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Review Article

Basic tools and continuity-like properties in relator spaces*

Michael Th. Rassias[1], Árpád Száz[2,†]

[1]*Institute of Mathematics, University of Zürich, CH–8057 Zürich, Switzerland*
[2]*Institute of Mathematics, University of Debrecen, H–4002 Debrecen, Hungary*

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Abstract

This paper provides the unification of several continuity-like properties of functions and relations in the framework of relator spaces. Motivated by Galois connections, we consider an ordered pair of relations instead of a single relation. A family $\mathcal{R}$ of relations on a set X to another set Y is called a relator on X to Y. All reasonable generalizations of the usual topological structures (such as proximities, closures, topologies, filters and convergences, for instance) can be derived from relators. Therefore, they should not be studied separately. From the various topological and algebraic structures (such as lower bounds, minimum and infimum, for instance) derived from relators, by using Pataki connections, we can obtain several closure and projection operations for relators. Each of them will lead to four continuity-like properties of an ordered pair of relators.

Keywords: generalized uniformities; continuous relations; Galois-type connections.

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1. A few basic facts on relations

A subset R of a product set $X \times Y$ is called a *relation* on X to Y. In particular, a relation R on X to itself is simply called a relation on X. And, $\Delta_X = \{(x, x) : x \in X\}$ is called the *identity relation* of X.

If R is a relation on X to Y, then by the above definitions we can also state that R is a relation on $X \cup Y$. However, for several purposes, the latter view of the relation R would be quite unnatural.

If R is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $R(x) = \{y \in Y : (x, y) \in R\}$ and $R[A] = \bigcup_{a \in A} R(a)$ are called the *images* or *neighbourhoods* of x and A under R, respectively.

If $(x, y) \in R$, then instead of $y \in R(x)$, we may also write $x\,R\,y$. However, instead of $R[A]$, we cannot write $R(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

Now, the sets $D_R = \{x \in X : R(x) \neq \emptyset\}$ and $R[X]$ are called the *domain* and *range* of R, respectively. If in particular $D_R = X$, then we say that R is a *relation of* X *to* Y, or that R is a *total (or non-partial) relation on* X *to* Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*(x, y)$.

If R is a relation on X to Y, then a function f of D_R to Y is called a *selection function* of R if $f(x) \in R(x)$ for all $x \in D_R$. Thus, by the Axiom of Choice, we can see that every relation is the union of its selection functions.

If R is a relation on X to Y, then we have $R = \bigcup_{x \in X} \{x\} \times R(x)$. Therefore, the values $R(x)$, where $x \in X$, uniquely determine R. Thus, a relation R on X to Y can be naturally defined by specifying $R(x)$ for all $x \in X$.

For instance, the *complement relation* R^c can be defined such that $R^c(x) = R(x)^c = Y \setminus R(x)$ for all $x \in X$. Thus, we also have $R^c = X \times Y \setminus R$. Moreover, we can note that $R^c[A]^c = \bigcap_{a \in A} R(a)$ for all $A \subseteq X$ [63].

While, the *inverse relation* R^{-1} can be defined such that $R^{-1}(y) = \{x \in X : y \in R(x)\}$ for all $y \in Y$. Thus, we also have $R^{-1} = \{(y, x) : (x, y) \in R\}$. And, we can note that $R^{-1}[B] = \{x \in X : R(x) \cap B \neq \emptyset\}$ for all $B \subseteq Y$.

Moreover, if in addition S is a relation on Y to Z, then the *composition relation* $S \circ R$ can be defined such that $(S \circ R)(x) = S[R(x)]$ for all $x \in X$. Thus, it can be easily seen that $(S \circ R)[A] = S\big[R[A]\big]$ for all $A \subseteq X$.

*A summary of the main ideas of researches on unified continuity-like properties of relations in relator spaces.
†Corresponding author (szaz@science.unideb.hu).

Now, a relation R on X may be briefly defined to be *reflexive* if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For any relation R on X, we may also define $R^0 = \Delta_X$, and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also define $R^\infty = \bigcup_{n=0}^\infty R^n$. Thus, it can be shown that R^∞ is the smallest preorder relation on X containing R [14].

Now, in contrast to $(R^c)^c = R$ and $(R^{-1})^{-1} = R$, we have $(R^\infty)^\infty = R^\infty$. Moreover, analogously to $(R^c)^{-1} = (R^{-1})^c$, we also have $(R^\infty)^{-1} = (R^{-1})^\infty$. Thus, in particular R^{-1} is also a preorder on X if R is a preorder on X.

For $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup A^c \times X$ is an important preorder on X [31]. While, for a *pseudometric* d on X, the *Weil surrounding* $B_r = \{(x, y) \in X^2 : \ d(x, y) < r\}$, with $r > 0$, is an important tolerance on X [81].

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cap (A^c)^2$ is already an equivalence relation on X. And, more generally if $\mathcal{A}$ is a *cover (partition)* of X, then $S_\mathcal{A} = \bigcup_{A \in \mathcal{A}} A^2$ is a tolerance (equivalence) relation on X.

As an important generalization of the Pervin relation R_A, for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally consider the *Hunsaker-Lindgren relation* $R_{(A,B)} = A \times B \cap A^c \times Y$ [15]. Namely, thus we evidently have $R_A = R_{(A,A)}$.

The Pervin relations R_A and the Hunsaker-Lindgren relations $R_{(A,B)}$ were actually first used by Davis [8] and Császár [6, pp. 42 and 351] in some less explicit and convenient forms, respectively.

2. The box product of relations

Notation 2.1. *In this section, we shall assume that F is a relation on X to Z and G is a relation on Y to W.*

If in particular $Y = Z$, then we may naturally consider the composition product $G \circ F$. However, instead of the composition product, it is frequently more convenient to consider the point-wise Cartesian product of relations.

Definition 2.1. *For any $x \in X$ and $y \in Y$, we define*

$$(F \boxtimes G)(x, y) = F(x) \times G(y).$$

Remark 2.1. Thus, $F \boxtimes G$ is a relation on $X \times Y$ to $Z \times W$, which has been called the *box product* of F and G in [63].

By a letter of B. M. Schein this product was already considered by some authors much before a thesis of J. Riquet in 1951 who named it tensor product.

The importance of the box product is already quite obvious from the following

Theorem 2.1. *For any $R \subseteq X \times Y$ we have*

$$(F \boxtimes G)[R] = G \circ R \circ F^{-1}.$$

Proof. For instance, if $(z, w) \in (F \boxtimes G)[R]$, then there exists $(x, y) \in R$ such that

$$(z, w) \in (F \boxtimes G)(x, y) = F(x) \times G(y),$$

and thus $z \in F(x)$ and $w \in G(y)$. Hence, by noticing that $x \in F^{-1}(z)$, we can already see that

$$y \in R(x) \subset R\left[F^{-1}(z)\right] = \left(R \circ F^{-1}\right)(y),$$

and thus

$$w \in G(y) \subseteq G\left[\left(R \circ F^{-1}\right)(z)\right] = \left(G \circ \left(R \circ F^{-1}\right)\right)(z).$$

Therefore, $(z, w) \in G \circ \left(R \circ F^{-1}\right) = G \circ R \circ F^{-1}$. This proves that $(F \boxtimes G)[R] \subseteq G \circ R \circ F^{-1}$.

From Theorem 2.1, we can immediately derive the following two corollaries.

Corollary 2.1. *For any $x \in X$ and $y \in Y$, we have $(F \boxtimes G)(x, y) = G \circ \{(x, y)\} \circ F^{-1}$.*

Corollary 2.2. *If in particular $Y = Z$, then $G \circ F = \left(F^{-1} \boxtimes G\right)[\Delta_Y]$.*

Remark 2.2. The above corollaries show that the box and composition products of relations are actually equivalent tools.

However, in contrast to the composition product, the box product of relations can be immediately defined for an arbitrary family of relations. Moreover, for the box product, we can prove a more direct inversion formula.

Theorem 2.2. *We have $(F \boxtimes G)^{-1} = F^{-1} \boxtimes G^{-1}$.*

Proof. For any $(x, y) \in X \times Y$ and $(z, w) \in Z \times W$, we have

$$(x, y) \in (F \boxtimes G)^{-1}(z, w) \iff (z, w) \in (F \boxtimes G)(x, y) \iff (z, w) \in F(x) \times G(y) \iff z \in F(x), \quad w \in G(y) \iff x \in F^{-1}(z), \quad y \in G^{-1}(w) \iff (x, y) \in F^{-1}(z) \times G^{-1}(w) \iff (x, y) \in \big(F^{-1} \boxtimes G^{-1}\big)(z, w).$$

Therefore, $(F \boxtimes G)^{-1}(z, w) = \big(F^{-1} \boxtimes G^{-1}\big)(z, w)$ for all $(z, w) \in Z \times W$, and thus the required equality is also true.

Now, by using Theorems 2.1 and 2.2, we can also easily establish the following

Theorem 2.3. *For any* $S \subseteq Z \times W$ *we have*

$$(F \boxtimes G)^{-1}[S] = G^{-1} \circ S \circ F.$$

Remark 2.3. Hence, by using the closure formula

$$\mathrm{cl}_R(B) = \big\{ x \in X : \quad R(x) \cap B \neq \emptyset \big\} = R^{-1}[B], \qquad \text{with} \qquad R \subseteq X \times Y \quad \text{and} \quad B \subseteq Y,$$

we can at once see that $\mathrm{cl}_{F \boxtimes G}(S) = G^{-1} \circ S \circ F$ for all $S \subseteq Z \times W$. (For some interesting applications, see [71].)

3. A few basic facts on ordered sets

If $\leq$ is a relation on X, then having in mind a widely used terminology of Birkhoff [3] the ordered pair $X(\leq) = (X, \leq)$ will be called a *goset* (generalized ordered set) [68].

In particular, the goset $X(\leq)$ will be called a *proset* (preordered set) if $\leq$ is a preorder on X. Moreover, $X(\leq)$ will be called a *poset* (partially ordered set) if $\leq$ is a partial order on X.

Thus, every set X is a poset with the identity relation Δ_X. Moreover, X is a proset with the *universal relation* X^2. And, the power set $\mathcal{P}(X) = \{A : \ A \subseteq X\}$ of X is a poset with the ordinary set inclusion $\subseteq$.

In this respect, it is also worth mentioning that if in particular X is a goset, then for any $A, B \subseteq X$ we may also naturally write $A \leq B$ if $a \leq b$ for all $a \in A$ and $b \in B$. Thus, $\mathcal{P}(X)$ is also a goset with this extended inequality.

Several definitions on posets can as well be applied to gosets. For instance, if $X(\leq)$ is goset, then for any $Y \subseteq X$ the goset $Y(\leq \cap Y^2)$ is called a *subgoset* of $X(\leq)$. While, the goset $X'(\leq') = X(\leq^{-1})$ is called the dual of $X(\leq)$.

Moreover, for any subset A of a goset X, we may naturally define $\mathrm{lb}(A) = \big\{ x \in X : \quad \forall\ a \in A : \quad x \leq a \big\}$ and $\mathrm{ub}(A) = \big\{ x \in X : \quad \forall\ a \in A : \quad a \leq x \big\}$. Thus, for any $A, B \subseteq X$, we have $A \subseteq \mathrm{lb}(B) \iff B \subseteq \mathrm{ub}(A)$.

Now, we may also naturally define $\min(A) = A \cap \mathrm{lb}(A)$, $\max(A) = A \cap \mathrm{ub}(A)$, $\inf(A) = \max\big(\mathrm{lb}(A)\big)$ and $\sup(A) = \min\big(\mathrm{ub}(A)\big)$. However, if in particular X is a poset, then each of these sets is either empty or a singleton.

Moreover, a goset X may, for instance, be naturally called *inf-complete* if $\inf(A) \neq \emptyset$ for all $A \subseteq X$. Thus, by using that $\sup(A) = \inf\big(\mathrm{ub}(A)\big)$ for all $A \subseteq X$, we can at once see that if X is inf-complete, then it is also *sup-complete* [4].

Now, a function f of one goset X to another Y may, for instance, be called *increasing* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in X$. Thus, we can note that f is *decreasing* if and only if it is increasing as a function of X to Y'.

Moreover, in particular a function φ of a goset X to itself may be naturally called *extensive, intensive, involutive and idempotent* if $x \leq \varphi(x)$, $\varphi(x) \leq x$, $\varphi\big(\varphi(x)\big) = x$ and $\varphi\big(\varphi(x)\big) = \varphi(x)$ for all $x \in X$, respectively.

Thus, in particular, an increasing idempotent function φ of X to itself may be naturally called a *projection operation* on X. While, an extensive projection operation on X may be naturally called a *closure operation* on X.

Moreover, an increasing extensive function may be called a *preclosure operation*. And, an extensive idempotent function may be called a *semiclosure operation*. The corresponding interior operations can be defined quite similarly.

In [68], we have established several properties of increasing functions. For instance, we have proved that if f is an increasing function of one complete poset X to another Y, then $\sup\big(f[A]\big) \leq f\big(\sup(A)\big)$ for all $A \subseteq X$.

Moreover, we have also established several properties of closure operations. For instance, we have proved that if φ is a closure operation on a complete poset X, then $\varphi\big(\sup(A)\big) = \varphi\big(\sup(\varphi[A])\big)$ for all $A \subseteq X$.

However, it is now more important to note that a function φ of a proset X to itself is a closure operation on X if and only if, for any $u, v \in X$, we have $\varphi(u) \leq \varphi(v) \iff u \leq \varphi(v)$. (For the origin, see [9, p. 50].)

Now, if f is a function of a set X to a goset Y, then according to [75], for any $x \in X$ and $y \in Y$, we may also naturally define $\mathrm{Ord}_f(x) = \big\{ v \in X : \quad f(x) \leq f(v) \big\}$ and $\mathrm{Int}_f(y) = \big\{ u \in X : \quad f(u) \leq y \big\}$.

Thus, it can be easily seen that Ord_f is the largest relation on X making f to be increasing. Moreover, we have $\mathrm{Ord}_f = \big(\mathrm{Int}_{\mathcal{R}} \circ f\big)^{-1}$ and $\mathrm{Int}_f(y) = f^{-1}\big[\mathrm{lb}\big(\{y\}\big)\big]$ for all $y \in Y$.

However, the importance of the relation Int_f lies mainly in the fact that if f is a function of one goset X to another Y such that $f[\sup(A)] \subseteq \operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big)$ for all $A \subseteq X$, then $\max\big(\operatorname{Int}_f(y)\big) = \sup\big(\operatorname{Int}_f(y)\big)$ for all $y \in Y$ [75].

If f is a function of one goset X to another Y and g is a function of Y to X such that $f(x) \leq y \iff x \leq g(y)$ for all $x \in X$ and $y \in Y$, then according to [7] we say that f and g form a *Galois connection* between X and Y.

While, if f is a function of one goset X to another Y and φ is a function of X to itself such that $f(u) \leq f(v) \iff u \leq \varphi(v)$ for all $u, v \in X$, then according to [55], we say that f and φ form a *Pataki connection* between X and Y.

Thus, if f and g form a Galois connection between X and Y, then by defining $\varphi = g \circ f$ we can at once see that $f(u) \leq f(v) \iff u \leq g\big(f(v)\big) \iff u \leq \varphi(v)$ for all $u, v \in X$. Therefore, f and φ form a Pataki connection.

While, if f and φ form a Pataki connection between X and Y such that $Y = f[X]$ and there exists a function g of Y to X such that $\varphi = g \circ f$, then we can quite similarly see that f and g form a Galois connection.

Hence, it is clear that Pataki connections can be used to establish several properties of Galois connections. Moreover, one can feel that, despite this, Galois connections are still somewhat more general objects than the Pataki ones.

Note that a function φ of a proset X to itself, then by our former observation φ is a closure operation on X if and only if φ and φ form a Pataki connection.

Moreover, if X is a goset, and $f(A) = \operatorname{ub}(A)$ and $g(B) = \operatorname{lb}(B)$ for all $A, B \subseteq X$, then f and g form a Galois connection between the poset $\mathcal{P}(X)$ and its dual.

In the practically important applications, we usually encounter with Galois and Pataki connections between power sets. In this case, in addition to our former definitions, we may naturally introduce several further useful definitions.

For instance, if F is a function of one power set $\mathcal{P}(X)$ to another $\mathcal{P}(Y)$, then F may, in addition, be naturally called *quasi-increasing* if $F(x) \subseteq F(A)$ for all $x \in A \subseteq X$, where $F(x) = F\big(\{x\}\big)$.

Moreover, the function F may be naturally called *union-preserving* if $F\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} F(A)$ for all $\mathcal{A} \subseteq \mathcal{P}(X)$. Thus, it can be shown that F is union-preserving if and only if $F(A) = \bigcup_{x \in A} F(x)$ for all $A \subseteq X$ [65].

4. Pataki connections between posets

Notation 4.1. *In this section, we shall assume that f is a function of one poset X to another Y and φ is a function of X to itself.*

However, the forthcoming theorems can be stated in more instructive forms in prosets (preordered sets) [55, 58]. (See [66, 75] for some less obvious generalizations.)

Definition 4.1. We say that

(1) f is *right φ–regular* if $f(u) \leq f(v) \implies u \leq \varphi(v)$ for all $u, v \in X$;

(2) f is *left φ–regular* if $u \leq \varphi(v) \implies f(u) \leq f(v)$ for all $u, v \in X$.

Remark 4.1. Quite similarly, the function f may, for instance, be naturally called *right g–normal*, for some function g of Y to X, if $f(x) \leq y \implies x \leq g(y)$ for all $x \in X$ and $y \in Y$. Thus, by taking $\varphi = g \circ f$, we can easily establish that right normal functions are in particular right regular.

Now, by calling the function f to be *φ–regular* if it is both right and left φ–regular, we can easily prove

Theorem 4.1. *If f is φ–regular, then f is increasing and $f = f \circ \varphi$.*

Proof. For any $u \in X$, we have $f(u) \leq f(u)$ and $\varphi(u) \leq \varphi(u)$. Hence, by using Definition 4.1, we can infer that $u \leq \varphi(u)$ and $f\big(\varphi(u)\big) \leq f(u)$.

Moreover, if $v \in X$ such that $u \leq v$, then because of $v \leq \varphi(v)$, we also have $u \leq \varphi(v)$. Hence, by using Definition 4.1, we can infer that $f(u) \leq f(v)$. Therefore, f is increasing. Now, from $u \leq \varphi(u)$, we can also see that $f(u) \leq f\big(\varphi(u)\big)$, and thus the corresponding equality is also true.

Remark 4.2. If the assertions of Theorem 4.1 hold, then we can only note that $u \leq \varphi(v) \implies f(u) \leq f\big(\varphi(v)\big) \implies f(u) \leq f(v)$ for all $u, v \in X$. Thus, f is left φ–regular.

Moreover, if in addition, X is linearly ordered and f is injective, then f^{-1} is also an increasing function. Therefore, we can also note that $f(u) \leq f(v) \implies f(u) \leq f\big(\varphi(v)\big) \implies f^{-1}\big(f(u)\big) \leq f^{-1}\big(f\big(\varphi(v)\big)\big) \implies u \leq \varphi(v)$ for all $u, v \in X$. Thus, f is also right φ–regular.

However, the latter fact is of no importance for us. Namely, by Theorem 4.1, we evidently have the following

Corollary 4.1. *If f is φ–regular, then f is injective if and only if $\varphi = \Delta_X$.*

The importance of regular functions is also apparent from the following

Theorem 4.2. *The following assertions are equivalent :*

(1) φ is a closure ; *(2) φ is φ–regular ;*

(3) there exists a φ–regular function h of X to another poset.

Proof. To prove the implication (3) $\Longrightarrow$ (1), note that if (3) holds, then because of $h(u) \leq h(u)$ we have $u \leq \varphi(u)$ for all $u \in X$. Therefore, φ is extensive. Thus, $\varphi(u) \leq \varphi\big(\varphi(u)\big)$ also holds for all $u \in X$.

Moreover, by Theorem 4.1, we have $h\big(\varphi(u)\big) = h(u)$ for all $u \in X$. Thus, $h\left(\varphi\big(\varphi(u)\big)\right) = h\big(\varphi(u)\big) = h(u) \leq h(u)$ also holds for all $u \in X$. Hence, by using (3), we can infer that $\varphi\big(\varphi(u)\big) \leq \varphi(u)$ for all $u \in X$. Therefore, φ is idempotent.

On the other hand, if $u, v \in X$ such that $u \leq v$, then by Theorem 4.1 we have $h\big(\varphi(u)\big) = h(u) \leq h(v)$. Hence, by using (3), we can infer that $\varphi(u) \leq \varphi(v)$. Therefore, φ is increasing, and thus (1) also holds.

Remark 4.3. In addition to this theorem, it is also worth mentioning that φ is an involution if and only if φ is φ–normal.

However, it is now more important to note that, as an immediate consequence of Theorem 4.2, we can also state.

