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Research Article

Some results on the Sombor indices of graphs

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Abstract

This paper is concerned with three recently introduced degree-based graph invariants; namely, the Sombor index, the reduced Sombor index and the average Sombor index. The first aim of the present paper is to give some results that may be helpful in proving a recently proposed conjecture concerning the Sombor index. Establishing inequalities related to the aforementioned three graph invariants is the second aim of this paper.

Keywords: graph invariant; topological index; Sombor indices; bounds; chemical graph theory.

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1. Introduction

The study of the mathematical aspects of the degree-based graph invariants (also known as topological indices) is considered to be one of the very active research areas within the field of chemical graph theory [17]. Recently, the mathematical chemist Ivan Gutman [18], one of the pioneers of chemical graph theory, proposed a geometric approach to interpret degree-based graph invariants and based on this approach, he devised three new graph invariants; namely the Sombor index, the reduced Sombor index and the average Sombor index. The Sombor index, being the simplest one among the aforementioned three invariants, has attracted a significant attention from researchers within a very short time [3, 7–9, 14, 15, 19, 23, 26–31, 34, 35, 39, 41, 42].

The first aim of this paper to give some results that may be helpful in proving a conjecture concerning the Sombor index posed in the reference [35]. In order to state this conjecture, we need some definitions first. An acyclic graph is the graph containing no cycle. For a graph G, its cyclomatic number $\nu(G)$ (or simply ν) is the least number of edges whose deletion makes the graph G as acyclic. A ν-cyclic graph is the one having the cyclomatic number ν. A pendent vertex of a graph is a vertex of degree 1. For $\nu \geq 1$, denote by $H_{n,\nu}$ the graph deduced from the star graph of order n by adding ν edge(s) between a fixed pendent vertex and ν other pendent vertices.

Conjecture 1.1. [35] *For the fixed integers n and ν with $6 \leq \nu \leq n-2$, $H_{n,\nu}$ is the only graph attaining the maximum Sombor index in the class of all ν-cyclic connected graphs of order n.*

Establishing inequalities related to the Sombor index, the reduced Sombor index and the average Sombor index is another aim of this paper.

2. Preliminaries

Let G be a graph. Denote by $E(G)$ and $V(G)$ the edge set and vertex set, respectively, of G. Denote by $i \sim j$ the edge connecting the vertices $v_i, v_j \in V(G)$. For a vertex $v_i \in V(G)$, its degree is denoted by $d_{v_i}(G)$ (or simply by $d_i(G)$). A regular graph is the one in which all of its vertices have the same degree. For an edge $e \in E(G)$, its degree is the number of edges adjacent to e. By an edge-regular graph, we mean a graph in which all of its edges have the same degree. A graph of order n is also known as an n-vertex graph. Denote by $G - v_i$ and $G - v_iv_j$ the graphs obtained from G by removing the vertex v_i and the edge v_iv_j, respectively. The n-vertex complete graph and the n-vertex star graph are denoted as K_n and $K_{1,n-1}$, respectively. From the notations $E(G)$, $V(G)$, $\nu(G)$ and $d_i(G)$, we remove "(G)" whenever the graph under consideration is clear. The graph-theoretical notation and terminology used in this paper but not defined here, may be found in some standard graph-theoretical books, like [4, 6, 10].

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If $V(G) = \{v_1, v_2, \dots, v_n\}$ and $|E(G)| = m$ then the Sombor index, the average Sombor index and the reduced Sombor index of the graph G are defined as

$$SO(G) = \sum_{i\sim j} \sqrt{d_i^2 + d_j^2}\,, \quad SO_{avr}(G) = \sum_{i\sim j} \sqrt{\left(d_i - \frac{2m}{n}\right)^2 + \left(d_j - \frac{2m}{n}\right)^2} \quad \text{and} \quad SO_{red}(G) = \sum_{i\sim j} \sqrt{(d_i-1)^2 + (d_j-1)^2},$$

respectively.

