s) < A(1) = \ln 2,$$

which completes the proof. □


## Acknowledgement

The author would like thank the anonymous referees for their comments and suggestions which helped to improve the quality and readability of this paper.


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