Corollary 4.2. *f is φ–regular if and only if φ is a closure and $\mathrm{Ord}_f = \mathrm{Ord}_\varphi$.*

Remark 4.4. Recall that $\mathrm{Ord}_f = \mathrm{Ord}_\varphi$ if and only if $f(u) \leq f(v) \iff \varphi(u) \leq \varphi(v)$ for all $u, v \in X$.

Concerning regular functions, we can also prove the following

Theorem 4.3. *The following assertions are equivalent :*

(1) f is φ–regular ;

(2) f is increasing and $\varphi(v) = \max\big(\mathrm{Ord}_f^{-1}(v)\big)$ for all $v \in X$;

(3) $\mathrm{lb}\big(\varphi(v)\big) = \mathrm{Ord}_f^{-1}(v)$ $\Big(\mathrm{lb}\big(\varphi(v)\big) = \mathrm{Int}_f\big(f(v)\big)\Big)$ for all $v \in X$.

Proof. To prove the equivalence of (1) and (3), note that, for any $u, v \in X$, we have

$$u \in \mathrm{lb}\big(\varphi(v)\big) \iff u \in \mathrm{lb}\big(\{\varphi(v)\}\big) \iff u \leq \varphi(v) \iff f(u) \leq f(v) \iff v \in \mathrm{Ord}_f(u) \iff u \in \mathrm{Ord}_f^{-1}(v).$$

Therefore, $\mathrm{lb}\big(\varphi(v)\big) = \mathrm{Ord}_f^{-1}(v)$. Moreover, we also have $\mathrm{Ord}_f^{-1}(v) = \big(\mathrm{Int}_f \circ f\big)(v) = \mathrm{Int}_f\big(f(v)\big)$.

While, to prove the equivalence of (1) and (2), note that if for instance (1) holds, then from Theorem 4.1 we can see that f is increasing. Moreover, for any $v \in X$ we have $\varphi(v) \in \mathrm{lb}\big(\varphi(v)\big)$, and thus by the implication (1) $\Longrightarrow$ (3) also $\varphi(v) \in \mathrm{Ord}_f^{-1}(v)$. Moreover, we can also easily see that

$$u \in \mathrm{Ord}_f^{-1}(v) \implies v \in \mathrm{Ord}_f^{-1}(u) \implies f(u) \leq f(v) \implies u \leq \varphi(v).$$

Therefore, $\varphi(v) \in \mathrm{ub}\big(\mathrm{Ord}_f^{-1}(v)\big)$, and thus $\varphi(v) \in \mathrm{Ord}_f^{-1}(v) \cap \mathrm{ub}\big(\mathrm{Ord}_f^{-1}(v)\big) = \max\big(\mathrm{Ord}_f^{-1}(u)\big)$. Consequently, (2) also holds

Remark 4.5. Hence, in particular we can see that if f is φ–regular, then φ is uniquely determined by f.

Moreover, from Theorem 4.3, by Theorem 4.2, for instance we can see that φ is closure if and only if φ is increasing and $\varphi(v) = \max\big(\mathrm{Ord}_\varphi^{-1}(v)\big)$ for all $v \in X$.

In this respect, it is also worth mentioning that φ is a semiclosure if and only if $\varphi(v) = \max\big(\mathrm{Ord}_\varphi^{-1}(v)\big)$ for all $v \in X$.

However, it is now more important to note that, by calling f *regular* if f is φ–regular for some function φ of X to itself, from Theorem 4.3 we can also easily derive the following

Corollary 4.3. *f is regular if and only if it is increasing and $\max\big(\mathrm{Ord}_f^{-1}(v)\big) \neq \emptyset$ for all $v \in X$.*

Remark 4.6. Hence, we can see that if in particular X is max–*complete*, then f is regular if and only if f is increasing.

Moreover, in addition to Corollary 4.3, it is also worth mentioning that if $Y = f\,[\,X\,]$, then f is regular if and only if f is normal [58, Theorem 7.5].

5. Regular functions and closure operations on power sets

Notation 5.1. *In this section, we shall assume that F is a function of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ and Φ is a function of $\mathcal{P}(X)$ to itself for some sets X and Y.*

However, the forthcoming definition and theorems can be easily generalized to the case when the second power set $\mathcal{P}(Y)$ is replaced by an arbitrary poset.

Now, if F is Φ–regular, then by Theorem 4.3 we can see that $\Phi(V) = \max\big\{U \subseteq X : \ F(U) \subseteq F(V)\big\}$ for all $V \subseteq X$. Therefore, we may also naturally introduce the following

Definition 5.1. For any $V \subseteq X$, we define

$$\Phi_F(V) = \big\{u \in X : \ F(u) \subseteq F(V)\big\},$$

where $F(u) = F\big(\{u\}\big)$.

The importance of the above *Pataki operation* Φ_F, generated by F, is already quite obvious from the following

Theorem 5.1. *If F is Φ–regular, then $\Phi = \Phi_F$.*

Proof. For any $u \in X$ and $V \subseteq X$, we have

$$u \in \Phi(V) \iff \{u\} \subseteq \Phi(V) \iff F\big(\{u\}\big) \subseteq F(V) \iff F(u) \subseteq F(V) \iff u \in \Phi_F(V).$$

Therefore, $\Phi(V) = \Phi_F(V)$ for all $V \subseteq X$, and thus $\Phi = \Phi_F$.

Hence, it is clear that in particular we also have the following

Corollary 5.1. *F is regular if and only if F is Φ_F–regular.*

Moreover, by Theorem 4.3, we can also at once state the following

Theorem 5.2. *The following assertions are equivalent:*

(1) F is Φ_F–regular;

(2) F is increasing and $\Phi_F(V) = \max\big(\operatorname{Ord}_F^{-1}(V)\big)$ for all $V \subseteq X$;

(3) $\mathcal{P}\big(\Phi_F(V)\big) = \operatorname{Ord}_F^{-1}(V)$ $\Big(\mathcal{P}\big(\Phi_F(V)\big) = F^{-1}\big[\mathcal{P}\big(F(V)\big)\big]\Big)$ for all $V \subseteq X$.

Proof. Note that $\operatorname{lb}\big(\Phi_F(V)\big) = \mathcal{P}\big(\Phi_F(V)\big)$ and $\operatorname{Int}_F\big(F(V)\big) = F^{-1}\big[\operatorname{lb}\big(F(V)\big)\big] = F^{-1}\big[\mathcal{P}\big(F(V)\big)\big]$ for all $V \subseteq X$.

Concerning the function Φ_F, we can also easily prove the following

Theorem 5.3. *The following assertions are equivalent:*

(1) Φ_F is extensive; *(2) F is quasi-increasing;* *(3) F is right Φ_F–semiregular.*

Proof. If $u \in V \subseteq X$ and (1) holds, then we have $V \subseteq \Phi_F(V)$, and thus also $u \in \Phi_F(V)$. Hence, we can already infer that $F(u) \subseteq F(V)$. Therefore, (2) also holds.

While, $U, V \subseteq X$ such that $F(U) \subseteq F(V)$ and (2) holds, then for any $u \in U$ we have $F(u) \subseteq F(U) \subseteq F(V)$, and thus $u \in \Phi_F(V)$. Therefore, $U \subseteq \Phi_F(V)$, and thus (3) also holds.

Finally, if $V \subseteq X$ and (3) holds, then from the trivial inclusion $F(V) \subseteq F(V)$ we can already infer that $V \subseteq \Phi_F(V)$. Therefore, (1) also holds.

Remark 5.1. If in particular F is increasing, then we can at once see that Φ_F is also increasing. Therefore, in this case, by the above theorem we can also state that Φ_F is a preclosure on $\mathcal{P}(X)$.

Theorem 5.4. *If F is union-preserving, then F is Φ_F–regular.*

Proof. Since F is now increasing, from Theorem 5.3, we know that $\mathcal{F}$ is right Φ_F–semiregular. Therefore, we need only show that now F is also left Φ_R–semiregular.

For this, assume that $U, V \subseteq X$ such that $U \subseteq \Phi_F(V)$ and $y \in F(U)$. Then, since $F(U) = \bigcup_{u \in U} F\big(\{u\}\big) = \bigcup_{u \in U} F(u)$, there exists $u \in U$ such that $y \in F(u)$. Hence, we can infer that $u \in \Phi_F(V)$, and thus $y \in F(u) \subseteq F(V)$. Therefore, $F(U) \subseteq F(V)$ also holds.

From this theorem, by using Theorems 4.1, 4.2 and 4.3, we can immediately infer the following

Corollary 5.2. *If F is union-preserving, then*

(1) $F = F \circ \Phi_F$; *(2)* Φ_F *is a closure ;* *(3)* $\Phi_F(V) = \max\big(\operatorname{Ord}_F^{-1}(V)\big)$ *for all* $V \subseteq X$.

Remark 5.2. By assertion (3), for any $V \subseteq X$, we have $\Phi_F(V) = \max\big\{ U \subseteq X : \ F(U) \subseteq F(V) \big\}$. Therefore, $U = \Phi_F(V)$ is the largest subset of X such that $F(U) \subseteq F(V)$.

Moreover, by assertion (1), we also have $F\big(\Phi_F(V)\big) = F(V)$. Therefore, in particular, we can also state that, $U = \Phi_F(V)$ is the largest subset of X such that $F(U) = F(V)$.

6. A few basic facts on relators

A family $\mathcal{R}$ of relations on one set X to another Y is called a *relator on* X *to* Y, and the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a *relator space*. For the origins of this notion, see [37, 50], and the references in [37].

If in particular $\mathcal{R}$ is a relator on X to itself, then $\mathcal{R}$ is simply called a *relator on* X. Thus, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ instead of $(X, X)(\mathcal{R})$. Namely, $(X, X) = \{\{X\}, \{X, X\}\} = \{\{X\}\}$.

Relator spaces of this simpler type are already substantial generalizations of the various *ordered sets* [7, 68] and *uniform spaces* [12, 16]. However, they are insufficient for some important purposes. (See, [13] and [50, 62, 66, 70, 77].)

A relator $\mathcal{R}$ on X to Y, or the relator space $(X, Y)(\mathcal{R})$, is called *simple* if $\mathcal{R} = \{R\}$ for some relation R on X to Y. Simple relator spaces $(X, Y)(R)$ and $X(R)$ were called *formal contexts* and *gosets* in [13] and [68], respectively.

Moreover, a relator $\mathcal{R}$ on X, or the relator space $X(\mathcal{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathcal{R}$ is reflexive on X. Thus, we may also naturally speak of *preorder, tolerance and equivalence relators*.

For instance, for a family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_\mathcal{A} = \{ R_A : \ A \in \mathcal{A} \}$, where $R_A = A^2 \cup A^c \times X$, is an important preorder relator on X. Such relators were first used by Pervin [31] and Levine [22].

While, for a family $\mathcal{D}$ of *pseudo-metrics* on X, the family $\mathcal{R}_\mathcal{D} = \{ B_r^d : \ r > 0, \ d \in \mathcal{D} \}$, where $B_r^d = \{ (x, y) : \ d(x, y) < r \}$, is an important tolerance relator on X. Such relators were first considered by Weil [81].

Moreover, if $\mathfrak{S}$ is a family of *covers (partitions)* of X, then the family $\mathcal{R}_\mathfrak{S} = \{ S_\mathcal{A} : \ \mathcal{A} \in \mathfrak{S} \}$, where $S_\mathcal{A} = \bigcup_{A \in \mathcal{A}} A^2$, is an important tolerance (equivalence) relator on X. Equivalence relators were first studied by Levine [21].

If $\star$ is a unary operation for relations on X to Y, then for any relator $\mathcal{R}$ on X to Y we may naturally define $\mathcal{R}^\star = \big\{ R^\star : \ R \in \mathcal{R} \big\}$. However, this plausible notation may cause confusions if $\star$ is a set-theoretic operation.

For instance, for any relator $\mathcal{R}$ on X to Y, we may naturally define the *elementwise complement* $\mathcal{R}^c = \{ R^c : \ R \in \mathcal{R} \}$, which may easily be confused with the *global complement* $\mathcal{R}^c = \mathcal{P}(X \times Y) \setminus \mathcal{R}$ of $\mathcal{R}$.

However, for instance, the practical notations $\mathcal{R}^{-1} = \{ R^{-1} : \ R \in \mathcal{R} \}$, and $\mathcal{R}^\infty = \{ R^\infty : \ R \in \mathcal{R} \}$ whenever $\mathcal{R}$ is only a relator on X, will certainly not cause confusions in the sequel.

In particular, for a relator $\mathcal{R}$ on X, we may also naturally define $\mathcal{R}^\partial = \big\{ S \subseteq X^2 : \ S^\infty \in \mathcal{R} \big\}$. Namely, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we evidently have $\mathcal{R}^\infty \subseteq \mathcal{S} \iff \mathcal{R} \subseteq \mathcal{S}^\partial$. That is, ∞ and ∂ form a Galois connection.

The operations ∞ and ∂ were first introduced by Mala [23, 25] and Pataki [29, 30], respectively. These two former PhD students of mine, together with János Kurdics [19, 20], made revolutionary changes in the theory of relators.

Moreover, if $*$ is a binary operation for relations, then for any two relators $\mathcal{R}$ and $\mathcal{S}$ we may naturally define $\mathcal{R} * \mathcal{S} = \big\{ R * S : \ R \in \mathcal{R}, \ S \in \mathcal{S} \big\}$. However, this notation may again cause confusions if $*$ is a set-theoretic operation

Therefore, in our former papers, we rather wrote $\mathcal{R} \wedge \mathcal{S} = \big\{ R \cap S : \ R \in \mathcal{R}, \ S \in \mathcal{S} \big\}$. Moreover, for instance, we also wrote $\mathcal{R} \triangle \mathcal{R}^{-1} = \big\{ R \cap R^{-1} : \ R \in \mathcal{R} \big\}$. Thus, $\mathcal{R} \triangle \mathcal{R}^{-1}$ is a symmetric relator such that $\mathcal{R} \triangle \mathcal{R}^{-1} \subseteq \mathcal{R} \wedge \mathcal{R}^{-1}$.

A function $\square$ of the family of all relators on X to Y is called a *direct (indirect) unary operation for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathcal{R}^\square = \square(\mathcal{R})$ is a relator on X to Y (on Y to X).

More generally, a function $\mathfrak{F}$ of the family of all relators on X to Y is called a *structure for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathfrak{F}_\mathcal{R} = \mathfrak{F}(\mathcal{R})$ is in a power set depending only on X and Y.

Concerning structures and operation for relators, we can freely use our former terminology on set-to-set functions. However, for closures and projections, we can now also use the terms *refinements and modifications*, respectively.

For instance, c and -1 are *involution operations* for relators. While, ∞ and ∂ are *projection operations* for relators. Moreover, the operation $\square = c$, ∞ or ∂ is *inversion compatible* in the sense that $\mathcal{R}^{\square\,-1} = \mathcal{R}^{-1\,\square}$.

While, if for instance $\operatorname{int}_\mathcal{R}(B) = \{ x \in X : \ \exists\, R \in \mathcal{R} : \ R(x) \subseteq B \}$ for every relator $\mathcal{R}$ on X to Y and $B \subseteq Y$, then the function $\mathfrak{F}$, defined by $\mathfrak{F}(\mathcal{R}) = \operatorname{int}_\mathcal{R}$, is a union-preserving structure for relators.

The first basic problem in the theory of relators is that, for any increasing structure $\mathfrak{F}$, we have to find an operation $\square$ for relators such that, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y we could have $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}} \iff \mathcal{S} \subseteq \mathcal{R}^{\square}$.

By using *Pataki connections* [29, 75], several closure operations can be derived from union-preserving structures. However, more generally, one can find first the *Galois adjoint* $\mathfrak{G}$ of such a structure $\mathfrak{F}$, and then take $\square_{\mathfrak{F}} = \mathfrak{G} \circ \mathfrak{F}$ [54].

By finding the Galois adjoint of the structure $\mathfrak{F}$, the second basic problem for relators, that which structures can be derived from relators, can also be solved. However, for this, some direct methods can also be well used [43, 56].

Now, for an operation $\square$ for relators, a relator $\mathcal{R}$ on X to Y may be naturally called $\square$*–fine* if $\mathcal{R}^{\square} = \mathcal{R}$. And, for some structure $\mathfrak{F}$ for relators, two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y may be naturally called $\mathfrak{F}$*–equivalent* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$.

Moreover, for a structure $\mathfrak{F}$ for relators, a relator $\mathcal{R}$ on X to Y may, for instance, be naturally called $\mathfrak{F}$*–simple* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{R}$ for some relation R on X to Y. Thus, in particular singleton relators have to be actually called *properly simple*.

Finally, we note that a relator $\mathcal{R}$ on X to Y will be called *non-partial* if each member of $\mathcal{R}$ is non-partial. Moreover, the relator $\mathcal{R}$, or the relator spaces $(X, Y)(\mathcal{R})$, will be called *non-degerated* if both X and $\mathcal{R}$ are nonvoid.

7. Structures derived from relators

Notation 7.1. *In this and the next two sections, we shall assume that $\mathcal{R}$ is a relator on X to Y.*

Definition 7.1. For any $x \in X$, $A \subseteq X$ and $B \subseteq Y$, we define:

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ *if* $R[A] \subseteq B$ *for some* $R \in \mathcal{R}$*;*

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ *if* $R[A] \cap B \neq \emptyset$ *for all* $R \in \mathcal{R}$*;*

(3) $x \in \operatorname{int}_{\mathcal{R}}(B)$ *if* $\{x\} \in \operatorname{Int}_{\mathcal{R}}(B)$*;* *(4)* $x \in \operatorname{cl}_{\mathcal{R}}(B)$ *if* $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(B)$*;*

(5) $B \in \mathcal{E}_{\mathcal{R}}$ *if* $\operatorname{int}_{\mathcal{R}}(B) \neq \emptyset$*;* *(6)* $B \in \mathcal{D}_{\mathcal{R}}$ *if* $\operatorname{cl}_{\mathcal{R}}(B) = X$.

Remark 7.1. The relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$ are called *the proximal and topological interiors* generated by $\mathcal{R}$, respectively. While, the members of the families, $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ are called the *fat and dense subsets* of the relator space $(X, Y)(\mathcal{R})$, respectively.

The origins of the relations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ go back to Efremović's proximity δ [10] and Smirnov's strong inclusion $\Subset$ [35], respectively. While, the convenient notations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$, and the family $\mathcal{E}_{\mathcal{R}}$, together with its dual $\mathcal{D}_{\mathcal{R}}$, were first explicitly used by the second author in [37, 41, 43, 53].

The following theorem shows that the corresponding closure and interior relations are equivalent tools. Moreover, in a relator space the closure of a set can be more nicely described than in a topological one.

Theorem 7.1. *For any $B \subseteq X$, we have*

(1) $\operatorname{Cl}_{\mathcal{R}}(B) = \mathcal{P}(X) \setminus \operatorname{Int}_{\mathcal{R}}\big(B^c\big)$*;* *(2)* $\operatorname{cl}_{\mathcal{R}}(B) = X \setminus \operatorname{int}_{\mathcal{R}}\big(B^c\big)$*;* *(3)* $\operatorname{cl}_{\mathcal{R}}(B) = \bigcap_{R \in \mathcal{R}} R^{-1}[B]$.

Remark 7.2. By using appropriate complementations, assertion (1) can be expressed in the more concise form that $\operatorname{Cl}_{\mathcal{R}} = \big(\operatorname{Int}_{\mathcal{R}} \circ \mathcal{C}_Y\big)^c$ or $\operatorname{Cl}_{\mathcal{R}} = \big(\operatorname{Int}_{\mathcal{R}}\big)^c \circ \mathcal{C}_Y$.

Moreover, by defining the *infinitesimal closure relation* $\rho_{\mathcal{R}}$ such that $\rho_{\mathcal{R}}(y) = \operatorname{cl}_{\mathcal{R}}(\{y\})$ for all $y \in Y$, from assertion (3) we can easily derive that $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = \big(\bigcap \mathcal{R}\big)^{-1}$.

In addition to Theorem 7.1, it is also worth noticing that the small closure and interior relations are usually much weaker tools than the big closure and interior ones. Namely, in general, we can only state the following

Theorem 7.2. *For any $A \subseteq X$ and $B \subseteq Y$*

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ *implies* $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$*;* *(2)* $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ *implies* $A \in \operatorname{Cl}_{\mathcal{R}}(B)$.

Remark 7.3. Later, we shall see that if in particular $\mathcal{R}$ is topologically fine, then the converse implications are also true.

The following theorem shows that, in contrast to their equivalence, the big closure relation is usually a more convenient tool than the big interior one.

Theorem 7.3. *We have*

(1) $\operatorname{Cl}_{\mathcal{R}^{-1}} = \operatorname{Cl}_{\mathcal{R}}^{-1}$*;* *(2)* $\operatorname{Int}_{\mathcal{R}^{-1}} = \mathcal{C}_Y \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ \mathcal{C}_X$.