Most of the degree-based graph invariants can be written [22,38] as:

$$BID(G) = \sum_{i\sim j} f(d_i, d_j), \tag{1}$$

where f is a symmetric non-negative real-valued function of d_i and d_j. The graph invariants having the form (1) are known as the *bond incident degree indices* [36], BID indices in short [2]. Those choices of the function f are given in Table 1 that correspond to the graph invariants used in the next sections.

Table 1: The graph invariants to be used in the next sections.

Function $f(d_i, d_j)$	Equation (1) corresponds to	Symbol
$d_i + d_j$	first Zagreb index [20,21]	M_1
$d_i\, d_j$	second Zagreb index [20]	M_2
$2(d_i + d_j)^{-1}$	harmonic index [13]	H
$d_i^{-2} + d_j^{-2}$	inverse degree [13]	ID
$\lvert d_i - d_j\rvert$	Albertson index [1]	Alb
$2\sqrt{d_i\, d_j}(d_i + d_j)^{-1}$	geometric-arithmetic index [37]	GA
$d_i\, d_j(d_i + d_j)^{-1}$	inverse sum indeg index [38]	ISI
$d_i(d_j)^{-1} + d_j(d_i)^{-1}$	symmetric division deg index [38]	SDD
$(d_i + d_j)(4d_i\, d_j)^{-1/2}$	arithmetic-geometric index [11]	AG
$d_i^2 + d_j^2$	forgotten topological index [16]	F

3. Towards the proof of Conjecture 1.1

The p-Sombor index of a graph G is denoted by $SO_p(G)$ and is defined [35] as the sum of the quantities $(d_i^p + d_j^p)^{1/p}$ over all edges $i \sim j$ of G, where p is not equal to 0. The first lemma (Lemma 3.1) of this section gives an upper bound on a generalized variant $SO_{p,q}$ of the p-Sombor index:

$$SO_{p,q}(G) = \sum_{i\sim j} [(d_i + q)^p + (d_j + q)^p]^{1/p} = \sum_{i\sim j} \varphi_{p,q}(i \sim j)\,,$$

where q is a real number provided that $\varphi_{p,q}(i \sim j)$ is a real number for every edge $i \sim j$ of G. The name (p, q)-Sombor index may be associated with the graph invariant $SO_{p,q}$.

Lemma 3.1. *If G is a graph of size $m \geq 1$ then for any real number q, it holds that*

$$SO_{2,q}(G) \leq \sqrt{m[F(G) + 2q \cdot M_1(G) + 2q^2 m]}$$

with equality if and only if there exist a fixed real number t such that $(d_i + q)^2 + (d_j + q)^2 = t$ for every edge $i \sim j$ of G, where $F(G)$ and $M_1(G)$ are the forgotten topological index and first Zagreb index of G, respectively; see Table 1.

Proof. From Cauchy-Bunyakovsky-Schwarz's inequality, it follows that

$$\left(\sum_{i\sim j} \sqrt{(d_i + q)^2 + (d_j + q)^2}\right)^2 \leq \left(\sum_{i\sim j} (1)\right)\left(\sum_{i\sim j} \left[(d_i + q)^2 + (d_j + q)^2\right]\right) \tag{2}$$

$$= m[F(G) + 2q \cdot M_1(G) + 2q^2 m].$$

Note that the equality sign in (2) holds if and only if there exist a fixed real number t' such that $\sqrt{(d_i+q)^2+(d_j+q)^2} = t'$ for every edge $i \sim j$ of G.

□

Next, the bound on the invariant $SO_{2,q}$ given in Lemma 3.1 is used to derive another bound on $SO_{2,q}$ in terms of the parameters m and q only (see Lemma 3.4); however, to proceed, bounds on the forgotten topological index F and first Zagreb index M_1 in terms of m are required first.

Lemma 3.2. *For any n-vertex graph G of size m with $0 \le m \le n-1$, it holds that*

$$F(G) \le m(m^2+1)$$

with equality if and only if the star S_{m+1} is a component of G.