Concerning the small interior and closure relations, we can also easily prove the following

Theorem 7.4. *If* $R \in \mathcal{R}$, *then for any* $A \subseteq X$ *and* $B \subseteq Y$ *we have*

$$A \subseteq \operatorname{int}_R(B) \quad \Longleftrightarrow \quad \operatorname{cl}_{R^{-1}}(A) \subseteq B .$$

Proof. For instance, if $A \subseteq \operatorname{int}_R(B)$ hods, then by the corresponding definitions, for each $x \in A$, we have $x \in \operatorname{int}_R(B)$, and thus $R(x) \subseteq B$. Moreover, by Theorem 7.1, we can see that $\operatorname{cl}_{R^{-1}}(A) = \operatorname{cl}_{\{R^{-1}\}}(A) = R(A) = \bigcup_{x \in A} R(x)$. Therefore, $\operatorname{cl}_{R^{-1}}(A) \subseteq B$ also holds.

Remark 7.4. This shows that the mappings $A \mapsto \operatorname{cl}_{R^{-1}}(A)$ and $B \mapsto \operatorname{int}_R(B)$ establish a Galois connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

Later, we shall see that the above closure-interior Galois connection, used first in [71], is not independent from the well-known upper and lower bound one [58].

By using Theorem 7.1 and Definition 7.1, we can easily establish the following

Theorem 7.5. *We have*

(1) $\mathcal{D}_{\mathcal{R}} = \big\{ B \subseteq Y : \ \forall\ R \in \mathcal{R} : \ X = R^{-1}[B] \big\}$; *(2)* $\mathcal{E}_{\mathcal{R}} = \bigcup_{x \in X} \mathcal{U}_{\mathcal{R}}(x)$, *where* $\mathcal{U}_{\mathcal{R}}(x) = \operatorname{int}_{\mathcal{R}}^{-1}(x)$.

Remark 7.5. Note that thus $\mathcal{U}_{\mathcal{R}}(x) = \operatorname{int}_{\mathcal{R}}^{-1}(x) = \big\{ B \subseteq Y : \ x \in \operatorname{int}_{\mathcal{R}}(B) \big\}$ is just the family of all neighbourhoods of the point x of X in Y.

The following theorem shows that the families of fat and dense sets are also equivalent tools.

Theorem 7.6. *We have*

(1) $\mathcal{D}_{\mathcal{R}} = \big\{ D \subseteq Y : \ D^c \notin \mathcal{E}_{\mathcal{R}} \big\}$; *(2)* $\mathcal{D}_{\mathcal{R}} = \big\{ D \subseteq Y : \ \forall\ E \in \mathcal{E}_{\mathcal{R}} : \ E \cap D \neq \emptyset \big\}$.

Remark 7.6. If $\leq$ is a relation relation on X, then for any $A \subseteq X$ we have

(1) $A \in \mathcal{E}_{\leq}$ if and only if there exists $x \in X$ such that $y \in A$ for all $y \geq x$;

(2) $A \in \mathcal{D}_{\leq}$ if and only if for each $x \in X$ there exists $y \in A$ such that $y \geq x$.

Therefore, $\mathcal{E}_{\leq}$ and $\mathcal{D}_{\leq}$ are just the families of all *residual* and *cofinal* subsetes of the *goset* (generalized ordered set) $X(\leq)$, respectively.

Finally, we note that, by Definition 7.1 and [65, Theorem 3], the following theorem is also true.

Theorem 7.7. *The structures* Int, int *and* $\mathcal{E}$ *are union-preserving.*

Remark 7.7. In the sequel, instead of the union-preservingness of Int, we shall only need that $\operatorname{Int}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \operatorname{Int}_R$, where $\operatorname{Int}_R = \operatorname{Int}_{\{R\}}$. Therefore, [65, Theorem 3] is only of some terminological importance for us.

8. Convergence and adherence relations derived from relators

Now, having in mind the convergence and adherence of ordinary and generalized sequences and an observation of Efremović and Švarc [11], we may also naturally introduce the following

Definition 8.1. If φ and ψ are functions of a relator space $\Gamma(\mathcal{U})$ to X and Y, respectively, and $(\varphi, \psi)(\gamma) = \big(\varphi(\gamma), \psi(\gamma)\big)$ for all $\gamma \in \Gamma$, then we also define:

(1) $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$ if $(\varphi, \psi)^{-1}[R] \in \mathcal{E}_{\mathcal{U}}$ for all $R \in \mathcal{R}$;

(2) $\varphi \in \operatorname{Adh}_{\mathcal{R}}(\psi)$ if $(\varphi, \psi)^{-1}[R] \in \mathcal{D}_{\mathcal{U}}$ for all $R \in \mathcal{R}$.

Moreover, if $x \in X$ and $x_{\Gamma}(\gamma) = x$ for all $\gamma \in \Gamma$, then we also define

(3) $x \in \operatorname{lim}_{\mathcal{R}}(\psi)$ if $x_{\Gamma} \in \operatorname{Lim}_{\mathcal{R}}(\psi)$; (4) $x \in \operatorname{adh}_{\mathcal{R}}(\psi)$ if $x_{\Gamma} \in \operatorname{Adh}_{\mathcal{R}}(\psi)$.

Remark 8.1. This definition can be immediately generalized to the case when Φ and Ψ are relations on Γ to X and Y, respectively, and $(\Phi \otimes \Psi)(\gamma) = \Phi(\gamma) \times \Psi(\gamma)$ for all $\gamma \in \Gamma$. Moreover, $A \subseteq X$ and $A_{\Gamma}(\gamma) = A$ for all $\gamma \in \Gamma$.

However, to make the above big limit and adherence relations to be stronger tools than the big closure and interior ones, it is sufficient to consider only a *proset (preordered set)* $\Gamma(\leq)$ instead of the relator space $\Gamma(\mathcal{U})$.

Theorem 8.1. *For any* $A \subseteq X$ *and* $B \subseteq Y$*, the following assertions are equivalent :*

(1) $A \in \mathrm{Cl}_{\mathcal{R}}(B)$*;*

(2) there exist a poset $\Gamma(\leq)$ *and functions* φ *and* ψ *of* Γ *to* A *and* B*, respectively, such that* $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$*;*

(3) there exist a non-partial relator space $\Gamma(\mathcal{U})$ *and functions* φ *and* ψ *of* Γ *to* A *and* B*, respectively, such that* $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$.

Proof. For instance, if (1) holds, then for each $R \in \mathcal{R}$ we have $R[A] \cap B \neq \emptyset$. Therefore, there exist $\varphi(R) \in A$ and $\psi(R) \in B$ such that $\psi(R) \in R\big(\varphi(R)\big)$. Hence, we can already infer that $(\varphi, \psi)(R) = \big(\varphi(R), \psi(R)\big) \in R$, and thus

$$R \in (\varphi, \psi)^{-1}[R].$$

Now, by taking $\Gamma = \mathcal{R}$ and $\leq = \supseteq$, we can see that $\Gamma(\leq)$ is poset (partially ordered set) such that $\Gamma \neq \emptyset$ if $\mathcal{R} \neq \emptyset$. Moreover, if $R \in \mathcal{R}$, then $R \in \Gamma$ such that for any $S \in \Gamma$, with $S \geq R$, we have $S \subseteq R$, and thus

$$S \in (\varphi, \psi)^{-1}[S] \subseteq (\varphi, \psi)^{-1}[R].$$

This shows that $(\varphi, \psi)^{-1}[R]$ is a residual, and thus fat subset of the poset $\Gamma(\leq)$. Therefore, $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$, and thus (2) also holds.

Theorem 8.2. *For any* $A \subseteq X$ *and* $B \subseteq Y$*, the following assertions are equivalent :*

(1) $A \in \mathrm{Cl}_{\mathcal{R}}(B)$*;*

(2) there exist a proset $\Gamma(\leq)$*, with* $\Gamma \neq \emptyset$ *if* $\mathcal{R} \neq \emptyset$*, and functions* φ *and* ψ *of* Γ *to* A *and* B*, such that* $\varphi \in \mathrm{Adh}_{\mathcal{R}}(\psi)$*;*

(3) there exist a non-denerated relator space $\Gamma(\mathcal{U})$ *and functions* φ *and* ψ *of* Γ *to* A *and* B*, respectively, such that* $\varphi \in \mathrm{Adh}_{\mathcal{R}}(\psi)$.

Remark 8.2. To prove the implication (1) $\Longrightarrow$ (2), on the set $\Gamma = \mathcal{R}$ we have to consider the preorder $\leq = \Gamma^2$. Therefore, posets are usually not sufficient for our present purposes.

However, if $\mathcal{R}$ is *uniformly filtered* in the sense that for every $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T \subseteq R \cap S$, then in the proof of the implication (1) $\Longrightarrow$ (2) we can also take $\leq = \supseteq$.

Remark 8.3. In this respect, it is also worth mentioning that if a relator space $\Gamma(\mathcal{U})$ is *semi-directed* in the sense that $U(\alpha) \cap V(\beta) \neq \emptyset$ for all $\alpha, \beta \in \Gamma$ and $U, V \in \mathcal{U}$, i. e., $\mathcal{U}^{-1} \circ \mathcal{U} = \{\Gamma^2\}$, then we have $\mathcal{E}_{\mathcal{U}} \subseteq \mathcal{D}_{\mathcal{U}}$.

Therefore, by Definition 8.1, $\mathrm{Lim}_{\mathcal{R}}(\psi) \subseteq \mathrm{Adh}_{\mathcal{R}}(\psi)$ for every function ψ of Γ to Y. However, to establish some deeper relationships between the above two relations, we have to consider some *subfunctions* of a function φ of $\Gamma(\mathcal{U})$ to X.

For instance, a function ψ of a relator space $\Lambda(\mathcal{V})$ to X may be naturally called a *fat subfunction* of φ if $\varphi^{-1}[A] \in \mathcal{E}_{\mathcal{U}}$ implies $\psi^{-1}[A] \in \mathcal{E}_{\mathcal{V}}$ for all $A \subseteq X$. That is, if φ is fatly in a subset A of X, then ψ is also fatly in A.

Thus, the function φ may be called *universal* if it is fatly in any subset A of X if and only if it densely in A. Clearly, if (φ, ψ) is a universal functions of $\Gamma(\mathcal{U})$ to $X \times Y$, then $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$ and $\varphi \in \mathrm{Adh}_{\mathcal{R}}(\psi)$ are equivalent.

9. Some further theorems on convergence and adherence relations

From Theorems 8.1 and 8.2, by letting φ to be a constant function, we can easily derive the next two theorems.

Theorem 9.1. *For any* $x \in X$ *and* $B \subseteq Y$*, the following assertions are equivalent :*

(1) $x \in \mathrm{cl}_{\mathcal{R}}(B)$*;*

(2) there exist a poset $\Gamma(\leq)$ *and a function* ψ *of* Γ *to* B *such that* $x \in \mathrm{lim}_{\mathcal{R}}(\psi)$*;*

(3) there exist a non-partial relator space $\Gamma(\mathcal{U})$ *and a function* ψ *of* Γ *to* B *such that* $x \in \mathrm{lim}_{\mathcal{R}}(\psi)$.

Theorem 9.2. *For any* $x \in X$ *and* $B \subseteq Y$*, the following assertions are equivalent :*

(1) $x \in \mathrm{cl}_{\mathcal{R}}(B)$*;*

(2) there exist a proset $\Gamma(\leq)$*, with* $\Gamma \neq \emptyset$ *if* $\mathcal{R} \neq \emptyset$*, and a function* ψ *of* Γ *to* B *such that* $x \in \mathrm{adh}_{\mathcal{R}}(\psi)$*;*

(3) there exist a non-degenerated relator space $\Gamma(\mathcal{U})$ *and a function* ψ *of* Γ *to* B *such that* $x \in \mathrm{adh}_{\mathcal{R}}(\psi)$.

In addition to these theorems, it is also worth proving the following

Theorem 9.3. *For any function* ψ *of a relator space* $\Gamma(\mathcal{U})$ *to* Y, *we have*

(1) $\lim_{\mathcal{R}}(\psi) = \bigcap_{D \in \mathcal{D}_{\mathcal{U}}} \operatorname{cl}_{\mathcal{R}}\big(\psi[D]\big)$ *(2)* $\operatorname{adh}_{\mathcal{R}}(\psi) = \bigcap_{E \in \mathcal{E}_{\mathcal{U}}} \operatorname{cl}_{\mathcal{R}}\big(\psi[E]\big)$.

Proof. To prove (1), suppose first that $x \in \lim_R(\psi)$ and $D \in D_{\mathcal{U}}$. Then, by Definition 8.1, for each $R \in \mathcal{R}$ we have $(x_\Gamma, \psi)^{-1}[R] \in \mathcal{E}_{\mathcal{U}}$. Hence, by defining $E = (x_\Gamma, \psi)^{-1}[R]$, we can see that $E \in \mathcal{E}_{\mathcal{U}}$ such that

$$\big(x, \psi(\alpha)\big) = \big(x_\Gamma(\alpha), \psi(\alpha)\big) = \big(x_\Gamma\, \psi\big)(\alpha) \in R,$$

and thus $x \in R^{-1}\big(\psi(\alpha)\big)$ for all $\alpha \in E$. Moreover, by Theorem 7.6, we can state that there exists $\alpha \in E$ such that $\alpha \in D$ also holds. Thus, we also have

$$x \in R^{-1}\big(\psi(\alpha)\big) \subseteq R^{-1}[D].$$

Hence, by using Theorem 7.1, we can infer that

$$x \in \bigcap_{R \in \mathcal{R}} R^{-1}[D] = \operatorname{cl}_{\mathcal{R}}(D), \qquad \text{and thus} \qquad x \in \bigcap_{D \in \mathcal{D}_{\mathcal{U}}} \operatorname{cl}_{\mathcal{R}}\big(\psi[D]\big)$$

also holds. This proves that $\lim_{\mathcal{R}}(\psi) \subseteq \bigcap_{D \in \mathcal{D}_{\mathcal{U}}} \operatorname{cl}_{\mathcal{R}}\big(\psi[D]\big)$.

To prove the converse inclusion, next suppose $x \in \bigcap_{D \in \mathcal{D}_{\mathcal{U}}} \operatorname{cl}_{\mathcal{R}}\big(\psi[D]\big)$, and assume on the contrary that $x \notin \lim_{\mathcal{R}}(\psi)$. Then, by Definition 8.1, there exists $R \in \mathcal{R}$ such that $(x_\Gamma, \psi)^{-1}[R] \notin \mathcal{E}_{\mathcal{U}}$. Then, by defining $D = (x_\Gamma, \psi)^{-1}[R]^c$ and and using Theorem 7.6, we can see that $D \in \mathcal{D}_{\mathcal{U}}$. Moreover, quite similarly as above, we can note that $x \notin R^{-1}\big(\psi(\alpha)\big)$ for all $\alpha \in D$, and thus $x \notin R^{-1}\big[\psi[D]\big]$. Hence, by using Theorem 7.1, we can infer that $x \notin \operatorname{cl}_{\mathcal{R}}\big(\psi[D]\big)$. This contradiction proves that $x \in \lim_{\mathcal{R}}(\psi)$, and thus $\bigcap_{D \in \mathcal{D}_{\mathcal{U}}} \operatorname{cl}_{\mathcal{R}}\big(\psi[D]\big) \subseteq \lim_{\mathcal{R}}(\psi)$ also holds.

Remark 9.1. The above theorems show that the relations $\operatorname{cl}_{\mathcal{R}}$, $\lim_{\mathcal{R}}$ and $\operatorname{adh}_{\mathcal{R}}$ are usually also equivalent tools in the relator space $(X, Y)(\mathcal{R})$.

By using the corresponding definitions, we can much more easily prove the following

Theorem 9.4. *For any function* ψ *of a goset* $\Gamma(\leq)$ *to* Y, *we have*

(1) $\lim_{\mathcal{R}}(\psi) = \bigcap_{R \in \mathcal{R}} \underline{\lim}_{\gamma \in \Gamma} R^{-1}\big(\psi(\gamma)\big) = \bigcap_{R \in \mathcal{R}} \bigcup_{\alpha \in \Gamma} \bigcap_{\beta \geq \alpha} R^{-1}\big(\psi(\beta)\big)$;

(2) $\operatorname{adh}_{\mathcal{R}}(\psi) = \bigcap_{R \in \mathcal{R}} \overline{\lim}_{\gamma \in \Gamma} R^{-1}\big(\psi(\gamma)\big) = \bigcap_{R \in \mathcal{R}} \bigcap_{\alpha \in \Gamma} \bigcup_{\beta \geq \alpha} R^{-1}\big(\psi(\beta)\big)$.

Remark 9.2. Beside assertion (3) of Theorem 7.1, this theorem also shows a remarkable advantage of relator spaces over the topological ones.

Finally, we note that, by Definition 8.1 and a dual of [65, Theorem 3], the following theorem is also true.

Theorem 9.5. *The structures* Lim, lim, Adh *and* adh *are intersection-preserving.*

Remark 9.3. Thus, in particular, for a function ψ of a relator space $\Gamma(\mathcal{U})$ to Y, we have $\lim_{\mathcal{R}}(\psi) = \bigcap_{R \in \mathcal{R}} \lim_R(\psi)$ and $\operatorname{adh}_{\mathcal{R}}(\psi) = \bigcap_{R \in \mathcal{R}} \operatorname{adh}_R(\psi)$.

Therefore, the function ψ may be naturally called *convergence (adherence) Cauchy* if $\lim_R(\psi) \neq \emptyset$ $\big(\operatorname{adh}_R(\psi) \neq \emptyset\big)$ for all $R \in \mathcal{R}$.

Thus, "convergent (adherent)" trivially implies "convergence (adherence) Cauchy". Moreover, if $\mathcal{R}$ is topologically fine, then it can be shown that the converse implication is also true. (See [44, 48].)

Analogously to *completeness and compactness*, the *Lebesgue and Baire properties* can also be most nicely treated in relator spaces [40, 52]. By [30, 34], the same is true for the *well-chainedness and connectedness properties*.

10. Some further structures derived from relators

Notation 10.1. *In this section, we shall assume that* $\mathcal{R}$ *is a relator on* X.

Definition 10.1. For any $A \subseteq X$, we define:

(1) $A \in \tau_{\mathcal{R}}$ *if* $A \in \operatorname{Int}_{\mathcal{R}}(A)$; *(2)* $A \in \tau^{-}_{\mathcal{R}}$ *if* $A^c \notin \operatorname{Cl}_{\mathcal{R}}(A)$;

(3) $A \in \mathcal{T}_{\mathcal{R}}$ *if* $A \subseteq \operatorname{int}_{\mathcal{R}}(A)$; *(4)* $A \in \mathcal{F}_{\mathcal{R}}$ *if* $\operatorname{cl}_{\mathcal{R}}(A) \subseteq A$;

(5) $A \in \mathcal{N}_{\mathcal{R}}$ *if* $\operatorname{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$; *(6)* $A \in \mathcal{M}_{\mathcal{R}}$ *if* $\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{D}_{\mathcal{R}}$.

Remark 10.1. The members of the families, $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{N}_{\mathcal{R}}$ are called the *proximally open, topologically open and rare (or nowhere dense) subsets* relator spaces $X(\mathcal{R})$, respectively.

The families $\tau_{\mathcal{R}}$ and $\bar{\tau}_{\mathcal{R}}$ were first explicitly used by the second author in [41,43]. In particular, the practical notation $\bar{\tau}_{\mathcal{R}}$ has been suggested by J. Kurdics who first noticed that connectedness is a particular case of well-chainedness [19,20].

Theorem 10.1. *We have*

(1) $\bar{\tau}_{\mathcal{R}} = \tau_{\mathcal{R}^{-1}}$; *(2)* $\bar{\tau}_{\mathcal{R}} = \{ A \subseteq X : \ A^c \in \tau_{\mathcal{R}} \}$;

(3) $\mathcal{F}_{\mathcal{R}} = \{ A \subseteq X : \ A^c \in \mathcal{T}_{\mathcal{R}} \}$; *(4)* $\mathcal{M}_{\mathcal{R}} = \{ A \subseteq X : \ A^c \in \mathcal{N}_{\mathcal{R}} \}$.

Theorem 10.2. *We have*

(1) $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$; *(2)* $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$; *(3)* $\mathcal{D}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{X\}$.

Remark 10.2. In addition to assertion (1), it also worth noticing that $\tau_R = \mathcal{T}_R$ for any $R \in \mathcal{R}$.

Moreover, from assertion (3), by using global complementations, we can easily infer that $\mathcal{F}_{\mathcal{R}} \subseteq \left(\mathcal{D}_{\mathcal{R}}\right)^c \cup \{X\}$ and $\mathcal{D}_{\mathcal{R}} \subseteq \left(\mathcal{F}_{\mathcal{R}}\right)^c \cup \{X\}$.

However, it is now more important to note that we also have the following

Theorem 10.3. *For any* $A \subseteq X$ *we have*

(1) $\mathcal{P}(A) \cap \left(\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}\right) \neq \emptyset$ *implies* $A \in \mathcal{E}_{\mathcal{R}}$;

(2) $\bigcup \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A) \subseteq \operatorname{int}_{\mathcal{R}}(A)$; *(3)* $\mathcal{P}\left[\tau_{\mathcal{R}} \cap \mathcal{P}(A)\right] \subseteq \operatorname{Int}_{\mathcal{R}}(A)$.