Proof. We fix n and use induction on m. For $m = 0, 1$, the lemma is obviously true; thus, the induction starts. Suppose that G is an n-vertex graph of size k such that $0 \le k \le n-1$ and $k \ge 2$. Take an edge $i \sim j$. Without loss of generality, assume that $d_j \le d_i$. Note that $d_i + d_j \le k+1$, which gives $d_i^2 + d_j^2 - (d_i + d_j) \le (k+1-d_j)^2 + d_j^2 - (k+1)$ and hence the equation $F(G) - F(G - v_iv_j) = 3(d_i^2 + d_j^2 - d_i - d_j) + 2$ gives

$$F(G) - F(G - v_iv_j) \le 3[(k+1-d_j)^2 + d_j^2 - (k+1)] + 2\,, \tag{3}$$

where the equality sign in (3) holds if and only if $d_i + d_j = k+1$. (It needs to be mentioned here that, throughout this proof, d_r denotes the degree of a vertex v_r in G not in $G - v_iv_j$.) The inequalities $d_j \le d_i$ and $d_i + d_j \le k+1$ confirm that $2\,d_j \le k+1$, which forces that the right hand side of (3) is maximum if and only if $d_j = 1$. Thus, (3) gives

$$F(G) - F(G - v_iv_j) \le 3(k^2 - k) + 2, \tag{4}$$

with equality if and only if $d_i = k$ and $d_j = 1$. Also, by inductive hypothesis, it holds that

$$F(G - v_iv_j) \le (k-1)[(k-1)^2 + 1], \tag{5}$$

with equality if and only if the star S_k is a component of $G - v_iv_j$. Thus, from (4) and (5), it follows that $F(G) \le k(k^2+1)$ with equality if and only if the star S_{k+1} is a component of G. This completes the induction and hence the proof.

□

Lemma 3.3. [40] *For any n-vertex graph G of size m with $0 \le m \le n-1$, it holds that*

$$M_1(G) \le m(m+1)$$

with equality if and only if

$$\begin{cases} \text{the star } S_{m+1} \text{ is a component of } G, & \text{if } m \neq 3, \\ \text{either the star } S_4 \text{ is a component of } G \text{ or the cycle } C_3 \text{ is a component of } G, & \text{if } m = 3. \end{cases}$$

The next result follows directly from Lemmas 3.1, 3.2 and 3.3.

Lemma 3.4. *For any n-vertex graph G of size m with $0 \le m \le n-1$ and for any non-negative real number q, it holds that*

$$SO_{2,q}(G) \le m\sqrt{m^2 + 2q(m+q+1) + 1}$$

with equality if and only if the star S_{m+1} is a component of G.

The next two results were proven in [35].

Lemma 3.5. [35] *If ν and n are fixed integers such that $0 \le \nu \le n-2$ then the graph attaining the maximum Sombor index in the class of all connected ν-cyclic graphs of order n has the maximum degree $n-1$.*

Lemma 3.6. [35] *Let ν and n be fixed integers such that $2 \le \nu \le n-2$. Let G be a graph with the maximum value of the Sombor index in the class of all connected ν-cyclic graphs of order n. If $(d_1, d_2, \ldots, d_n)$ is the vertex-degree sequence of G such that $d_1 \ge d_2 \ge \cdots \ge d_n$, then the vertex v_2 is adjacent to all non-pendent vertices of G, where $d_i = d_{v_i}$ for $v_i \in V(G)$.*

For a real number α, define the graph invariant $SO_{V,\alpha}$ as follows

$$SO_{V,\alpha}(G) = \sum_{v_i \in V(G)} \sqrt{(d_i+1)^2+\alpha^2}\,.$$

Towards the Proof of Conjecture 1.1. Let G be a ν-cyclic connected graph of order n with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $6 \le \nu \le n-2$. Let $(d_1, d_2, \ldots, d_n)$ be the vertex-degree sequence of G such that $d_1 \ge d_2 \ge \cdots \ge d_n$, where $d_i = d_{v_i}$ for $v_i \in V(G)$. If either $d_1 \le n-2$ or v_2 is not adjacent to any non-pendent vertex of G then from either Lemma 3.5 or Lemma 3.6, respectively, it follows that G does not have the maximum value of SO in the class of all ν-cyclic connected graphs of order n. In what follows, assume that $d_1 = n-1$ and that v_2 is adjacent to all non-pendent vertices of G. Note that the graph $G - v_1$ has exactly one connected non-trivial component C and the subgraph induced by $V(C)$ (the vertex set of C) has a vertex of degree $|V(C)|-1$, and that $G-v_1$ has the size $m' = \nu$, where $6 \le m' \le n-2 = |V(G-v_1)|-1$. Thus, from Lemma 3.4, it follows that