Remark 10.3. The fat sets are frequently more convenient tools than the open ones. For instance, if $\leq$ is a relation on X, then $\mathcal{T}_{\leq}$ and $\mathcal{E}_{\leq}$ are the families of all *ascending and residual subsets* of the goset $X(\leq)$, respectively.

This fact, stressed first by the second author at the Seventh Prague Topological Symposium in 1991, can also be well seen from the following

Example 10.1. If in particular $X = \mathbb{R}$ and $R(x) = \{x-1\} \cup [x, +\infty[$ for all $x \in X$, then R is a reflexive relation on X such that $\mathcal{T}_R = \{\emptyset, X\}$, but $\mathcal{E}_R$ is quite a large family.

Remark 10.4. However, if the relator is *topological or proximal* in the sense that:

(1) for each $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$;

(2) for each $A \subseteq X$ and $R \in \mathcal{R}$ there exists $V \in \tau_{\mathcal{R}}$ such that $A \subseteq V \subseteq R[A]$;

respectively, then the converses of the assertions (1)–(3) of Theorem 10.3 can also be proved. Therefore, in these cases, the families $\mathcal{T}_{\mathcal{R}}$ and $\tau_{\mathcal{R}}$ are also quite powerful tools.

Finally, we note that, by Definition 10.1 and [65, Theorem 3], the following theorem is also true.

Theorem 10.4. *The structure* τ *is also union-preserving.*

The following simple example shows that the increasing structure $\mathcal{T}$ need not be union-preserving. This is a serious disadvantage of the topologically open sets.

Example 10.2. If $\operatorname{card}(X) > 2$ and $x_1, x_2 \in X$ such that $x_1 \neq x_2$, and

$$R_i = \{x_i\}^2 \cup \left(\{x_i\}^c\right)^2$$

for all $i = 1, 2$, then $\mathcal{R} = \{R_1, R_2\}$ is an equivalence relator on X such that $\{x_1, x_2\} \in \mathcal{T}_{\mathcal{R}} \setminus \left(\mathcal{T}_{R_1} \cup \mathcal{T}_{R_2}\right)$, and thus $\mathcal{T}_{\mathcal{R}} \not\subseteq \mathcal{T}_{R_1} \cup \mathcal{T}_{R_2}$.

Remark 10.5. Later, by using the *topological closure (refinement)* $\mathcal{R}^\wedge$ of $\mathcal{R}$, we can see that $\mathcal{T}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}^\wedge} \mathcal{T}_{\mathcal{R}}$.

11. Some algebraic structures derived from relators

Notation 11.1. *In this section, we shall assume that* $\mathcal{R}$ *is a relator on* X *to* Y.

According to [51], we may also naturally introduce the following counterpart of Definition 7.1.

Definition 11.1. For any A, B, $x \in X$ and $y \in Y$, we define

(1) $A \in \mathrm{Lb}_{\mathcal{R}}(B)$ *and* $B \in \mathrm{Ub}_{\mathcal{R}}(A)$ *if* $A \times B \subseteq R$ *for some* $R \in \mathcal{R}$;

(2) $x \in \mathrm{lb}_{\mathcal{R}}(B)$ *if* $\{x\} \in \mathrm{Lb}_{\mathcal{R}}(B)$; *(3)* $y \in \mathrm{ub}_{\mathcal{R}}(A)$ *if* $\{y\} \in \mathrm{Ub}_{\mathcal{R}}(A)$;

(4) $A \in \mathfrak{L}_{\mathcal{R}}$ *if* $\mathrm{lb}_{\mathcal{R}}(A) \neq \emptyset$; *(5)* $B \in \mathfrak{U}_{\mathcal{R}}$ *if* $\mathrm{ub}_{\mathcal{R}}(A) \neq \emptyset$.

Thus, we evidently have the following

Theorem 11.1. *We have*

(1) $\mathrm{Ub}_{\mathcal{R}} = \mathrm{Lb}_{\mathcal{R}^{-1}} = \mathrm{Lb}_{\mathcal{R}}^{-1}$; *(2)* $\mathrm{ub}_{\mathcal{R}} = \mathrm{lb}_{\mathcal{R}^{-1}}$.

Moreover, we can also easily prove the following

Theorem 11.2. *We have*

(1) $\mathrm{Lb}_{\mathcal{R}} = \mathrm{Cl}_{\mathcal{R}^c}^{c} = \mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}_Y$; *(2)* $\mathrm{lb}_{\mathcal{R}} = \mathrm{cl}_{\mathcal{R}^c}^{c} = \mathrm{int}_{\mathcal{R}^c} \circ \mathcal{C}_Y$.

Proof. By Definitions 11.1 and 7.1, for any $A \subseteq X$ and $B \subseteq Y$ we have

$$\begin{aligned} A \in \mathrm{Lb}_{\mathcal{R}}(B) \iff \exists\, R \in \mathcal{R}: \ A \times B \subseteq R \iff \exists\, R \in \mathcal{R}: \ \forall\, (a, b) \in A \times B: \ (a, b) \notin R^c \\ \iff \exists\, R \in \mathcal{R}: \ \forall\, a \in A,\ b \in B: \ b \notin R^c(a) \iff \exists\, R \in \mathcal{R}: \ R^c[A] \cap B = \emptyset \\ \iff A \notin \mathrm{Cl}_{\mathcal{R}^c}(B) \iff A \in \mathrm{Cl}_{\mathcal{R}^c}(B)^c \iff A \in \mathrm{Cl}_{\mathcal{R}^c}^{c}(B). \end{aligned}$$

Therefore, $\mathrm{Lb}_{\mathcal{R}}(B) = \mathrm{Cl}_{\mathcal{R}^c}^{c}(B)$ for all $B \subseteq Y$, and thus the first part of (1) is true. The second part of (1) can be derived from the first part of (1) by using Theorem 7.1.

Remark 11.1. The above two theorem show that, for instance, the relations $\mathrm{Lb}_{\mathcal{R}}$, $\mathrm{Ub}_{\mathcal{R}}$ and $\mathrm{Cl}_{\mathcal{R}}$ and $\mathrm{Int}_{\mathcal{R}}$ are also equivalent tools in the relator space $(X, Y)(\mathcal{R})$.

Thus, in contrast to a common belief, some algebraic and topological structures are just as closely related to each other by the above equalities as the exponential and trigonometric functions are by the famous Euler formulas.

By using the corresponding definitions, we can also prove the following

Theorem 11.3. *For any* $A \subseteq X$, *we have*

(1) $\mathrm{ub}_{\mathcal{R}}(A) = \bigcup_{R \in \mathcal{R}} R^c(A)^c = \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} R(a)$.

(2) $\mathrm{Ub}_{\mathcal{R}}(A) = \bigcup_{R \in \mathcal{R}} \mathcal{P}\big(R^c(A)^c\big) = \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} \mathcal{P}\big(R(a)\big)$;

(3) $\mathrm{Ub}_{\mathcal{R}}(A) \subseteq \mathcal{P}\big(\mathrm{ub}_{\mathcal{R}}(A)\big)$; *(4)* $\mathrm{Ub}_{\mathcal{R}}(A) = \big\{ B \subseteq Y : \ \mathcal{P}(A) \subseteq \mathrm{Lb}_{\mathcal{R}}(B) \big\}$.

Remark 11.2. Moreover, we can also easily prove that $\mathrm{Ub}_{\mathcal{R}} = \mathcal{P} \circ \mathrm{Ub}_{\mathcal{R}} = \mathrm{Ub}_{\mathcal{R}} \circ \mathcal{P}^{-1}$.

However, it is now more important to note that, by the corresponding definitions, we also have

Theorem 11.4. *For any* $R \in \mathcal{R}$ *and* $A \subseteq X$ *and* $B \subseteq Y$, *we have*

$$A \subseteq \mathrm{lb}_R(B) \iff B \subseteq \mathrm{ub}_R(A).$$

Remark 11.3. This shows that the mappings $A \mapsto \mathrm{ub}_R(A)$ and $B \mapsto \mathrm{lb}_R(B)$ establish a Galois connection between the poset $\mathcal{P}(X)$ and the dual of $\mathcal{P}(Y)$.

Finally, we note that, by Definition 10.1 and [65, Theorem 3], the following theorem is also true.

Theorem 11.5. *The structures* Lb, lb *and* $\mathfrak{L}$ *are union-preserving.*

12. Some further algebraic structures derived from relators

Notation 12.1. *In this section, we shall assume that* $\mathcal{R}$ *is a relator on* X.

Now, in addition to Definition 11.1, we may also naturally introduce the following

Definition 12.1. For any $A \subseteq X$, we define

(1) $\min_{\mathcal{R}}(A) = A \cap \mathrm{lb}_{\mathcal{R}}(A)$; *(2)* $\max_{\mathcal{R}}(A) = A \cap \mathrm{ub}_{\mathcal{R}}(A)$;

(3) $\mathrm{Min}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \mathrm{Lb}_{\mathcal{R}}(A)$; *(4)* $\mathrm{Max}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \mathrm{Ub}_{\mathcal{R}}(A)$;

(5) $\inf_{\mathcal{R}}(A) = \max_{\mathcal{R}}\big(\mathrm{lb}_{\mathcal{R}}(A)\big)$; *(6)* $\sup_{\mathcal{R}}(A) = \min_{\mathcal{R}}\big(\mathrm{ub}_{\mathcal{R}}(A)\big)$;

(7) $\mathrm{Inf}_{\mathcal{R}}(A) = \mathrm{Max}_{\mathcal{R}}\big[\mathrm{Lb}_{\mathcal{R}}(A)\big]$; *(8)* $\mathrm{Sup}_{\mathcal{R}}(A) = \mathrm{Min}_{\mathcal{R}}\big[\mathrm{Ub}_{\mathcal{R}}(A)\big]$;

(9) $A \in \ell_{\mathcal{R}}$ *if* $A \in \mathrm{Lb}_{\mathcal{R}}(A)$; *(10)* $A \in \mathcal{L}_{\mathcal{R}}$ *if* $A \subseteq \mathrm{lb}_{\mathcal{R}}(A)$.

Remark 12.1. Thus, for instance, for any $x \in X$ and $A \subseteq X$ we also have

$$x \in \min_{\mathcal{R}}(A) \iff x \in A,\ x \in \mathrm{lb}_{\mathcal{R}}(A) \iff \{x\} \in \mathcal{P}(A),\ \{x\} \in \mathrm{Lb}_{\mathcal{R}}(A) \iff \{x\} \in \mathrm{Min}_{\mathcal{R}}(A).$$

However, a similar statement for the relations $\sup_{\mathcal{R}}$ and $\mathrm{Sup}_{\mathcal{R}}$ seems not to be true. Therefore, the definition of $\mathrm{Sup}_{\mathcal{R}}$, and thus also that of $\mathrm{Inf}_{\mathcal{R}}$, is perhaps not the most convenient one.

By using Definition 12.1, in [51], the second author proved the following theorems.

Theorem 12.1. *We have*

(1) $\max_{\mathcal{R}} = \min_{\mathcal{R}^{-1}}$; *(2)* $\mathrm{Max}_{\mathcal{R}} = \mathrm{Min}_{\mathcal{R}^{-1}}$.

Theorem 12.2. *For any* $A \subseteq X$, *we have*

(1) $\mathrm{Max}_{\mathcal{R}}(A) \subseteq \mathcal{P}\big(\max_{\mathcal{R}}(A)\big)$; *(2)* $\mathrm{Max}_{\mathcal{R}}(A) = \big\{ B \subseteq X :\ \mathcal{P}(A) \subseteq \mathrm{Lb}_{\mathcal{R}}(B)\big\}$.

Remark 12.2. Concerning the relation $\mathrm{Max}_{\mathcal{R}}$, it also worth mentioning that $\mathrm{Max}_{\mathcal{R}} = \mathcal{P} \circ \mathrm{Max}_{\mathcal{R}}$.

Theorem 12.3. *For any* $A \subseteq X$, *we have*

(1) $\min_{\mathcal{R}}(A) = A \setminus \mathrm{cl}_{\mathcal{R}^c}(A) = A \cap \mathrm{int}_{\mathcal{R}^c}(A^c)$; *(2)* $\mathrm{Min}_{\mathcal{R}}(A) = \mathcal{P}(A) \setminus \mathrm{Cl}_{\mathcal{R}^c}(A) = \mathcal{P}(A) \cap \mathrm{Int}_{\mathcal{R}^c}(A^c)$.

Remark 12.3. The latter assertion can be expressed in the more concise form that $\mathrm{Min}_{\mathcal{R}} = \mathcal{P} \setminus \mathrm{Cl}_{\mathcal{R}^c} = \mathcal{P} \cap \big(\mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}_X\big)$.

Theorem 12.4. *For any* $\emptyset \neq A \subseteq X$, *we have*

(1) $\max_{\mathcal{R}}(A) = \bigcup_{R \in \mathcal{R}} \big(A \setminus R^c[A]\big) = \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} A \cap R(a)$;

(2) $\mathrm{Max}_{\mathcal{R}}(A) = \bigcup_{R \in \mathcal{R}} \mathcal{P}\big(A \setminus R^c[A]\big) = \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} \mathcal{P}\big(A \cap R(a)\big)$.

Theorem 12.5. *For any* $A \subseteq X$, *the following assertions are equivalent:*

(1) $A \in \ell_{\mathcal{R}}$; *(2)* $A \in \mathrm{Ub}_{\mathcal{R}}(A)$; *(3)* $A \in \mathrm{Min}_{\mathcal{R}}(A)$; *(4)* $A \in \mathrm{Max}_{\mathcal{R}}(A)$.

Remark 12.4. By using Theorem 10.1, assertion (1) can also be reformulated in the form that $A \notin \mathrm{Cl}_{\mathcal{R}^c}(A)$, or equivalently $A \in \mathrm{Int}_{\mathcal{R}^c}(A^c)$.

Moreover, by using the corresponding definitions, we can also easily prove that $\ell_{\mathcal{R}} = \mathrm{Min}_{\mathcal{R}}[\mathcal{P}(X)] = \mathrm{Max}_{\mathcal{R}}[\mathcal{P}(X)]$.

Theorem 12.6. *For any* $A \subseteq X$, *the following assertions are equivalent:*

(1) $A \in \mathcal{L}_{\mathcal{R}}$; *(2)* $A = \min_{\mathcal{R}}(A)$; *(3)* $A \in \bigcap_{a \in A} \mathrm{lb}_{\mathcal{R}}^{-1}\big(\{a\}\big)$.

Remark 12.5. By using Theorem 11.2, assertion (1) can also be reformulated in the form that $\mathrm{cl}_{\mathcal{R}^c}(A) \subseteq A^c$, or equivalently $A \subseteq \mathrm{int}_{\mathcal{R}^c}(A^c)$.

Theorem 12.7. *We have*

(1) $\ell_{\mathcal{R}} = \ell_{\mathcal{R}^{-1}}$; *(2)* $\ell_{\mathcal{R}} \subseteq \mathcal{L}_{\mathcal{R}} \cap \mathcal{L}_{\mathcal{R}^{-1}}$; *(3)* $\mathcal{L}_{\mathcal{R}} = \{\min_{\mathcal{R}}(A) :\ A \subseteq X\}$.

13. Closure operations for relators

Notation 13.1. *In this and the next section, we shall assume that* $\mathcal{R}$ *is a relator on* X *to* Y.

Some of the following operations were already considered by Kenyon [17] and Nakano and Nakano [26].

Definition 13.1. The relators

$$\mathcal{R}^{*} = \big\{ S \subseteq X \times Y : \quad \exists\ R \in \mathcal{R} : \quad R \subseteq S \big\},$$
$$\mathcal{R}^{\#} = \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad \exists\ R \in \mathcal{R} : \quad R[A] \subseteq S[A] \big\},$$
$$\mathcal{R}^{\wedge} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad \exists\ R \in \mathcal{R} : \quad R(x) \subseteq S(x) \big\},$$

$$\mathcal{R}^{\triangle} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad \exists\ u \in X : \quad \exists\ R \in \mathcal{R} : \quad R(u) \subseteq S(x) \big\}$$

are called the *uniform, proximal, topological and paratopological closures (or refinements)* of the relator $\mathcal{R}$, respectively.

Remark 13.1. Thus, we evidently have $\mathcal{R} \subseteq \mathcal{R}^{*} \subseteq \mathcal{R}^{\#} \subseteq \mathcal{R}^{\wedge} \subseteq \mathcal{R}^{\triangle}$. Moreover, if $\mathcal{R}$ is a relator on X, then we can easily prove that $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{*\infty} \subseteq \mathcal{R}^{\infty *} \subseteq \mathcal{R}^{*}$.

Remark 13.2. Moreover, by Definitions 13.1 and 7.1, we have

$$\mathcal{R}^{\#} = \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad A \in \operatorname{Int}_{\mathcal{R}}\big(S[A] \big) \big\};$$
$$\mathcal{R}^{\wedge} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad x \in \operatorname{int}_{\mathcal{R}}\big(S(x) \big) \big\};$$
$$\mathcal{R}^{\triangle} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad S(x) \in \mathcal{E}_{\mathcal{R}} \big\}.$$

However, the importance of the operation $*$ can only be seen from the following

Theorem 13.1. *$*$ is a closure operation for relators on X to Y such that, for any relator $\mathcal{S}$ on X to Y, we have*

(1) $\mathcal{S} \subseteq \mathcal{R}^{*} \iff \operatorname{Lim}_{\mathcal{R}} \subseteq \operatorname{Lim}_{\mathcal{S}}$; *(2)* $\mathcal{S} \subseteq \mathcal{R}^{*} \iff \operatorname{Adh}_{\mathcal{R}} \subseteq \operatorname{Adh}_{\mathcal{S}}$ *if* $\mathcal{R} \neq \emptyset$.

Proof. For instance, if $\mathcal{S} \not\subseteq \mathcal{R}^{*}$, then there exists $S \in \mathcal{S}$ such that $S \notin \mathcal{R}^{*}$. Thus, by Definition 13.1, for any $R \in \mathcal{R}$ we have $R \not\subseteq S$. Therefore, there exists $\big(\varphi(R),\ \psi(R)\big) \in R$ such that $\big(\varphi(R),\ \psi(R)\big) \notin S$. Hence, as in the proof of Theorem 8.1, we can infer that

$$R \in (\varphi,\ \psi)^{-1}[R] \qquad \text{and} \qquad (\varphi,\ \psi)^{-1}[S] = \emptyset.$$

Now, by taking $\Gamma = \mathcal{R}$ and $\leq\, = \,\supseteq$, we can see that $\Gamma(\leq)$ is a poset. Moreover, as in the proof of Theorem 8.1, we can show that $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$.

Furthermore, we can note that $\emptyset \notin \mathcal{E}_{\leq}$, and thus $(\varphi,\ \psi)^{-1}[S] \notin \mathcal{E}_{\leq}$. Therefore, by Definition 8.1, we have $\varphi \notin \operatorname{Lim}_{\mathcal{S}}(\psi)$, and thus $\operatorname{Lim}_{\mathcal{S}} \not\subseteq \operatorname{Lim}_{\mathcal{R}}$. This proves that $\operatorname{Lim}_{\mathcal{S}} \subseteq \operatorname{Lim}_{\mathcal{R}}$ implies $\mathcal{S} \subseteq \mathcal{R}^{*}$.

Remark 13.3. To prove the implication $\operatorname{Adh}_{\mathcal{R}} \subseteq \operatorname{Adh}_{\mathcal{S}} \implies \mathcal{S} \subseteq \mathcal{R}^{*}$, on the set $\Gamma = \mathcal{R}$ we have again to consider the preorder $\leq\, = \Gamma^{2}$. Moreover, we have to note that $\emptyset \notin \mathcal{D}_{\leq}$ if $\mathcal{R} \neq \emptyset$.

Corollary 13.1. *The following assertion are true :*

(1) $\mathcal{S} = \mathcal{R}^{*}$ *is the largest relator on X to Y such that* $\operatorname{Lim}_{\mathcal{S}} = \operatorname{Lim}_{\mathcal{R}}$;

(2) if $\mathcal{R} \neq \emptyset$, *then* $\mathcal{S} = \mathcal{R}^{*}$ *is the largest relator on X to Y such that* $\operatorname{Adh}_{\mathcal{S}} = \operatorname{Adh}_{\mathcal{R}}$.

Proof. To prove (1), note that, by Theorem 13.1, the inclusion $\mathcal{R}^{*} \subseteq \mathcal{R}^{*}$ implies $\operatorname{Lim}_{\mathcal{R}} \subseteq \operatorname{Lim}_{\mathcal{R}^{*}}$, and $\mathcal{R} \subseteq \mathcal{R}^{**}$ implies $\operatorname{Lim}_{\mathcal{R}^{*}} \subseteq \operatorname{Lim}_{\mathcal{R}}$. Therefore, we actually have $\operatorname{Lim}_{\mathcal{R}^{*}} = \operatorname{Lim}_{\mathcal{R}}$.