$$SO(G) = SO_{V,n-1}(G-v_1) + SO_{2,1}(G-v_1) \tag{6}$$

$$\le SO_{V,n-1}(G-v_1) + \nu\sqrt{(\nu+1)^2+4}, \tag{7}$$

where the equality sign in (7) holds if and only if the star $S_{\nu+1}$ is a component of $G - v_1$.

We believe that the next result concerning the invariant $SO_{V,\alpha}$ is true. However, at the present moment, we do not have its proof; thus, we state it as a conjecture (if one proves this conjecture then from (7), the proof of Conjecture 1.1 follows directly).

Conjecture 3.1. *For any n-vertex graph G of size m with $6 \le m \le n-1$, it holds that*

$$SO_{V,n-1}(G) \le m\sqrt{(n-1)^2+4} + \sqrt{(n-1)^2+(m+1)^2} + (n-m-1)\sqrt{(n-1)^2+1}$$

with equality if and only if the star S_{m+1} is a component of G.

4. Some relations between Sombor indices and other degree-based graph invariants

Before establishing the main results of this section, we first recall an inequality for the real number sequences reported in [33].

Lemma 4.1. [33] *Let $x = (x_i)$, $i = 1, 2, \ldots, n$, be a sequence of non-negative real numbers and $a = (a_i)$, $i = 1, 2, \ldots, n$, a sequence of positive real numbers. Then, for any $r \ge 0$ holds*

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \ge \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^{r}}\,. \tag{8}$$

Equality holds if and only if $r = 0$ or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

In the next theorem we determine a relationship between $SO(G)$ and $M_1(G)$ and $ISI(G)$.

Theorem 4.1. *Let G be a connected graph. Then*

$$SO(G) \le \sqrt{M_1(G)(M_1(G) - 2ISI(G))}\,. \tag{9}$$

Equality holds if and only if G is an edge-regular graph.

Proof. From the definitions of $M_1(G)$ and $ISI(G)$ we have that

$$M_1(G) - 2ISI(G) = \sum_{i\sim j}(d_i+d_j) - \sum_{i\sim j}\frac{2d_id_j}{d_i+d_j} = \sum_{i\sim j}\frac{d_i^2+d_j^2}{d_i+d_j}\,. \tag{10}$$

On the other hand, for $r = 1$, $x_i := \sqrt{d_i^2+d_j^2}$, $a_i := d_i + d_j$, with summation performed over all adjacent vertices v_i and v_j in G, the inequality (8) becomes

$$\sum_{i\sim j}\frac{d_i^2+d_j^2}{d_i+d_j} = \sum_{i\sim j}\frac{\left(\sqrt{d_i^2+d_j^2}\right)^2}{d_i+d_j} \ge \frac{\left(\sum_{i\sim j}\sqrt{d_i^2+d_j^2}\right)^2}{\sum_{i\sim j}(d_i+d_j)}\,,$$

that is

$$\sum_{i\sim j} \frac{d_i^2 + d_j^2}{d_i + d_j} \geq \frac{SO(G)^2}{M_1(G)}\,. \tag{11}$$

From the above and inequality (10) we arrive at (9).

Equality in (11) holds if and only if $\frac{\sqrt{d_i^2+d_j^2}}{d_i+d_j}$ is constant for any pair of adjcent vertices v_i and v_j in G. Suppose that v_j and v_k are adjacent to vertex v_i. Then

$$\frac{\sqrt{d_i^2 + d_j^2}}{d_i + d_j} = \frac{\sqrt{d_i^2 + d_k^2}}{d_i + d_k}\,,$$

that is

$$2d_i(d_i^2 - d_j d_k)(d_j - d_k) = 0\,.$$

From the above identity it follows that equality in (9) holds if and only if G is an edge-regular graph. □

Corollary 4.1. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$SO(G) \leq \sqrt{M_1(G)\left(M_1(G) - \frac{2m^2}{n}\right)}\,. \tag{12}$$

Equality holds if and only if G is an edge-regular graph.