Moreover, if $\mathcal{S}$ is a relator on X to Y such that $\operatorname{Lim}_{\mathcal{R}} \subseteq \operatorname{Lim}_{\mathcal{S}}$ then by Theorem 13.1 we have $\mathcal{S} \subseteq \mathcal{R}^{*}$. Therefore, $\mathcal{S} = \mathcal{R}^{*}$ is actually the largest relator on X to Y such that $\operatorname{Lim}_{\mathcal{R}} \subseteq \operatorname{Lim}_{\mathcal{S}}$.

Remark 13.4. Note that the implication $\mathcal{S} \subseteq \mathcal{R}^{*} \implies \operatorname{Adh}_{\mathcal{R}} \subseteq \operatorname{Adh}_{\mathcal{S}}$ is always true. Therefore, by the first part of the above proof, the equality $\operatorname{Adh}_{\mathcal{R}^{*}} = \operatorname{Adh}_{\mathcal{R}}$ is also always true.

Now, analogously to Theorem 13.1 and it corollary, we can also prove the following theorem and its corollary.

Theorem 13.2. *For any relator $\mathcal{S}$ on X to Y, we have*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\wedge} \iff \operatorname{lim}_{\mathcal{R}} \subseteq \operatorname{lim}_{\mathcal{S}}$; *(2)* $\mathcal{S} \subseteq \mathcal{R}^{\wedge} \iff \operatorname{adh}_{\mathcal{R}} \subseteq \operatorname{adh}_{\mathcal{S}}$ *if* $\mathcal{R} \neq \emptyset$.

Corollary 13.2. *The following assertion are true :*

(1) $\mathcal{S} = \mathcal{R}^{\wedge}$ *is the largest relator on X to Y such that* $\operatorname{lim}_{\mathcal{S}} = \operatorname{lim}_{\mathcal{R}}$;

(2) if $\mathcal{R} \neq \emptyset$, *then* $\mathcal{S} = \mathcal{R}^{\wedge}$ *is the largest relator on X to Y such that* $\operatorname{adh}_{\mathcal{S}} = \operatorname{adh}_{\mathcal{R}}$.

14. Some basic theorems on the operations $\wedge$, $\#$ and $\triangle$

Analogously to Theorem 13.1 and its corollary, one can also prove the following theorem and its corollary. However, these statements can be more briefly proved by using the results of Sections 4 and 5.

Theorem 14.1. *$\#$, $\wedge$ and $\triangle$ are closure operations for relators on X to Y such that, for any relator $\mathcal{S}$ on X to Y,*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\#} \iff \mathrm{Int}_{\mathcal{S}} \subseteq \mathrm{Int}_{\mathcal{R}} \iff \mathrm{Cl}_{\mathcal{R}} \subseteq \mathrm{Cl}_{\mathcal{S}}$*;*

(2) $\mathcal{S} \subseteq \mathcal{R}^{\wedge} \iff \mathrm{int}_{\mathcal{S}} \subseteq \mathrm{int}_{\mathcal{R}} \iff \mathrm{cl}_{\mathcal{R}} \subseteq \mathrm{cl}_{\mathcal{S}}$*;*

(3) $\mathcal{S} \subseteq \mathcal{R}^{\triangle} \iff \mathcal{E}_{\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}} \iff \mathcal{D}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{S}}$.

Proof. To prove the first part of assertion (1), note that, by the notation of Definition 5.1, we now have

$$\Phi_{\mathrm{Int}}(\mathcal{R}) = \{ S \subseteq X \times Y : \quad \mathrm{Int}_{S} \subseteq \mathrm{Int}_{\mathcal{R}} \}.$$

Therefore, if for instance $S \in \Phi_{\mathrm{Int}}(\mathcal{R})$, then for any $A \subseteq X$ and $B \subseteq Y$ the implication $A \in \mathrm{Int}_{S}(B) \implies A \in \mathrm{Int}_{\mathcal{R}}(B)$ holds. Hence, by noticing that $A \in \mathrm{Int}_{S}\big(S[A]\big)$, we can infer that $A \in \mathrm{Int}_{\mathcal{R}}\big(S[A]\big)$. Thus, by Remark 13.2, we also have $S \in \mathcal{R}^{\#}$.

Remark 14.1. Note that here, for instance, we do not need to prove directly that $\#$ is a closure operation for relators. Namely, it is an immediate consequence of the first part of assertion (1) and our former Theorem 4.2.

Corollary 14.1. *The following assertions are true:*

(1) $\mathcal{S} = \mathcal{R}^{\#}$ *is the largest relator on X to Y such that* $\mathrm{Int}_{\mathcal{S}} = \mathrm{Int}_{\mathcal{R}}$ $\big(\mathrm{Cl}_{\mathcal{S}} = \mathrm{Cl}_{\mathcal{R}}\big)$*;*

(2) $\mathcal{S} = \mathcal{R}^{\wedge}$ *is the largest relator on X to Y such that* $\mathrm{int}_{\mathcal{S}} = \mathrm{int}_{\mathcal{R}}$ $\big(\mathrm{cl}_{\mathcal{S}} = \mathrm{cl}_{\mathcal{R}}\big)$*;*

(3) $\mathcal{S} = \mathcal{R}^{\triangle}$ *is the largest relator on X to Y such that* $\mathcal{E}_{\mathcal{S}} = \mathcal{E}_{\mathcal{R}}$ $\big(\mathcal{D}_{\mathcal{S}} = \mathcal{D}_{\mathcal{R}}\big)$.

Concerning the above basic closure operations, we can also easily prove the following

Theorem 14.2. *We have*

(1) $\mathcal{R}^{\#} = \mathcal{R}^{*\#} = \mathcal{R}^{\#*}$*;*

(2) $\mathcal{R}^{\wedge} = \mathcal{R}^{\Diamond\wedge} = \mathcal{R}^{\wedge\Diamond}$ *with* $\Diamond = *$ *and* $\#$*;* *(3)* $\mathcal{R}^{\triangle} = \mathcal{R}^{\Diamond\triangle} = \mathcal{R}^{\triangle\Diamond}$ *with* $\Diamond = *, \#$ *and* $\wedge$.

Proof. To prove (1), note that, by Remark 13.1 and the closure properties, we have $\mathcal{R}^{\#} \subseteq \mathcal{R}^{\#*} \subseteq \mathcal{R}^{\#\#} = \mathcal{R}^{\#}$ and $\mathcal{R}^{\#} \subseteq \mathcal{R}^{*\#} \subseteq \mathcal{R}^{\#\#} = \mathcal{R}^{\#}$.

Remark 14.2. By using Remark 13.1 or some former results on Galois connections, we can also easily prove that $\mathcal{R}^{*\infty} = \mathcal{R}^{\infty*\infty}$ and $\mathcal{R}^{\infty*} = \mathcal{R}^{*\infty*}$.

However, it is now more important to note that now we also have the following

Theorem 14.3. *We have*

(1) $\mathcal{R}^{*-1} = \mathcal{R}^{-1*}$*;* *(2)* $\mathcal{R}^{\#-1} = \mathcal{R}^{-1\#}$.

Proof. To prove (2), note that by Theorems 7.3 and Corollary 14.1 we have $\mathrm{Cl}_{\mathcal{R}^{\#-1}} = \mathrm{Cl}_{\mathcal{R}^{\#}}^{-1} = \mathrm{Cl}_{\mathcal{R}}^{-1} = \mathrm{Cl}_{\mathcal{R}^{-1}}$, and thus in particular $\mathrm{Cl}_{\mathcal{R}^{-1}} \subseteq \mathrm{Cl}_{\mathcal{R}^{\#-1}}$. Hence, by using Theorem 14.1, we can infer that $\mathcal{R}^{\#-1} \subseteq \mathcal{R}^{-1\#}$. Now, by writing $\mathcal{R}^{-1}$ in place of $\mathcal{R}$, we can see that assertion (2) is also true.

Remark 14.3. Note that the elementwise operations c and ∞ are also inversion compatible. Moreover, the operation ∂ is also inversion compatible.

Namely, for any relator $\mathcal{R}$ and relation S on X, we have

$$S \in \mathcal{R}^{-1\partial} \iff S^{\infty} \in \mathcal{R}^{-1} \iff S^{\infty-1} \in \mathcal{R} \iff S^{-1\infty} \in \mathcal{R} \iff S^{-1} \in \mathcal{R}^{\partial} \iff S \in \mathcal{R}^{\partial-1}.$$

However, for instance, the operations $\wedge$ and $\triangle$ are not inversion compatible. Therefore, in addition to Definition 13.1, we must also introduce the following

Definition 14.1. We define $\mathcal{R}^{\vee} = \mathcal{R}^{\wedge-1}$ and $\mathcal{R}^{\triangledown} = \mathcal{R}^{\triangle-1}$.

Remark 14.4. The latter operations have very curious properties. For instance, if $\mathcal{R} \neq \emptyset$, then $\mathcal{R}^{\vee\wedge} = \{ \rho_{\mathcal{R}} \}^{\wedge}$, and thus in particular the relator $\mathcal{R}^{\vee}$ is topologically simple. (For some generalizations, see [24].)

Moreover, the operations $\vee\vee$ and $\triangledown\triangledown$, ,already coincide with the extremal closure operations $\bullet$ and $\blacklozenge$, defined for any relator $\mathcal{R}$ on X to Y such that $\mathcal{R}^{\bullet} = \{ \bigcap \mathcal{R} \}^*$ and

$$\mathcal{R}^{\blacklozenge} = \mathcal{R} \quad \text{if} \quad \mathcal{R} = \{ X \times Y \} \qquad \text{and} \qquad \mathcal{R}^{\blacklozenge} = \mathcal{P}(X \times Y) \quad \text{if} \quad \mathcal{R} \neq \{ X \times Y \}.$$

The importance of the operation $\blacklozenge$ lies in the fact that it is the ultimate stable unary operation for relators on X to Y.

15. Some further theorems on the operations $\wedge$ and $\vartriangle$

A preliminary form of the following fundamental theorem was already proved in [37].

Theorem 15.1. *If $\mathcal{R}$ is a nonvoid relator on X to Y, then for any $B \subseteq Y$ we have:*

(1) $\operatorname{Int}_{\mathcal{R}^{\wedge}}(B) = \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$; *(2)* $\operatorname{Cl}_{\mathcal{R}^{\wedge}}(B) = \mathcal{P}\big(\operatorname{cl}_{\mathcal{R}}(B)^c\big)^c$.

Proof. To prove (1), note that if $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(B)$, then by Theorem 7.2 and Corollary 14.1 we have $A \subseteq \operatorname{int}_{\mathcal{R}^{\wedge}}(B) = \operatorname{int}_{\mathcal{R}}(B)$, and thus $A \in \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$. Therefore, $\operatorname{Int}_{\mathcal{R}^{\wedge}}(B) \subseteq \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$.

While, if $A \in \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$, then $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$. Therefore, for each $x \in A$, there exists $R_x \in \mathcal{R}$ such that $R_x(x) \subseteq B$. Now, by defining

$$S(x) = R_x(x) \quad \text{for all} \quad x \in A \qquad \text{and} \qquad S(x) = Y \quad \text{for all} \quad x \in A^c,$$

we can at once state that $S[A] \subseteq B$. Moreover, by using that $\mathcal{R} \neq \emptyset$, we can also easily note that $S \in \mathcal{R}^{\wedge}$. Therefore, we also have $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(B)$. Consequently, $\mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big) \subseteq \operatorname{Int}_{\mathcal{R}^{\wedge}}(B)$, and thus (1) also holds.

Remark 15.1. By assertion (2), for any $A \subseteq X$, we have $A \in \operatorname{Cl}_{\mathcal{R}^{\wedge}}(B)$ if and only if $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$.

From the above theorem, by using Definition 10.1, we can immediately derived

Corollary 15.1. *If $\mathcal{R}$ is a nonvoid relator on X, then*

(1) $\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$; *(2)* $\tau\!\!\!\!\!-_{\mathcal{R}^{\wedge}} = \mathcal{F}_{\mathcal{R}}$.

Remark 15.2. Recall that, by Theorem 10.4 and Remark 10.2, we have $\tau_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \tau_R = \bigcup_{R \in \mathcal{R}} \mathcal{T}_R$. Hence, by writing $\mathcal{R}^{\wedge}$ in place of $\mathcal{R}$ and using Corollary 15.1, we can immediately infer that $\mathcal{T}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}^{\wedge}} \mathcal{T}_R$.

From Corollary 15.1, by writing $\mathcal{R}^{\vartriangle}$ and using Theorem 14.3, we can also immediately derive

Corollary 15.2. *If $\mathcal{R}$ is a nonvoid relator on X, then*

(1) $\tau_{\mathcal{R}^{\vartriangle}} = \mathcal{T}_{\mathcal{R}^{\vartriangle}}$; *(2)* $\tau\!\!\!\!\!-_{\mathcal{R}^{\vartriangle}} = \mathcal{F}_{\mathcal{R}^{\vartriangle}}$.

Concerning the operation $\vartriangle$, we can also prove the following

Theorem 15.2. *If $\mathcal{R}$ is a nonvoid relator on X to Y, then for any $B \subseteq Y$ we have:*

(1) $\operatorname{Int}_{\mathcal{R}^{\vartriangle}}(B) = \{\emptyset\}$ *if* $B \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{Int}_{\mathcal{R}^{\vartriangle}}(B) = \mathcal{P}(X)$ *if* $B \in \mathcal{E}_{\mathcal{R}}$;

(2) $\operatorname{Cl}_{\mathcal{R}^{\vartriangle}}(B) = \emptyset$ *if* $B \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{Cl}_{\mathcal{R}^{\vartriangle}}(B) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if* $B \in \mathcal{D}_{\mathcal{R}}$.

Proof. If $A \in \operatorname{Int}_{\mathcal{R}^{\vartriangle}}(B)$, then there exists $S \in \mathcal{R}^{\vartriangle}$ such that $S[A] \subseteq B$. Therefore, if $A \neq \emptyset$, then there exists $x \in X$ such that $S(x) \subseteq B$. Hence, by using that $S(x) \in \mathcal{E}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ is ascending, we can infer that $B \in \mathcal{E}_{\mathcal{R}}$. Therefore, if $B \notin \mathcal{E}_{\mathcal{R}}$, then we necessarily have $\operatorname{Int}_{\mathcal{R}^{\vartriangle}}(B) \subseteq \{\emptyset\}$. Moreover, since $\mathcal{R} \neq \emptyset$, we can also note that $\mathcal{R}^{\vartriangle} \neq \emptyset$, and thus $\emptyset \in \operatorname{Int}_{\mathcal{R}^{\vartriangle}}(B)$. Therefore, the first part of assertion (1) is true.

On the other hand, if $B \in \mathcal{E}_{\mathcal{R}}$, then by defining $R = X \times B$ and using Remark 13.2, we can see that $R \in \mathcal{R}^{\vartriangle}$. Moreover, we can also note that $R[A] \subseteq B$, and thus $A \in \operatorname{Int}_{\mathcal{R}^{\vartriangle}}(B)$ for all $A \subseteq X$. Therefore, the second part of assertion (1) is also true.

From this theorem, by Definition 7.1, it is clear that in particular we also have

Corollary 15.3. *If $\mathcal{R}$ is a nonvoid relator on X to Y, then for any $B \subseteq Y$:*

(1) $\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(B) = \emptyset$ *if* $B \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(B) = X$ *if* $B \in \mathcal{D}_{\mathcal{R}}$;

(2) $\operatorname{int}_{\mathcal{R}^{\vartriangle}}(B) = \emptyset$ *if* $B \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{int}_{\mathcal{R}^{\vartriangle}}(B) = X$ *if* $B \in \mathcal{E}_{\mathcal{R}}$.

Hence, by using Definitions 7.1 and 10.1, we can immediately derive

Corollary 15.4. *If* $\mathcal{R}$ *is a relator on* X, *then*

(1) $\mathcal{T}_{\mathcal{R}^{\triangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$; *(2)* $\mathcal{F}_{\mathcal{R}^{\triangle}} = \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}$.

Remark 15.3. Note that if in particular $\mathcal{R} = \emptyset$, then $\mathcal{E}_{\mathcal{R}} = \emptyset$. Moreover, $\mathcal{R}^{\triangle} = \emptyset$ if $X \neq \emptyset$, and $\mathcal{R}^{\triangle} = \{\emptyset\}$ if $X = \emptyset$. Therefore, $\mathcal{T}_{\mathcal{R}^{\triangle}} = \{\emptyset\}$, and thus assertion (1) is still true.

Now, since $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ if $\mathcal{R}$ is non-partial, we can also state

Corollary 15.5. *If* $\mathcal{R}$ *is a non-partial relator on* X, *then*

(1) $\mathcal{E}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\triangle}} \setminus \{\emptyset\}$, *(2)* $\mathcal{D}_{\mathcal{R}} = \big(\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}^{\triangle}}\big) \cup \{X\}$.

16. Projection operations for relators

Notation 16.1. *In this section, we shall assume that* $\mathcal{R}$ *is a relator on* X.

By using some useful properties of the operation ∞, we can also prove the following analogue of our former results.

Theorem 16.1. ∞ *is a closure operation for relations on* X *such that, for any* $R, S \in \mathcal{R}$, *we have*

$$S \subseteq R^{\infty} \iff \tau_R \subseteq \tau_S \iff \tau\!\!\!\tau_R \subseteq \tau\!\!\!\tau_S .$$

Proof. If $x \in X$, then because of the inclusion $R \subseteq R^{\infty}$ and the transitivity of R^{∞} we have

$$R\,[\,R^{\infty}(x)\,] \subseteq R^{\infty}\,[\,R^{\infty}(x)\,] = \big(R^{\infty} \circ R^{\infty}\big)(x) \subseteq R^{\infty}(x) .$$

Thus, by the definition of τ_R, we have $R^{\infty}(x) \in \tau_{\mathcal{R}}$. Now, if $\tau_R \subseteq \tau_S$ holds, then we can see that $R^{\infty}(x) \in \tau_S$, and thus $S\,[\,R^{\infty}(x)\,] \subseteq R^{\infty}(x)$. Hence, by using the reflexivity of R^{∞}, we can already infer that $S(x) \subseteq R^{\infty}(x)$. Therefore, $S \subseteq R^{\infty}$ also holds.

While, if $A \in \tau_R$, then by the definition of $\tau_{\mathcal{R}}$ we have $R\,[\,A\,] \subseteq A$. Hence, by induction, we can see that $R^n\,[\,A\,] \subseteq A$ for all $n \in \mathbb{N}$. Now, since $R^0\,[\,A\,] = \Delta_X\,[\,A\,] = A$ also holds, we can already state that

$$R^{\infty}\,[\,A\,] = \Big(\bigcup_{n=0}^{\infty} R^n\Big)\,[\,A\,] = \bigcup_{n=0}^{\infty} R^n\,[\,A\,] \subseteq \bigcup_{n=0}^{\infty} A = A .$$

Therefore, if $S \subseteq R^{\infty}$ holds, then we have $S\,[\,A\,] \subseteq R^{\infty}\,[\,A\,] \subseteq A$, and thus $A \in \tau_S$. Consequently, $\tau_R \subseteq \tau_S$ also holds.

Now, analogously to our former similar results, we can also state

Corollary 16.1. *For any* $R \in \mathcal{R}$, $S = R^{\infty}$ *is the largest relation on* X *such that* $\tau_R = \tau_S$ $\big(\tau\!\!\!\tau_R = \tau\!\!\!\tau_S\big)$.

Remark 16.1. Preliminary forms the above theorem and its corollary were first proved by Mala [23]. Moreover, he also proved that $R^{\infty}(x) = \bigcap\,\{\,A \in \tau_R : \; x \in A\,\}$ for all $x \in X$, and thus $R^{\infty} = \bigcap\,\{\,R_A : \; A \in \tau_R\}$.

By using Theorem 16.1, as an analogue of Theorem 14.1, we can also prove

Theorem 16.2. $\#\partial$ *is a closure operation for relators on* X *such that, for any relator* $\mathcal{S}$ *on* X, *we have*

$$\mathcal{S} \subseteq \mathcal{R}^{\#\partial} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \tau\!\!\!\tau_{\mathcal{S}} \subseteq \tau\!\!\!\tau_{\mathcal{R}} .$$

Thus, analogously to Corollary 14.1, we can also state

Corollary 16.2. $\mathcal{S} = \mathcal{R}^{\#\partial}$ *is the largest relator on* X *such that* $\tau_{\mathcal{S}} = \tau_{\mathcal{R}}$ $\big(\tau\!\!\!\tau_{\mathcal{S}} = \tau\!\!\!\tau_{\mathcal{R}}\big)$.

By using the Galois property of the operation ∂, Theorem 16.2 can be reformulated in a more convenient form.

Theorem 16.3. $\#\infty$ *is a projection operation for relators on* X *such that, for any relator* $\mathcal{S}$ *on* X, *we have*

$$\mathcal{S}^{\infty} \subseteq \mathcal{R}^{\#} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \tau\!\!\!\tau_{\mathcal{S}} \subseteq \tau\!\!\!\tau_{\mathcal{R}}.$$

Remark 16.2. Moreover, it can be easily shown that that the inclusions $\mathcal{S}^{\infty} \subseteq \mathcal{R}^{\#}$, $\mathcal{S}^{\#\infty} \subseteq \mathcal{R}^{\#\infty}$, $\mathcal{S}^{\#\infty} \subseteq \mathcal{R}^{\#}$ and $\mathcal{S}^{\infty\#} \subseteq \mathcal{R}^{\infty\#}$ are also equivalent.