Proof. The inequality (12) is obtained from (9) and

$$ISI(G) \geq \frac{m^2}{n}\,,$$

which was proven in [12] □

Corollary 4.2. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$SO(G) \leq \frac{m}{n-1}\sqrt{(2m + (n-1)(n-2))\left(\frac{2m}{n} + (n-1)(n-2)\right)}\,. \tag{13}$$

Equality holds if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Proof. The inequality (13) is obtained from (12) and

$$M_1(G) \leq m\left(\frac{2m}{n-1} + n - 2\right)\,,$$

which was proven in [5]. □

Corollary 4.3. *Let T be a tree with $n \geq 2$ vertices. Then*

$$SO(T) \leq (n-1)\sqrt{n^2 - 2n + 2}\,. \tag{14}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

The inequality (14) was proven in [18].

Proofs of the following theorems are analogous to that of Theorem 4.1, thus omitted.

Theorem 4.2. *Let G be a connected graph with $m \geq 1$ edges. Then*

$$SO_{red}(G) \leq \sqrt{M_1(G)\,(M_1(G) - 2ISI(G) + H(G) - 2m)}\,.$$

Equality holds if and only if G is an edge-regular graph.

Theorem 4.3. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$SO_{avr}(G) \leq \sqrt{M_1(G)\left(M_1(G) - 2ISI(G) + \frac{4m^2}{n^2}H(G) - \frac{4m^2}{n}\right)}\,.$$

Equality holds if and only if G is an edge-regular graph.

Theorem 4.4. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$SO(\overline{G}) \leq \sqrt{M_1(G)\left(M_1(G) - 2ISI(G) + \frac{1}{2}(n-1)^2 H(G) - m(n-1)\right)}.$$

Equality holds if and only if G is an edge-regular graph.

The next theorem reveals a connection between Sombor index and indices $F(G)$, $M_2(G)$, $AG(G)$ and $GA(G)$.

Theorem 4.5. *Let G be a connected graph. Then*

$$SO(G) \leq \sqrt{\frac{1}{2}(F(G) + 2M_2(G))(2AG(G) - GA(G))}. \tag{15}$$

Equality holds if and only if G is regular.

Proof. The following identity holds

$$2AG(G) - GA(G) = \sum_{i\sim j}\left(\frac{d_i + d_j}{\sqrt{d_i d_j}} - \frac{2\sqrt{d_i d_j}}{d_i + d_j}\right) = \sum_{i\sim j} \frac{d_i^2 + d_j^2}{\sqrt{d_i d_j}(d_i + d_j)}. \tag{16}$$

By the arithmetic–geometric mean inequality (see e.g. [32]) we have that

$$\sqrt{d_i d_j} \leq \frac{1}{2}(d_i + d_j). \tag{17}$$

Combining (16) and (17) gives

$$2AG(G) - GA(G) \geq 2\sum_{i\sim j} \frac{d_i^2 + d_j^2}{(d_i + d_j)^2}. \tag{18}$$

On the other hand, for $r = 1$, $x_i := \sqrt{d_i^2 + d_j^2}$, $a_i := (d_i + d_j)^2$, with summation performed over all adjacent vertices v_i and v_j in G, the inequality (8) transforms into

$$\sum_{i\sim j} \frac{d_i^2 + d_j^2}{(d_i + d_j)^2} = \sum_{i\sim j} \frac{\left(\sqrt{d_i^2 + d_j^2}\right)^2}{(d_i + d_j)^2} \geq \frac{\left(\sum_{i\sim j}\sqrt{d_i^2 + d_j^2}\right)^2}{\sum_{i\sim j}(d_i + d_j)^2},$$

that is

$$\sum_{i\sim j} \frac{d_i^2 + d_j^2}{(d_i + d_j)^2} \geq \frac{SO(G)^2}{F(G) + 2M_2(G)}. \tag{19}$$

Now, from the above and (18) we arrive at (15).

Equality in (17) holds if and only if $d_i = d_j$ for any pair of adjacent vertices v_i and v_j in G, which implies that equality in (15) holds if and only if G is regular. □

Corollary 4.4. *Let G be a connected graph. Then*

$$SO(G) \leq \sqrt{F(G)(2AG(G) - GA(G))}. \tag{20}$$

Equality holds if and only if G is regular.

Proof. By the AM–GM inequality we have that

$$F(G) = \sum_{i\sim j}(d_i^2 + d_j^2) \geq \sum_{i\sim j} 2d_i d_j = 2M_2(G).$$

The inequality (20) is obtained from the above and (15). □

Corollary 4.5. *Let G be a connected graph and Δ be its maximum vertex degree. Then*

$$SO(G) \leq \sqrt{\Delta M_1(G)(2AG(G) - GA(G))}.$$

Equality holds if and only if G is regular.

Proof. The following is valid

$$F(G) = \sum_{i=1}^{n} d_i^3 \leq \Delta \sum_{i=1}^{n} d_i^2 = \Delta M_1(G).$$

From the above and inequality (20) we obtain the required result. □

The following inequality was proven in [24] for the real number sequences.

Lemma 4.2. [24] *Let $p=(p_i)$, $i=1,2,\dots,n$ be a sequence of non–negative real numbers and $a=(a_i)$, $i=1,2,\dots,n$, positive real number sequence.Then, for any real r, $r\le 0$ or $r\ge 1$, holds*

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1}\sum_{i=1}^{n} p_i a_i^r \ge \left(\sum_{i=1}^{n} p_i a_i\right)^{r}. \tag{21}$$

When $0\le r\le 1$ the opposite inequality is valid.

Equality holds if and only if either $r=0$, or $r=1$, or $a_1=a_2=\cdots=a_n$, or $p_1=p_2=\cdots=p_t=0$ and $a_{t+1}=\cdots=a_n$, for some t, $1\le t\le n-1$.

In the next theorem we determine a relationship between $SO(G)$ and $ID(G)$, $F(G)$ and $M_2(G)$.

Theorem 4.6. *Let G be a connected graph. Then*

$$SO(G)\le \sqrt[4]{ID(G)F(G)M_2(G)^2}\,. \tag{22}$$

Equality holds if and only if G is an edge-regular graph.

Proof. For $r=2$, $p_i:=d_i^2+d_j^2$, $a_i:=\frac{1}{d_id_j}$, with summation performed over all pairs of adjacent vertices v_i and v_j in G, the inequality (21) becomes

$$\sum_{i\sim j}(d_i^2+d_j^2)\sum_{i\sim j}\frac{d_i^2+d_j^2}{(d_id_j)^2}\ge\left(\sum_{i\sim j}\frac{d_i^2+d_j^2}{d_id_j}\right)^2,$$

that is

$$ID(G)F(G)\ge\left(\sum_{i\sim j}\frac{d_i^2+d_j^2}{d_id_j}\right)^2. \tag{23}$$

On the other hand, for $r=1$, $x_i:=\sqrt{d_i^2+d_j^2}$, $a_i:=d_id_j$, with summation performed over all pairs of adjacent vertices v_i and v_j in G, the inequality (8) becomes

$$\sum_{i\sim j}\frac{d_i^2+d_j^2}{d_id_j}=\sum_{i\sim j}\frac{\left(\sqrt{d_i^2+d_j^2}\right)^2}{d_id_j}\ge\frac{\left(\sum_{i\sim j}\sqrt{d_i^2+d_j^2}\right)^2}{\sum_{i\sim j}d_id_j},$$

that is

$$\sum_{i\sim j}\frac{d_i^2+d_j^2}{d_id_j}\ge\frac{SO(G)^2}{M_2(G)}. \tag{24}$$

Now, from (23) and (24) we arrive at (22).