Now, analogously to our former corollaries, we can also state

Corollary 16.3. *$\mathcal{S} = \mathcal{R}^{\#\infty}$ is the largest preorder relator on X such that $\tau_{\mathcal{S}} = \tau_{\mathcal{R}}$ $\big(\tau_{\mathcal{S}} = \tau_{\mathcal{R}}\big)$.*

Remark 16.3. The advantage of the projection operation $\#\infty$ over the closure operation $\#\partial$ lies mainly in the fact that, in contrast to $\#\partial$, the operation $\#\infty$ is *stable* in the sense $\{X^2\}^{\#\infty} = \{X^2\}$.

Since the structure $\mathcal{T}$ is not union-preserving, by using some parts of the theory of Pataki connections [29, 34, 75], we can only prove the following

Theorem 16.4. *$\wedge\partial$ is a preclosure operation for relators such that, for any relator $\mathcal{S}$ on X, we have*

$$\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}} \implies \mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge\partial}.$$

Remark 16.4. If $\operatorname{card}(X) > 2$, then by using the equivalence relator $\mathcal{R} = \{X^2\}$ Mala [23, Example 5.3] proved that there does not exist a largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$.

Moreover, Pataki [29, Example 7.2] proved that $\mathcal{T}_{\mathcal{R}^{\wedge\partial}} \not\subseteq \mathcal{T}_{\mathcal{R}}$ and $\wedge\partial$ is not idempotent. (Actually, it can be proved that $\mathcal{R}^{\wedge\partial\wedge} \not\subseteq \mathcal{R}^{\wedge\partial}$ also holds [54, Example 10.11].)

Fortunately, as an analogue of Theorem 16.3, we can also prove

Theorem 16.5. *$\wedge\infty$ is a projection operation for relators on X such that if $\mathcal{R} \neq \emptyset$, then for any nonvoid relator $\mathcal{S}$ on X, we have*

$$\mathcal{S}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge} \iff \mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}}.$$

Thus, in particular, we can also state

Corollary 16.4. *If $\mathcal{R} \neq \emptyset$, then $\mathcal{S} = \mathcal{R}^{\wedge\infty}$ is the largest preorder relator on X such that $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}$ $\big(\mathcal{F}_{\mathcal{S}} = \mathcal{F}_{\mathcal{R}}\big)$.*

Remark 16.5. In the light of the several disadvantages of the structure $\mathcal{T}$, it is rather curious that most of the works in general topology and abstract analysis have been based on open sets suggested by Tietze [80] and Alexandroff [1], and standardized by Bourbaki [5] and Kelley [16]. (See Thron [79, p. 18].)

Moreover, it a striking fact that, despite the results of Davis [8], Pervin [31], Hunsaker and Lindgren [15] and the second author [43, 56], generalized proximities and closures, minimal structures, generalized topologies and stacks (ascending systems) are still intensively investigated by a great number of mathematicians without using generalized uniformities.

17. Some general theorems on closure and projection operations

Notation 17.1. *In this section, we shall assume that $\square$ and $\lozenge$ are unary operations for relators.*

However, the forthcoming definitions and theorems can be easily extended to some more general settings.

Definition 17.1. The operation $\square$ will be called is *$\lozenge$–dominating, $\lozenge$–invariant, $\lozenge$–absorbing, resp. $\lozenge$–compatible* if, for any relator $\mathcal{R}$, we have

$$\mathcal{R}^{\lozenge} \subseteq \mathcal{R}^{\square}, \qquad \mathcal{R}^{\square} = \mathcal{R}^{\square\lozenge}, \qquad \mathcal{R}^{\square} = \mathcal{R}^{\lozenge\square}, \qquad \textit{resp.} \qquad \mathcal{R}^{\square\lozenge} = \mathcal{R}^{\lozenge\square}.$$

Remark 17.1. From Theorem 14.2, we can see that if $\lozenge, \square \in \{*, \#, \wedge, \triangle\}$ such that $\lozenge$ precedes $\square$ in the above list, then $\square$ is both $\lozenge$–invariant and $\lozenge$–absorbing. Thus, in particular it is also $\lozenge$–compatible.

Remark 17.2. Moreover, from Theorem 14.3 and Remark 14.3, we know that the operations c, ∞, ∂, $*$, $\#$ are inversion-compatible. However, the important closure operations $\wedge$ and $\triangle$ are not inversion-compatible.

Concerning the operations $\square$ and $\lozenge$, for instance, we can also easily prove the following three theorems.

Theorem 17.1. *If $\lozenge$ is extensive and $\square$ is $\lozenge$–dominating and idempotent, then $\square$ is $\lozenge$–invariant. Moreover, if in addition $\square$ is increasing, then $\square$ is $\lozenge$–absorbing and $\lozenge$–compatible.*

Remark 17.3. In this respect, it is also worth mentioning that if $\lozenge$ is extensive and $\square$ is $\lozenge$–dominating then $\square$ is also extensive. Moreover, if $\lozenge$ is increasing and $\square$ is extensive such that $\mathcal{R}^{\square\lozenge} \subseteq \mathcal{R}^{\square}$ for every relator $\mathcal{R}$, then $\square$ is $\lozenge$–dominating.

Theorem 17.2. *If $\square$ and $\lozenge$ are inversion compatible, then their compositions $\square\lozenge$ and $\lozenge\square$ are also inversion-compatible.*

Remark 17.4. Note that if $\square$ is an inversion compatible operation for relations, then the elementwise operation defined by it for relators is also inversion-compatible. Or, somewhat differently, if $\square$ is a union-preserving operations for relators, then $\square$ is inversion-compatible if and only if $\{R\}^{\square\,-1} = \{R^{-1}\}^{\square}$ for every relation R.

Theorem 17.3. *If $\square$ and $\lozenge$ are compatible closure (projection) operations for relators on X to Y, then $\square\lozenge$ is also a closure (projection) operation for relators on X to Y.*

Proof. To prove that $\square\lozenge = \lozenge \circ \square$ is also idempotent, note that

$$(\square\lozenge)(\square\lozenge) = (\lozenge\square)(\square\lozenge) = \lozenge(\square\square)\lozenge = \lozenge\square\lozenge = (\lozenge\square)\lozenge = (\square\lozenge)\lozenge = \square(\lozenge\lozenge) = \square\lozenge.$$

Remark 17.5. In this respect, it is also worth noticing that the composition of two union-preserving operations for relators is also union-preserving.

It can be easily seen that the operations c, -1, ∞, ∂, and $*$ are union-preserving. However, the important closure operations $\#$, $\wedge$, and $\vartriangle$ are not union-preserving. Concerning them, we can only make use of the following

Theorem 17.4. *If $\square$ is a closure operation, then for any family $(\mathcal{R}_i)_{i\in I}$ of relators we have*

(1) $\bigcap_{i\in I}\mathcal{R}_i^{\square} = \big(\bigcap_{i\in I}\mathcal{R}_i^{\square}\big)^{\square}$; *(2)* $\big(\bigcup_{\imath\in I}\mathcal{R}_i\big)^{\square} = \big(\bigcup_{i\in I}\mathcal{R}_i^{\square}\big)^{\square}$.

Proof. If $\mathcal{R} = \bigcap_{i\in I}\mathcal{R}_i$, then for each $\in I$ we have $\mathcal{R}\subseteq\mathcal{R}_i$, and hence also $\mathcal{R}^{\square}\subseteq\mathcal{R}_i^{\square}$. Therefore, $\big(\bigcap_{i\in I}\mathcal{R}_i\big)^{\square} = \mathcal{R}^{\square}\subseteq \bigcap_{i\in I}\mathcal{R}_i^{\square}$.

Hence, by taking $\mathcal{R}_i^{\square}$ in place of $\mathcal{R}_i$, we can already infer that $\big(\bigcap_{i\in I}\mathcal{R}_i^{\square}\big)^{\square}\subseteq\bigcap_{i\in I}\mathcal{R}_i^{\square\square} = \bigcap_{i\in I}\mathcal{R}_i^{\square}\subseteq\big(\bigcap_{i\in I}\mathcal{R}_i^{\square}\big)^{\square}$. Therefore, equality is true.

While, if $\mathcal{R} = \bigcup_{i\in I}\mathcal{R}_i$, then for each $i\in I$ we have $\mathcal{R}_i\subseteq\mathcal{R}$. Hence, we can infer that $\mathcal{R}_i^{\square}\subseteq\mathcal{R}^{\square}$. Therefore, we have $\bigcup_{i\in I}\mathcal{R}_i^{\square}\subseteq\mathcal{R}^{\square}$. Hence, we can already see that $\big(\bigcup_{i\in I}\mathcal{R}_i^{\square}\big)^{\square}\subseteq\mathcal{R}^{\square\square} = \mathcal{R}^{\square}$.

On the other hand, for each $i\in I$ we have $\mathcal{R}_i\subseteq\mathcal{R}_i^{\square}$, and hence also $\mathcal{R}_i\subseteq\bigcup_{i\in I}\mathcal{R}_i^{\square}$. Therefore, $\mathcal{R} = \bigcup_{i\in I}\mathcal{R}_i\subseteq\bigcup_{i\in I}\mathcal{R}_i^{\square}$. Hence, we can already infer that $\mathcal{R}^{\square}\subseteq\big(\bigcup_{i\in I}\mathcal{R}_i^{\square}\big)^{\square}$. Therefore, equality (2) is also true.

Remark 17.6. From (2), we can see that $\big(\bigcup_{i\in I}\mathcal{R}_i\big)^{\square} = \bigcup_{i\in I}\mathcal{R}_i^{\square}$ if and only the relator $\bigcup_{i\in I}\mathcal{R}_i^{\square}$ is $\square$–invariant.

While, from (1), we can see that the relator $\bigcap_{i\in I}\mathcal{R}_i^{\square}$ is always $\square$–invariant. Moreover, if each $\mathcal{R}_i$ is $\square$-invariant, then the relator $\bigcap_{i\in I}\mathcal{R}_i$ is also $\square$–invariant.

Note that the proofs of the above three theorems can also be used to establish some useful statements on preclosure, semiclosure and modification operations.

In addition to Theorem 17.4, we can also easily prove the following

Theorem 17.5. *If $\square$ is a closure (projection) and $\diamond$ is an involution operation for relators on X to Y, then $\lozenge = \diamond\square\diamond$ is also a closure (projection) operation for relators.*

Proof. To prove that $\lozenge = \diamond\circ\square\circ\diamond$ is also idempotent, note that

$$\lozenge\lozenge = (\diamond\square\diamond)(\diamond\square\diamond) = (\diamond\square)\big((\diamond\diamond)(\square\diamond)\big) = (\diamond\square)\big(\Delta(\square\diamond)\big) = (\diamond\square)\big(\square\diamond\big) = \diamond\big((\square\square)\diamond\big) = \diamond(\square\diamond) = \lozenge,$$

where Δ is the identity operation for relators on X to Y.

Because of this theorem, we may also naturally introduce the following

Definition 17.2. For the operation $\square$, we define

$$\circledcirc = c\,\square\,c \qquad \textit{and} \qquad \boxdot = -1\,\square\,-1.$$

Remark 17.7. Thus, by Theorem 17.5, for instance $\circledast$ is also a closure operation for relators. However, this is also quite obvious from the fact that, for any relator $\mathcal{R}$ on X to Y, we have

$$\mathcal{R}^{\circledast} = \bigcup_{R\in\mathcal{R}}\mathcal{P}(R) = \{\,S\subseteq X\times Y: \quad \exists\ R\in\mathcal{R}: \quad S\subseteq R\,\}.$$

Namely, if for instance $S\in\mathcal{R}^{\circledast}$, then $S\in\mathcal{R}^{c*c}$, and thus $S^c\in\mathcal{R}^{c*}$. Therefore, there exists $R\in\mathcal{R}$ such that $R^c\subseteq S^c$. Hence, it follows that $S\subseteq R$, and thus $S\in\mathcal{P}(R)$. Therefore, $S\in\bigcup_{R\in\mathcal{R}}\mathcal{P}(R)$ also holds.

However, the importance of the first part of Definition 17.2 lies mainly in the following

Theorem 17.6. *$\text{\textcircled{\#}}$ and $\text{\textcircled{$\wedge$}}$ are closure operations for relators such that, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y,*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\text{\textcircled{\#}}} \iff \mathrm{Lb}_{\mathcal{S}} \subseteq \mathrm{Lb}_{\mathcal{R}} \iff \mathrm{Ub}_{\mathcal{S}} \subseteq \mathrm{Ub}_{\mathcal{R}}$*;* *(2)* $\mathcal{S} \subseteq \mathcal{R}^{\text{\textcircled{$\wedge$}}} \iff \mathrm{lb}_{\mathcal{S}} \subseteq \mathrm{lb}_{\mathcal{R}}$.

Proof. By the corresponding definitions and Theorems 14.1, 11.2 and 11.1, we have

$$\mathcal{S} \subseteq \mathcal{R}^{\text{\textcircled{\#}}} \iff \mathcal{S} \subseteq \mathcal{R}^{c\#c} \iff \mathcal{S}^{c} \subseteq \mathcal{R}^{c\#} \iff \mathrm{Int}_{\mathcal{S}^c} \subseteq \mathrm{Int}_{\mathcal{R}^c}$$
$$\iff \mathrm{Int}_{\mathcal{S}^c} \circ \mathcal{C}_Y \subseteq \mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}_Y \iff \mathrm{Lb}_{\mathcal{S}} \subseteq \mathrm{Lb}_{\mathcal{R}} \iff \mathrm{Lb}_{\mathcal{S}}^{-1} \subseteq \mathrm{Lb}_{\mathcal{R}}^{-1} \iff \mathrm{Ub}_{\mathcal{S}} \subseteq \mathrm{Ub}_{\mathcal{R}}\,.$$

Therefore, assertion (1) is true. The proof of assertion (2) is quite similar.

Now, analogously to Corollary 14.1, we can also state the following

Corollary 17.1. *For a relator $\mathcal{R}$ on X to Y, the following assertions are true:*

(1) $\mathcal{S} = \mathcal{R}^{\text{\textcircled{\#}}}$ *is the largest relator on X to Y such that* $\mathrm{Lb}_{\mathcal{S}} = \mathrm{Lb}_{\mathcal{R}}$ $\big(\mathrm{Ub}_{\mathcal{S}} = \mathrm{Ub}_{\mathcal{R}}\big)$*;*

(2) $\mathcal{S} = \mathcal{R}^{\wedge}$ *is the largest relator on X to Y such that* $\mathrm{lb}_{\mathcal{S}} = \mathrm{lb}_{\mathcal{R}}$.

Remark 17.8. Concerning the structure ub, by using Theorems 11.2 and 17.6, we can only note that

$$\mathrm{ub}_{\mathcal{S}} \subseteq \mathrm{ub}_{\mathcal{R}} \iff \mathrm{lb}_{\mathcal{S}^{-1}} \subseteq \mathrm{lb}_{\mathcal{R}^{-1}} \iff \mathcal{S}^{-1} \subseteq \mathcal{R}^{-1\,\text{\textcircled{$\wedge$}}} \iff \mathcal{S} \subseteq \mathcal{R}^{-1\,\text{\textcircled{$\wedge$}}\,-1} \iff \mathcal{S} \subseteq \mathcal{R}^{\boxed{\text{\textcircled{$\wedge$}}}}$$

In this respect, it is also worth mentioning that, by using the associativity of composition and the inversion compatibility of c, we can also easily see that

$$\boxed{\text{\textcircled{$\wedge$}}} = -1\,\text{\textcircled{$\wedge$}}\, -1 = -1\,c \wedge c\,-1 = c\,-1 \wedge -1\,c = c\,\boxed{\wedge}\,c = \text{\textcircled{$\boxed{\wedge}$}}\,.$$

18. Composition-compatible unary operations for relators

Notation 18.1. *In this section, we shall assume that $\square$ is a unary operation for relators.*

Composition compatibility properties of $\square$ have been first considered in [61] in somewhat different forms.

Definition 18.1. We say that

(1) $\square$ is *left composition-compatible* if $\big(\mathcal{S} \circ \mathcal{R}\big)^{\square} = \big(\mathcal{S} \circ \mathcal{R}^{\square}\big)^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z;

(2) $\square$ is *right composition-compatible* if $\big(\mathcal{S} \circ \mathcal{R}\big)^{\square} = \big(\mathcal{S}^{\square} \circ \mathcal{R}\big)^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Remark 18.1. Now, the operation $\square$ may be naturally called *composition-compatible* if it is both left and right composition-compatible.

Note that, actually, this is also very weak composition compatibility property. However, by the subsequent theorems, it will be sufficient for our subsequent purposes.

Theorem 18.1. *If $\square$ is left (right) composition-compatible, then $\square$ is, in a certain sense, idempotent.*

Proof. If $\square$ is left composition-compatible, then for any relator $\mathcal{R}$ on X to Y we have

$$\mathcal{R}^{\square\square} = \big(\mathcal{R}^{\square}\big)^{\square} = \big(\{\Delta_Y\} \circ \mathcal{R}^{\square}\big)^{\square} = \big(\{\Delta_Y\} \circ \mathcal{R}\big)^{\square} = \mathcal{R}^{\square}\,.$$

Theorem 18.2. *If $\square$ is a composition-compatible, then for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z we have*

$$\big(\mathcal{S} \circ \mathcal{R}\big)^{\square} = \big(\mathcal{S}^{\square} \circ \mathcal{R}\big)^{\square} = \big(\mathcal{S} \circ \mathcal{R}^{\square}\big)^{\square} = \big(\mathcal{S}^{\square} \circ \mathcal{R}^{\square}\big)^{\square}.$$

Proof. Namely, for instance, we evidently have $\big(\mathcal{S} \circ \mathcal{R}\big)^{\square} = \big(\mathcal{S} \circ \mathcal{R}^{\square}\big)^{\square} = \big(\mathcal{S}^{\square} \circ \mathcal{R}^{\square}\big)^{\square}$.

Theorem 18.3. *If $\square$ is composition-compatible, then for any three relators $\mathcal{R}$ on X to Y, $\mathcal{S}$ on Y to Z, and $\mathcal{T}$ on Z to W we have*

$$\big(\mathcal{T} \circ \mathcal{S} \circ \mathcal{R}\big)^{\square} = \big(\mathcal{T} \circ \mathcal{S} \circ \mathcal{R}\big)^{\square} = \big(\mathcal{T}^{\square} \circ \mathcal{S} \circ \mathcal{R}\big)^{\square} = \big(\mathcal{T} \circ \mathcal{S}^{\square} \circ \mathcal{R}\big)^{\square} = \big(\mathcal{T} \circ \mathcal{S} \circ \mathcal{R}^{\square}\big)^{\square} = \big(\mathcal{T}^{\square} \circ \mathcal{S}^{\square} \circ \mathcal{R}^{\square}\big)^{\square}.$$

Proof. By using Theorem 18.2, for instance, we can see that

$$(\mathcal{T}\circ(\mathcal{S}\circ\mathcal{R}))^{\square}=\Big(\mathcal{T}^{\square}\circ(\mathcal{S}\circ\mathcal{R})^{\square}\Big)^{\square}=\Big(\mathcal{T}^{\square}\circ(\mathcal{S}^{\square}\circ\mathcal{R}^{\square})^{\square}\Big)^{\square}=(\mathcal{T}^{\square}\circ(\mathcal{S}^{\square}\circ\mathcal{R}^{\square}))^{\square}.$$

Theorem 18.4. *If $\square$ is a preclosure operation, then*

(1) $\square$ is left composition compatible if and only if $(\mathcal{S}\circ\mathcal{R}^{\square})^{\square}\subseteq(\mathcal{S}\circ\mathcal{R})^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z;

(2) $\square$ is right composition-compatible if and only if $(\mathcal{S}^{\square}\circ\mathcal{R})^{\square}\subseteq(\mathcal{S}\circ\mathcal{R})^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Proof. If $\mathcal{R}$ and S are as above, then we have $\mathcal{R}\subseteq\mathcal{R}^{\square}$, and thus $\mathcal{S}\circ\mathcal{R}\subseteq\mathcal{S}\circ\mathcal{R}^{\square}$, and thus $(\mathcal{S}\circ\mathcal{R})^{\square}\subseteq(\mathcal{S}\circ\mathcal{R}^{\square})^{\square}$.

Corollary 18.1. *If $\square$ is a closure operation, then*

(1) $\square$ is left composition-compatible if and only if $\mathcal{S}\circ\mathcal{R}^{\square}\subseteq(\mathcal{S}\circ\mathcal{R})^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z;

(2) $\square$ is right composition-compatible if and only if $\mathcal{S}^{\square}\circ\mathcal{R}\subseteq(\mathcal{S}\circ\mathcal{R})^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Remark 18.2. In addition to the above results, it is also worth noticing that if $\square$ is an involution operation, then $\square$ is left composition-compatible if and only if $\mathcal{S}\circ\mathcal{R}=\mathcal{S}\circ\mathcal{R}^{\square}$ for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Moreover, since $\mathcal{S}\circ\mathcal{R}=\bigcup_{S\in\mathcal{S}}S\circ\mathcal{R}$ holds, we can also at once state that if $\square$ is an involution operation, then $\square$ is left composition-compatible if and only if $S\circ\mathcal{R}=S\circ\mathcal{R}^{\square}$ for any relator $\mathcal{R}$ on X to Y and relation S on Y to Z.

Now, by using Corollary 18.1 and Theorem 17.4, we can also prove the following

Theorem 18.5. *If $\square$ is a closure operation, then*

(1) $\square$ is left composition-compatible if and only if $S\circ\mathcal{R}^{\square}\subseteq(S\circ\mathcal{R})^{\square}$ for any relator $\mathcal{R}$ on X to Y and relation S on Y to Z,

(2) $\square$ is right composition-compatible if and only if $\mathcal{S}^{\square}\circ R\subseteq(\mathcal{S}\circ R)^{\square}$ for any relation R on X to Y and relator $\mathcal{S}$ on Y to Z.

Proof. If $\square$ is left composition-compatible, then by Corollary 18.1, for any relator $\mathcal{R}$ and relation S on Y to Z, we have $\{S\}\circ\mathcal{R}^{\square}\subseteq(\{S\}\circ\mathcal{R})^{\square}$, and thus $S\circ\mathcal{R}^{\square}\subseteq(S\circ\mathcal{R})^{\square}$. Therefore, the "only if part" of (1) is true.

Conversely, if $\mathcal{R}$ is a relator on X to Y and $\mathcal{S}$ is a relator on Y to Z, and the inclusion $S\circ\mathcal{R}^{\square}\subseteq(S\circ\mathcal{R})^{\square}$ holds for any relation S on Y to Z, then by using the corresponding definitions and Theorem 17.4 we can see that

$$\mathcal{S}\circ\mathcal{R}^{\square}=\bigcup_{S\in\mathcal{S}}S\circ\mathcal{R}^{\square}\subset\bigcup_{S\in\mathcal{S}}(S\circ\mathcal{R})^{\square}\subseteq\Big(\bigcup_{S\in\mathcal{S}}(S\circ\mathcal{R})^{\square}\Big)^{\square}=\big(\bigcup_{S\in\mathcal{S}}S\circ\mathcal{R}\big)^{\square}=(\mathcal{S}\circ\mathcal{R})^{\square}.$$

Therefore, by Corollary 18.1, the "if part" of (1) is also true.

By using this theorem, we can somewhat more easily establish the composition compatibility properties of the basic closure operations considered in Section 13.

Theorem 18.6. *The operations $*$ and $\#$ are composition-compatible.*

Proof. To prove right composition compatibility of $\#$, by Theorem 18.5, it is enough to prove only that, for any relation R on X to Y and relator $\mathcal{S}$ on Y to Z, we have $\mathcal{S}^{\#}\circ R\subseteq(\mathcal{S}\circ R)^{\#}$.

For this, suppose that $W\in\mathcal{S}^{\#}\circ R$ and $A\subset X$. Then, there exists $V\in\mathcal{S}^{\#}$ such that $W=V\circ R$. Moreover, there exists $S\in\mathcal{S}$ such that $S\,[\,R\,[\,A\,]\,]\subseteq V\,[\,R\,[\,A\,]\,]$, and thus $(S\circ R)[\,A\,]\subseteq(V\circ R)[\,A\,]=W\,[\,A\,]$. Hence, by taking $U=S\circ R$, we can see that $U\in\mathcal{S}\circ R$ such that $U\,[\,A\,]\subseteq W\,[\,A\,]$. Therefore, $W\in(\mathcal{S}\circ R)^{\#}$ also holds.

Theorem 18.7. *The operations $\wedge$ and $\triangle$ are left composition-compatible.*

Proof. To prove left composition compatibility of $\triangle$, by Theorem 18.5, it is enough to prove only that, for any relator $\mathcal{R}$ on X to Y and relation S on Y to Z, we have $S\circ\mathcal{R}^{\triangle}\subseteq(S\circ\mathcal{R})^{\triangle}$.

For this, suppose that $W\in S\circ\mathcal{R}^{\triangle}$ and $x\in X$. Then, there exists $V\in\mathcal{R}^{\triangle}$ such that $W=S\circ V$. Moreover, there exist $u\in X$ and $R\in\mathcal{R}$ such that $R(u)\subseteq V(x)$. Hence, we can infer that

$$(S\circ R)(u)=S\,[\,R(u)\,]\subseteq S\,[\,V(x)\,]=(S\circ V)(x)=W(x)\,.$$

Now, by taking $U=S\circ R$, we can see that $U\in S\circ\mathcal{R}$ such that $U(u)\subset W(x)$. Therefore, $W\in(S\circ R)^{\triangle}$ also holds.

Instead of the right composition compatibility of the operations $\wedge$ and $\vartriangle$, we can only prove the following

Theorem 18.8. *For any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z, we have*

(1) $\left(\mathcal{S}\circ\mathcal{R}\right)^{\wedge}=\left(\mathcal{S}^{\#}\circ\mathcal{R}\right)^{\wedge}$, *(2)* $\left(\mathcal{S}\circ\mathcal{R}\right)^{\vartriangle}=\left(\mathcal{S}^{\#}\circ\mathcal{R}\right)^{\vartriangle}$.

Proof. By the extensivity of $\#$, we have $\mathcal{S}\subset\mathcal{S}^{\#}$. Hence, by the elementwise definition of composition of relators, we can see that $\mathcal{S}\circ\mathcal{R}\subseteq\mathcal{S}^{\#}\circ\mathcal{R}$. Thus, by the increasingness of $\wedge$, we also have $\left(\mathcal{S}\circ\mathcal{R}\right)^{\wedge}\subseteq\left(\mathcal{S}^{\#}\circ\mathcal{R}\right)^{\wedge}$.

To get the converse inclusion, since $\wedge$ is a closure, it is enough to prove only that $\mathcal{S}^{\#}\circ\mathcal{R}\subseteq\left(\mathcal{S}\circ\mathcal{R}\right)^{\wedge}$. For this, suppose that $W\in\mathcal{S}^{\#}\circ\mathcal{R}$ and $x\in X$. Then, there exists $V\in\mathcal{S}^{\#}$ and $R\in\mathcal{R}$ such that $W=V\circ R$. Moreover, there exists $S\in\mathcal{S}$, such that $S\left[R(x)\right]\subseteq V\left[R(x)\right]$, and thus $(S\circ R)(x)\subseteq(V\circ R)(x)=W(x)$. Hence, by taking $U=S\circ R$, we can see that $U\in\mathcal{S}\circ\mathcal{R}$ such that $U(x)\subseteq W(x)$. Therefore, $W\in\left(\mathcal{S}\circ\mathcal{R}\right)^{\wedge}$ also holds.

Thus, we have proved (1). Assertion (2) can now be immediately derived from (1) by using that $\mathcal{U}^{\wedge\vartriangle}=\mathcal{U}^{\vartriangle}$ for any relator $\mathcal{U}$ on X to Z.

From this theorem, by using Theorem 18.7, we can immediately derive

Corollary 18.2. *For any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z, we have*

(1) $\left(\mathcal{S}\circ\mathcal{R}\right)^{\wedge}=\left(\mathcal{S}^{\#}\circ\mathcal{R}^{\wedge}\right)^{\wedge}$; *(2)* $\left(\mathcal{S}\circ\mathcal{R}\right)^{\vartriangle}=\left(\mathcal{S}^{\#}\circ\mathcal{R}^{\vartriangle}\right)^{\vartriangle}$.

Remark 18.3. By using Theorem 18.5, we can also somewhat more easily prove that the operation $\circledast$, considered in Remark 17.7, is also composition-compatible.

19. Reformulations of increasingness and uniform continuity

Example 19.1. A function f of one ordered or generalized ordered set $X(\leq_X)$ to another $Y(\leq_Y)$ has been called *increasing* if

$$u\leq_X v\quad\Longrightarrow\quad f(u)\leq_Y f(v).$$

By using the notations $R=\leq_X$ and $S=\leq_Y$, the above implication can be reformulated in the equivalent forms that:

(1) $v\in R(u)\implies f(v)\in S\left(f(u)\right)$; *(2)* $(u,v)\in R\implies\left(f(u),f(v)\right)\in S$.

Example 19.2. A function f of one metric or generalized metric space $X(d_X)$ to another $Y(d_Y)$ has been called *uniformly continuous* if for each $\varepsilon>0$ there exists $\delta>0$ such that

$$d_X(u,v)<\delta\quad\Longrightarrow\quad d_Y\left(f(u),f(v)\right)<\varepsilon.$$

By using the notations

$$R=\left\{(u,v)\in X^2:\quad d_X(u,v)<\delta\right\}\qquad\text{and}\qquad S=\left\{(w,z)\in Y^2:\quad d_Y(w,z)<\varepsilon\right\},$$

the above implication can also be reformulated in the equivalent forms (1) and (2).

The above examples naturally lead to the following simple unifying definition for increasingness and continuity.

Definition 19.1. A function f of one relational (simple relator) space $X(R)$ to another $Y(S)$ will be called *relation-preserving* if implication (2) holds.

Remark 19.1. Having in mind Example 19.2, implication (2) can be expressed by saying that if u and v are *R–near*, then $f(u)$ and $f(v)$ are *S–near*.

While, its equivalent (1) can be expressed by saying that if v is in the *R–neighbourhood* of u, then $f(v)$ is in the *S–neighbourhood* of $f(u)$.

By using our former definitions and theorems on relations, the above implications (2) and (1) can be reformulated in several equivalent forms.

For instance, for a *function f of one relational space $X(R)$ to another $Y(S)$*, we can easily prove the following theorems.

Theorem 19.1. *The following assertions are equivalent:*

(1) *f is relation-preserving;* *(2)* $(f\boxtimes f)[R]\subseteq S$, *(3)* $R\subseteq(f\boxtimes f)^{-1}[S]$.

Proof. Because of the usual identification of singletons with their elements, we have $(f\boxtimes f)(u,v)=\left(f(u),f(v)\right)$ for all $u,v\in X$. Hence, by Definition 19.1, it is clear that assertions (1) and (2) are equivalent.

Remark 19.2. The equivalence of assertions (2) and (3), can be derived from the fact that if f is a function of X to Y, then for any $A \subseteq X$ and $B \subseteq Y$, we have

$$f\,[A] \subseteq B \quad \Longleftrightarrow \quad A \subseteq f^{-1}[B]\,.$$

This shows that the set-to-set functions associated with the function f and the relation f^{-1} form a *Galois connection* between the power sets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

From Theorem 19.1, by using Definition 7.1, we can immediately derive the following

Corollary 19.1. *The following assertions are equivalent :*

(1) f *relation-preserving ;* *(2)* $R \subseteq \mathrm{cl}_{f \boxtimes f}(S)$, *(3)* $R \subseteq \mathrm{int}_{f \boxtimes f}(S)$.

Remark 19.3. Note that if f is a function of X to Y, then by Remark 19.2, for any $A \subseteq Y$ and $B \subseteq Y$, we have $A \subseteq \mathrm{cl}_f(B) \iff A \subseteq \mathrm{int}_f(B)$.

Hence, by taking $x \in X$ and $A = \{x\}$, we can see that $\mathrm{cl}_f(B) = \mathrm{int}_f(B)$ for all $B \subseteq Y$, and thus in particular $\mathrm{cl}_f = \mathrm{int}_f$ also holds.

However, it is now more important to note that, by using the inverses and compositions of relations, we can also easily prove the following

Theorem 19.2. *The following assertions are equivalent :*

(1) f *relation-preserving ;* *(2)* $f \circ R \subseteq S \circ f$;

(3) $R \subseteq f^{-1} \circ S \circ f$; *(4)* $f \circ R \circ f^{-1} \subseteq S$; *(5)* $R \circ f^{-1} \subseteq f^{-1} \circ S$.

Proof. From Theorem 19.1, by Theorems 2.1 and 2.3, it is clear that assertions (4) and (3) are equivalent to assertion (1).

Moreover, if for instance (2) holds, then we can see that $R = \Delta_X \circ R \subseteq f^{-1} \circ f \circ R \subseteq f^{-1} \circ S \circ f$, and thus (3) also holds.

While, if (3) holds, then we can quite similarly see that $f \circ R \subseteq f \circ f^{-1} \circ S \circ f \subseteq \Delta_Y \circ S \circ f = S \circ f$, and thus (2) also holds. Therefore, assertions (2) and (3) are also equivalent.

From this theorem, by using Corollary 2.2 and Theorem 2.2, we can immediately derive the following

Corollary 19.2. *The following assertions are equivalent :*

(1) f *is relation-preserving ;* *(2)* $\big(f \boxtimes R\big)[\Delta_X] \subseteq (S \boxtimes f)^{-1}[\Delta_Y]$; *(3)* $\big(R^{-1} \boxtimes f\big)[\Delta_X] \subseteq \big(f^{-1} \boxtimes S\big)[\Delta_Y]$.

20. Reformulations of left regularity and normality

Analogously to Example 19.1, we can also easily establish the following

Example 20.1. If f is a function of one ordered or generalized ordered set $X(\leq_X)$ to another $Y(\leq_Y)$ and φ is a function of X to itself, then f has been called *left* φ*–regular* if

$$u \leq_X \varphi(v) \quad \Longrightarrow \quad f(u) \leq_Y f(v)\,.$$

By using the notations $R = \leq_X$ and $S = \leq_Y$, the above implication can be reformulated in the equivalent forms that:

(1) $\varphi(v) \in R(u) \implies f(v) \in S\big(f(u)\big)$; *(2)* $\big(u,\, \varphi(v)\big) \in R \implies \big(f(u),\, f(v)\big) \in S$.

Hence, by noticing that $\varphi(v) \in R(u) \iff v \in \varphi^{-1}[R(u)] \iff v \in \big(\varphi^{-1} \circ R\big)(u)$, we can immediately derive

Theorem 20.1. *If* f *is a function of one relational space* $X(R)$ *to another* $Y(S)$ *and* φ *is a function of* X *to itself, then the following assertions are equivalent :*

(1) f *is left* φ*–regular ;* *(2)* f *is a relation-preserving function of* $X\big(\varphi^{-1} \circ R\big)$ *to* $Y\big(S\big)$.

Thus, for instance, by Theorem 19.2, we can at once state the following

Theorem 20.2. *If* f *and* φ *are as in Theorem 20.1, then the following assertions are equivalent :*

(1) f *is left* φ*–regular ;* *(2)* $f \circ \varphi^{-1} \circ R \subseteq S \circ f$;

(3) $\varphi^{-1} \circ R \subseteq f^{-1} \circ S \circ f$; *(4)* $f \circ \varphi^{-1} \circ R \circ f^{-1} \subseteq S$; *(5)* $\varphi^{-1} \circ R \circ f^{-1} \subseteq f^{-1} \circ S$.

Remark 20.1. The right φ–regularity f has been defined by reversing the defining implication of the left φ–regularity of f. Therefore, to obtain characterizations of the right φ–regularity of f we have to reverse the inclusions in the characterizations of the left φ–regularity of f.

In addition Example 20.1, we can also easily establish the following

Example 20.2. If f is as in Example 19.1 and g is a function of Y to X, then f has been called *left g–normal* if

$$x \leq_X g(y) \quad\Longrightarrow\quad f(x) \leq_Y y.$$

By defining $\varphi = g \circ f$, we can see that

$$u \leq_X \varphi(v) \quad\Longrightarrow\quad u \leq_X (g \circ f)(v) \quad\Longrightarrow\quad u \leq_X g\big(f(v)\big) \quad\Longrightarrow\quad f(u) \leq_Y f(v).$$

Therefore, f is, in particular left φ–regular.

Moreover, by using the notations $R = \leq_X$ and $S = \leq_Y$, the defining implication of left g–normality of f can be reformulated in the equivalent forms that:

(1) $g(y) \in R(x) \;\Longrightarrow\; y \in S\big(f(x)\big)$; *(2)* $\big(x,\, g(y)\big) \in R \;\Longrightarrow\; \big(f(x),\, y\big) \in S$.

Remark 20.2. Thus, for instance, from Theorem 20.2, we can get some properties of left normal functions.

Moreover, analogously to Theorem 19.1 and its corollary we can also prove the following theorem and its corollary.

Theorem 20.3. *If f is a function of one relational space $X(R)$ to another $Y(S)$ and g is a function of Y to X, then the following assertions are equivalent:*

(1) f *is left* g*–normal;* *(2)* $(f \boxtimes \Delta_Y)[g^{-1} \circ R] \subseteq S$; *(3)* $g^{-1} \circ R \subseteq (f \boxtimes \Delta_Y)^{-1}[S]$.

Corollary 20.1. *If f and g are as in Theorem 20.3, then the following assertions are equivalent:*

(1) f *is left* g*–normal;* *(2)* $g^{-1} \circ R \subseteq \operatorname{cl}_{f \boxtimes \Delta_Y}(S)$; *(3)* $g^{-1} \circ R \subseteq \operatorname{int}_{f \boxtimes \Delta_Y}(S)$.

However, instead of Theorem 19.2 and its corollary, we can only prove the following

Theorem 20.4. *If f and g are as in Theorem 20.3, then the following assertions are equivalent:*

(1) f *is left* g*–normal;* *(2)* $g^{-1} \circ R \subseteq S \circ f$; *(3)* $g^{-1} \circ R \circ f^{-1} \subseteq S$.

Remark 20.3. The right g–normality of f has been defined by reversing the defining implication of the left g–normalty of f. Therefore, to obtain characterizations of the right g–normality of f we have to reverse the inclusions in the characterizations of the left g–normality of f.

21. Reformulations of proximal continuity

Notation 21.1. *In this section, we shall assume that f is a function of one relator spaces $X(\mathcal{R})$ to another $Y(\mathcal{S})$.*

Analogously to a localized version of Example 19.2 and its relational reformulation, for instance, we may also naturally introduce the following

Definition 21.1. The function f will be called *proximally continuous* if for each $A \subseteq X$ and $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that

$$f\big[R[A]\big] \subseteq S\big[f[A]\big].$$

By using the corresponding properties of the operation $\#$, the above inclusion can be reformulated in some less graphic, but more instructive forms.

Theorem 21.1. *The following assertions are equivalent:*

(1) f *is proximally continuous;* *(2)* $\mathcal{S} \circ f \subseteq (f \circ \mathcal{R})^{\#}$;

(3) $\big(\mathcal{S}^{\#} \circ f\big)^{\#} \subseteq \big(f \circ \mathcal{R}^{\#}\big)^{\#}$; *(4)* $\big(\mathcal{S}^{\#} \circ \{f\}^{\#}\big)^{\#} \subseteq \big(\{f\}^{\#} \circ \mathcal{R}^{\#}\big)^{\#}$.

Proof. If (1) holds, then by Definition 21.1, for each $A \subseteq X$ and $S \in \mathcal{S}$, there exists $R \in \mathcal{R}$ such that

$$(f \circ R)[A] = f\big[R[A]\big] \subseteq S\big[f[A]\big] = (S \circ f)[A].$$

Hence, since $f \circ R \in f \circ \mathcal{R}$, by Definition 13.1 we can see that $S \circ f \in (f \circ \mathcal{R})^{\#}$. Therefore, (2) also holds.

By reversing the above argument, we can easily see that (2) also implies (1). Moreover, since $\#$ is a closure, it is clear that inclusion (2) is equivalent to the inclusion $\big(\mathcal{S} \circ f\big)^{\#} \subseteq \big(f \circ \mathcal{R}\big)^{\#}$. Furthermore, from Theorems 18.6 and **??**, we can see that the latter inclusion is equivalent to inclusions (3) and (4).

Remark 21.1. By Theorem 14.3, the operation $\#$ is also inversion compatible. Therefore, inclusion (2) can be reformulated in the form that $f^{-1} \circ \mathcal{S}^{-1} \subseteq \big(\mathcal{R}^{-1} \circ f^{-1}\big)^{\#}$.

The latter apparently uninteresting observation will help us to keep in mind a forthcoming definition of the proximal lower semicontinuity of a relation F on $X(\mathcal{R})$ to $Y(\mathcal{S})$.

Note that in Definition 21.1 and Theorem 21.1, we can write the relation F instead of the function f. However, the following theorems would not be true for the relation F instead of the function f.

Theorem 21.2. *The following assertions are equivalent:*

(1) f *is proximally continuous;* *(2)* $f^{-1} \circ \mathcal{S} \circ f \subseteq \mathcal{R}^{\#}$*;*

(3) $\big(f^{-1} \circ \mathcal{S}^{\#} \circ f\big)^{\#} \subseteq \mathcal{R}^{\#}$*;* *(4)* $\big(\{f^{-1}\}^{\#} \circ \mathcal{S}^{\#} \circ \{f\}^{\#}\big)^{\#} \subseteq \mathcal{R}^{\#}$.

Proof. If (1) holds, then by Theorem 21.1 and Definition 13.1, for each $A \subseteq X$ and $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that $(f \circ R)[A] \subseteq (S \circ f)[A]$. Hence, we can infer that

$$\big(f^{-1} \circ (f \circ R)\big)[A] = f^{-1}\big[(f \circ R)[A]\big] \subseteq f^{-1}\big[(S \circ f)[A]\big] = \big(f^{-1} \circ (S \circ f)\big)[A].$$

Now, by using that $\Delta_X \subseteq f^{-1} \circ f$, and thus $R = \Delta_X \subseteq \big(f^{-1} \circ f\big) \circ R = f^{-1} \circ (f \circ R)$, we can see that $R[A] \subseteq \big(f^{-1} \circ (S \circ f)\big)[A]$. Therefore, by Definition 13.1, we have $f^{-1} \circ (S \circ f) \in \mathcal{R}^{\#}$, and thus inclusion (2) also holds.

The converse implication (2) $\Longrightarrow$ (1) can be proved quite similarly by using a reverse argument and the inclusion $f \circ f^{-1} \subseteq \Delta_Y$. Moreover, analogously to the proof of Theorem 21.1, we can also easily establish that innclusions (3) and (4) are also equivalent to (2).

Remark 21.2. Note that, in the above two theorems, we may write $\{f\}^{*}$ and $\big\{f^{-1}\big\}^{*}$ instead of $\{f\}^{\#}$ and $\big\{f^{-1}\big\}^{\#}$, respectively. However, the basic inclusions $\Delta_X \subseteq f^{-1} \circ f$ and $f \circ f^{-1} \subseteq \Delta_Y$ strongly need that $D_f = X$ and f is a function, respectively.

From Theorem 21.1, by using Theorems 2.3 and 7.1 , we can immediately derive

Corollary 21.1. *The following assertions are equivalent:*

(1) f *is proximally continuous;* *(2)* $(f \boxtimes f)^{-1}[S] \subseteq \mathcal{R}^{\#}$*;* *(3)* $\operatorname{cl}_{f \boxtimes f}(S) \subseteq \mathcal{R}^{\#}$.

Remark 21.3. Note that, in this corollary, we may again write $\mathcal{S}^{\#}$ in place of $\mathcal{S}$.

However, it is now more important to prove that the function f is proximally continuous if and only if it is *proximal closure preserving*.

Theorem 21.3. *The following assertions are equivalent:*

(1) f *is proximally continuous;*

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ *implies* $f[A] \in \operatorname{Cl}_{\mathcal{S}}\big(f[B]\big)$ *for all* $A, B \subseteq X$*;*

(3) $f[A] \in \operatorname{Int}_{\mathcal{S}}(B)$ *implies* $A \in \operatorname{Int}_{\mathcal{R}}\big(f^{-1}[B]\big)$ *for all* $A \subseteq X$ *and* $B \subseteq Y$.

Proof. By defining $\mathcal{U} = f^{-1} \circ \mathcal{S} \circ f$ and using Theorems 21.1 and 14.1, we can see that $\mathcal{U}$ is a relator on X such that

$$(1) \iff \mathcal{U} \subseteq \mathcal{R}^{\#} \iff \operatorname{Cl}_{\mathcal{R}} \subseteq \operatorname{Cl}_{\mathcal{U}}.$$

Therefore, to prove the equivalence of assertions (1) and (2), we need only show that, for any $A, B \subseteq X$, we have

$$A \in \operatorname{Cl}_{\mathcal{U}}(B) \iff f[A] \in \operatorname{Cl}_{\mathcal{S}}\big(f[B]\big).$$

However, by Definition 7.1, for this it is enough to note only that, for any $S \in \mathcal{S}$, we have

$$\big(f^{-1} \circ S \circ f\big)[A] \cap B \neq \emptyset \iff S\big[f[A]\big] \cap f[B] \neq \emptyset.$$

The equivalence of assertions (2) and (3) can, in principle, be easily proved with the helpf of Theorem 7.1.

Remark 21.4. This, theorem, in contrast to Theorems 21.1 and 21.2, can be easily generalized to relations. Moreover, particular cases of this generalization has several interesting reformulations [41].

In this respect, it is also worth mentioning that a relation F on $X(\mathcal{R})$ to $Y(\mathcal{S})$ is *proximal interior preserving* if and only if $F \circ \mathcal{R} \subseteq (\mathcal{S} \circ F)^{\#}$. In [70], proximal interior reversing relations have also been characterized.

Having in mind the widely investigated notion of contra continuity, the relation F may be naturally called *proximal interior reversing* if $A \in \operatorname{Int}_{\mathcal{R}}(B)$ implies $F[A] \in \operatorname{Cl}_{\mathcal{S}}(F[B])$ for all $A, B \subseteq X$ with $A \neq \emptyset$.

22. Some general continuity properties of relators

Notation 22.1. *In this section, we shall assume that $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ are relator spaces. Moreover, we shall assume that $\mathcal{F}$ is a relator on X to Z and $\mathcal{G}$ is a relator on Y to W.*

Remark 22.1. To keep in mind the above assumptions, for any $R \in \mathcal{R}$, $S \in \mathcal{S}$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$, one can consider the following illustrating diagram:

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "R"'] & Z \arrow[d, "S"] \\ Y \arrow[r, "G"] & W \end{tikzcd}$$

Now, by pexiderizing several former compositional inclusions on functions and relations, and using relators instead of functions and relations, we may naturally introduce the following general unifying definition which has been mainly developed by the second author [32, 37, 49, 50, 62, 69, 72, 77].

Definition 22.1. For a family $\square = (\square_i)_{i=1}^{6}$ of direct unary operations for relators, we shall say that the ordered pair

(1) $(\mathcal{F}, \mathcal{G})$ is *upper right $\square$–continuous* if $\left(\mathcal{S}^{\square_1} \circ \mathcal{F}^{\square_2}\right)^{\square_3} \subseteq \left(\mathcal{G}^{\square_4} \circ \mathcal{R}^{\square_5}\right)^{\square_6}$;

(2) $(\mathcal{F}, \mathcal{G})$ is *mildly right $\square$–continuous* if $\left(\mathcal{G}^{\square_1 -1} \circ \mathcal{S}^{\square_2} \circ \mathcal{F}^{\square_3}\right)^{\square_4} \subseteq \mathcal{R}^{\square_5 \square_6}$;

(3) $(\mathcal{F}, \mathcal{G})$ is *vaguely right $\square$–continuous* if $\mathcal{S}^{\square_1 \square_2} \subseteq \left(\mathcal{G}^{\square_3} \circ \mathcal{R}^{\square_4} \circ \mathcal{F}^{\square_5 -1}\right)^{\square_6}$;

(4) $(\mathcal{F}, \mathcal{G})$ is *lower right $\square$–continuous* if $\left(\mathcal{G}^{\square_1 -1} \circ \mathcal{S}^{\square_2}\right)^{\square_3} \subseteq \left(\mathcal{R}^{\square_4} \circ \mathcal{F}^{\square_5 -1}\right)^{\square_6}$.

Remark 22.2. Here, according to the corresponding definitions in topological spaces [18, 36], we should use the term "upper and lower semicontinuous" instead of "upper and lower continuous", respectively.

Moreover, according to the former definitions, the term "right" should be deleted. However, having in mind Galois connections, it seems convenient to define the corresponding left continuity properties by reversing the above inclusions.

Remark 22.3. Now, for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the pair (F, G) may, for instance, be naturally called *upper right $\square$–continuous*, if the pair $(\{F\}, \{G\})$ is upper right $\square$–continuous. That is,

$$\left(\mathcal{S}^{\square_1} \circ \{F\}^{\square_2}\right)^{\square_3} \subseteq \left(\{G\}^{\square_4} \circ \mathcal{R}^{\square_5}\right)^{\square_6}.$$

Unfortunately, this condition may differ from the property that $\left(\mathcal{S}^{\square_1} \circ F\right)^{\square_3} \subseteq \left(G \circ \mathcal{R}^{\square_5}\right)^{\square_6}$, which in the particular case $\square = \square_1 = \square_3 = \square_5 = \square_6$ has been given the name "upper $\square$–continuity".

In this respect, it is worth noticing that, for instance, for any $F \in \mathcal{F}$ we have

$$\{F\}^{\#} = \{F\}^{*}, \qquad \{F\}^{\wedge} = \{F\}^{*} \qquad \textit{and} \qquad \{F\}^{\vartriangle} = \left\{F \circ X^{X}\right\}^{*}.$$

Remark 22.4. Now, the pair $(\mathcal{F}, \mathcal{G})$, for the operation $\square$ considered in Definition 22.1, may be naturally called *right $\square$–continuous* if it is both upper and lower right $\square$–countinuous.

Thus, the pair (F, G) may, for instance, be naturally called *selectionally right $\square$–continuous* if for any selection functions f of F and g of G the pair (f, g) is right $\square$–continuous.

Moreover, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *elementwise right $\square$–continuous* if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the pair (F, G) is right $\square$–continuous.

Remark 22.5. If in particular $\square$ is a direct unary operation for relators, then the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be also naturally called upper right $\square$–continuous if it is upper right $(\square)_{i=1}^{6}$–continuous. That is,

$$\left(\mathcal{S}^{\square} \circ \mathcal{F}^{\square}\right)^{\square} \subseteq \left(\mathcal{G}^{\square} \circ \mathcal{R}^{\square}\right)^{\square}.$$

Remark 22.6. Thus, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *properly upper right continuous* if it is upper right $\square$–continuous with $\square$ being the identity operation for relators. That is, $\mathcal{S} \circ \mathcal{F} \subseteq \mathcal{G} \circ \mathcal{R}$.

Moreover, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be also naturally called *uniformly, proximally, topologically and paratopologically upper right continuous* if it is upper right $\square$–continuous with $\square = \ast, \#, \wedge$ and $\triangle$, respectively.

Thus, by using the operations $\square\infty$ and $\square\partial$ instead of $\square$, we can quite similarly speak of the corresponding *quasi-countinuity and pseudo-continuity properties* of $(\mathcal{F}, \mathcal{G})$.

Remark 22.7. Furthermore, if $X = Y$ and $Z = W$, then the relator $\mathcal{F}$ and a relation $F \in \mathcal{F}$ may, for instance, be naturally called *upper right $\square$–continuous* if the pairs $(\mathcal{F}, \mathcal{F})$ and (F, F) are upper right $\square$–continuous, respectively.

Concerning Definition 22.1, we can only mention here the following easy, but important observation which again shows a remarkable advantage of relator spaces over the topological ones.

Theorem 22.1. *If the operations $\square_i$, with $i = 2, 3, 4, 6$ are inversion compatible, and in addition to $\square = (\square_i)_{i=1}^6$, we also define $\Diamond = (\square_2, \square_1, \square_3, \square_5, \square_4, \square_6)$, then the following assertions are equivalent:*

(1) $(\mathcal{F}, \mathcal{G})$ is lower right $\square$–continuous with respect to the relators $\mathcal{R}$ and $\mathcal{S}$;

(2) $(\mathcal{G}, \mathcal{F})$ is upper right $\Diamond$–continuous with respect to the relators $\mathcal{R}^{-1}$ and $\mathcal{S}^{-1}$.

Remark 22.8. Note that if $X = Y$ and $Z = W$, and instead of the inversion compatibility of the operations $\square_2$ and $\square_4$, we assume that the relators $\mathcal{R}$ and $\mathcal{S}$ are $\square_4$–symmetric and $\square_2$–symmetric [66] in the sense that $\mathcal{R}^{\square_4 -1} = \mathcal{R}^{\square_4}$ and $\mathcal{S}^{\square_2 -1} = \mathcal{S}^{\square_2}$, respectively, then in assertion (2) we may simply write $\mathcal{R}$ and $\mathcal{S}$ instead of $\mathcal{R}^{-1}$ and $\mathcal{S}^{-1}$, respectively.

Remark 22.9. It is also worth mentioning that if $\square = \square_i$ for all $i = 1, \cdots, 6$, and $\square$ is a closure, or even a composition compatible closure, then Definition 22.1 can be greatly simplified.

For instance, in the latter particular case, we can easily establish that the ordered pair $(\mathcal{F}, \mathcal{G})$ is upper right $\square$–continuous if and only if, in accordance with Theorem 21.1, we have $\mathcal{S} \circ \mathcal{F} \subseteq (\mathcal{G} \circ \mathcal{R})^{\square}$.

Remark 22.10. Finally, we note that, because of the observations of Section 19, in Definition 22.1 we may sometimes also naturally write "increasing" instead of "continuous".

However, having in mind set-valued functions, a relation F on a goset $X(\leq)$ to a set Y may be naturally called increasing if $u \leq v$ implies $F(u) \subseteq F(v)$ for all $u, v \in X$.

Thus, it can be easily shown that the relation F is increasing if and only if its inverse F^{-1} is *ascending-valued* in the sense that $F^{-1}(y)$ is an ascending subset of $X(\leq)$ for all $y \in Y$.

By using the better notation $R = \leq$, the latter statement can be reformulated in the form that $R[F^{-1}(y)] \subseteq F^{-1}(y)$ for all $y \in Y$. That is, $R \circ F^{-1} \subseteq F^{-1}$, and thus $F^{-1} \circ \Delta_Y \in \{R \circ F^{-1}\}^{\ast}$.

23. Some general normality and regularity properties of relators

Now, analogously to Definition 22.1, we may also naturally introduce the following

Definition 23.1. Suppose that $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ are relator spaces. Moreover, assume that $\mathcal{F}$ is a relator on X to Z and $\mathcal{G}$ is a relator on W to Y.

Then, for a family $\square = (\square_i)_{i=1}^6$ of direct unary operations for relators, we say that:

(1) $\mathcal{F}$ is *upper right $\square$–$\mathcal{G}$–normal* if $(\mathcal{S}^{\square_1} \circ \mathcal{F}^{\square_2})^{\square_3} \subseteq (\mathcal{G}^{\square_4 -1} \circ \mathcal{R}^{\square_5})^{\square_6}$;

(2) $\mathcal{F}$ is *mildly right $\square$–$\mathcal{G}$–normal* if $(\mathcal{G}^{\square_1} \circ \mathcal{S}^{\square_2} \circ \mathcal{F}^{\square_3})^{\square_4} \subseteq \mathcal{R}^{\square_5 \square_6}$;

(3) $\mathcal{F}$ is *vaguely right $\square$–$\mathcal{G}$–normal* if $\mathcal{S}^{\square_1 \square_2} \subseteq (\mathcal{G}^{\square_3 -1} \circ \mathcal{R}^{\square_4} \circ \mathcal{F}^{\square_5 -1})^{\square_6}$;

(4) $\mathcal{F}$ is *lower right $\square$–$\mathcal{G}$–normal* if $(\mathcal{G}^{\square_1} \circ \mathcal{S}^{\square_2})^{\square_3} \subseteq (\mathcal{R}^{\square_4} \circ \mathcal{F}^{\square_5 -1})^{\square_6}$.

Remark 23.1. The corresponding left $\square$–normality properties have to be defined by reversing the above inclusions. Thus, concerning the use of the terms "left" and "right", there seems to be some inevitable troubles.

Namely, if f is a function of one simple relator space $X(R)$ to another $Y(S)$ and g is a function of Y to X, then by using Theorem 20.4 and its dual, and the composition compatibility of the closures $\ast$ and $\circledast$, we can easily establish that

(1) f is left g–normal $\iff$ $g^{-1} \circ R \subseteq S \circ f$ $\iff$ $S \circ f \in \{g^{-1} \circ R\}^{\ast}$ $\iff$ $(\{S\}^{\ast} \circ \{f\}^{\ast})^{\ast} \subseteq (\{g\}^{\ast -1} \circ \{R\})^{\ast}$;

(2) f is right g–normal $\iff S\circ f\subseteq g^{-1}\circ R \iff S\circ f\in\{g^{-1}\circ f\}^{\circledast} \iff (\{S\}^{\circledast}\circ\{f\}^{\circledast})^{\circledast}\subseteq(\{g\}^{\circledast\,-1}\circ\{R\})^{\circledast}$.

Now, concerning Definition 23.1, we can make some similar remarks as we did in connection with Definition 22.1. Moreover, for instance, we can at once state the following theorem which indicates that, analogously to increasingness, normality properties are, to some extent, also particular cases of continuity properties.

Theorem 23.1. *If the operation $\square_4$ is inversion compatible, then the following assertions are equivalent:*
(1) $\mathcal{F}$ is upper right $\square$–$\mathcal{G}$–normal; *(2) $(\mathcal{F},\mathcal{G}^{-1})$ is upper right $\square$–continuous.*

Now, having in mind Theorem 20.1 and Definition 22.1, instead of an analogue of Definition 23.1, we shall only consider here the following

Definition 23.2. Suppose that $(X,Y)(\mathcal{R})$ and $Z(\mathcal{S})$ are relator spaces, Moreover, assume that $\mathcal{F}$ is a relator on X to Z, and Φ is a relator on X to Y.

Then, for a family $\square=(\square_i)_{i=1}^7$ of direct unary operations for relators, we say that $\mathcal{F}$ is *mildly right $\square$–Φ–regular* with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if

$$\left(\mathcal{F}^{\square_1\,-1}\circ\mathcal{S}^{\square_2}\circ\mathcal{F}^{\square_3}\right)^{\square_4}\subseteq\left(\Phi^{\square_5\,-1}\circ\mathcal{R}^{\square_6}\right)^{\square_7}.$$

Thus, for instance, we can easily establish the following

Theorem 23.2. *If in addition to the latter notations, $\square_5=\square_6=\square_7$ is an inversion and composition compatible closure operation, then under the notation $\lozenge=(\square_i)_{i=1}^6$ the following assertions are equivalent:*

(1) $\mathcal{F}$ is mildly right $\lozenge$–Φ–regular with respect to the relators $\mathcal{R}$ and $\mathcal{S}$;

(2) $\mathcal{F}$ is mildly right $\lozenge$–continuous with respect to the relators $\Phi^{-1}\circ\mathcal{R}$ and $\mathcal{S}$.

Moreover, as a generalization a dual of Example 20.2, we can also prove the following

Theorem 23.3. *If in addition to the notations of Definition 23.1, we have $Z=W$, and $\square$ is an increasing, inversion and composition compatible unary operation for relators such that $\mathcal{F}$ is upper right $\square$–$\mathcal{G}$–normal, then under the notation $\Phi=\mathcal{G}\circ\mathcal{F}$ the relator $\mathcal{F}$ is upper right $\square$–Φ–regular.*

Proof. Now, by Definition 23.1 and the inversion compatibility of $\square$, we have

$$\left(\mathcal{S}^{\square}\circ\mathcal{F}^{\square}\right)^{\square}\subseteq\left(\mathcal{G}^{\square\,-1}\circ\mathcal{R}^{\square}\right)^{\square}=\left(\mathcal{G}^{-1\,\square}\circ\mathcal{R}^{\square}\right)^{\square}.$$

Hence, by using Theorem 18.2, we can infer that $(\mathcal{S}\circ\mathcal{F})^{\square}\subseteq(\mathcal{G}^{-1}\circ\mathcal{R})^{\square}$. Now, by the corresponding definitions and the increasingness of $\square$, it is clear that we also have

$$\left(\mathcal{F}^{-1}\circ(\mathcal{S}\circ\mathcal{F})^{\square}\right)^{\square}\subseteq\left(\mathcal{F}^{-1}\circ(\mathcal{G}^{-1}\circ\mathcal{R})^{\square}\right)^{\square}.$$

Hence, by using the left composition compatibility of $\square$ and the definition of Φ, we can infer that

$$\left(\mathcal{F}^{-1}\circ\mathcal{S}\circ\mathcal{F}\right)^{\square}\subseteq\left(\mathcal{F}^{-1}\circ\mathcal{G}^{-1}\circ\mathcal{R}\right)^{\square}=\left(\Phi^{-1}\circ\mathcal{R}\right)^{\square}.$$

Now, by Theorems 18.3 and 18.2 and the inversion compatibility of $\square$, it is clear that we also have

$$\left(\mathcal{F}^{\square\,-1}\circ\mathcal{S}^{\square}\circ\mathcal{F}^{\square}\right)^{\square}\subseteq\left(\Phi^{\square\,-1}\circ\mathcal{R}^{\square}\right)^{\square},$$

and thus by Definition 23.2 the required assertion is also true.

Remark 23.2. Finally, we note that by considering, instead of an ordinary relator space $(X,Y)(\mathcal{R})$, a *nonconventional birelator space* $(X,Y)(\mathcal{R},\mathcal{U})$, where $\mathcal{U}$ is an ordinary relator on $\mathcal{P}(X)$ to Y, the definitions and theorems of the present paper can be greatly generalized.

For this note that, following the ideas of [73,76], for any $A\subseteq X$ and $B\subseteq Y$ we may naturally define $A\in\mathrm{Cl}_{(\mathcal{R},\mathcal{U})}(B)$ if $U(A)\cap B\in\mathcal{D}_{\mathcal{R}}$ for all $U\in\mathcal{U}$. However, it seems now even more important to investigate possible applications of several reasonable particular cases of Definition 22.1.

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