Equality in (23) holds if and only if d_id_j is constant for any pairs of adjacent vertices v_i and v_j in G. Suppose that vertices v_j and v_k are adjacent to v_i. In that case, we have that $d_id_j=d_id_k$, that is $d_j=d_k$. This means that equality (23) holds if and only if G is an edge-regular graph. Equality in (24) holds if and only if $\frac{\sqrt{d_i^2+d_j^2}}{d_id_j}$ is constant for any pair of adjacent vertices v_i and v_j in G. Suppose that vertices v_j and v_k are adjacent to v_i. In that case holds $\frac{\sqrt{d_i^2+d_j^2}}{d_id_j}=\frac{\sqrt{d_i^2+d_k^2}}{d_id_k}$, that is $d_j=d_k$. This means that equality in (24) holds if and only if G is an edge-regular graph, which means that equality in (22) holds if and only if G is an edge-regular graph. □

One can easily verify that from (24) the inequality

$$SO(G)\le\sqrt{M_2(G)SDD(G)}\,,$$

(which was proven in [31]) follows.

Corollary 4.6. *Let G be a connected graph. Then*

$$SO(G)\le\sqrt[4]{\frac{1}{4}ID(G)F(G)^3}\,.$$

Equality holds if and only if G is regular.

Theorem 4.7. *Let G be a connected graph. Then*

$$SO(G) \geq \sqrt{\frac{M_1(G)^2 + Alb(G)^2}{2}}\,. \tag{25}$$

Equality holds if and only if G is an edge-regular graph.

Proof. The following identities are valid

$$SO(G) - \sum_{i\sim j} \frac{2d_i d_j}{\sqrt{d_i^2 + d_j^2}} = \sum_{i\sim j} \frac{(d_i - d_j)^2}{\sqrt{d_i^2 + d_j^2}}\,,$$

and

$$SO(G) + \sum_{i\sim j} \frac{2d_i d_j}{\sqrt{d_i^2 + d_j^2}} = \sum_{i\sim j} \frac{(d_i + d_j)^2}{\sqrt{d_i^2 + d_j^2}}\,.$$

Taking $r = 1$, $x_i := |d_i - d_j|$, and $a_i := \sqrt{d_i^2 + d_j^2}$ in inequality (8) with summation performed over all pairs of adjacent vertices v_i and v_j in G, we obtain

$$SO(G) - \sum_{i\sim j} \frac{2d_i d_j}{\sqrt{d_i^2 + d_j^2}} \geq \frac{Alb(G)^2}{SO(G)}\,.$$

Similarly, taking $r = 1$, $x_i := d_i + d_j$, and $a_i := \sqrt{d_i^2 + d_j^2}$ in inequality (8) with summation performed over all pairs of adjacent vertices v_i and v_j in G,we obtain

$$SO(G) + \sum_{i\sim j} \frac{2d_i d_j}{\sqrt{d_i^2 + d_j^2}} \geq \frac{M_1(G)^2}{SO(G)}\,.$$

From the above inequalities we obtain the assertion of the Theorem 4.7. □

Corollary 4.7. *Let G be a connected graph. Then*

$$SO(G) \geq \frac{\sqrt{2}}{2} M_1(G)\,. \tag{26}$$

Equality holds if and only if G is regular.

Proof. Since $Alb(G)^2 \geq 0$, the inequality (26) is obtained from (25). □

The inequality (26) was proven in [15, 31] (see also [19]). By a similar arguments, the following results can be proven.

Theorem 4.8. *Let G be a graph with $m \geq 1$ edges. Then*

$$SO_{red}(G) \geq \sqrt{\frac{(M_1(G) - 2m)^2 + Alb(G)^2}{2}}\,.$$

Equality holds if and only if G is an edge-regular graph.

Theorem 4.9. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$SO_{avr}(G) \geq \sqrt{\frac{\left(M_1(G) - \frac{4m^2}{n}\right)^2 + Alb(G)^2}{2}}\,.$$

Equality holds if and only if G is an edge-regular graph.

From Theorems 4.8 and 4.9 we have the following corollaries.

Corollary 4.8. *Let G be a graph with $m \geq 1$ edges. Then*

$$SO_{red}(G) \geq \frac{\sqrt{2}}{2}(M_1(G) - 2m)\,. \tag{27}$$

Equality holds if and only if G is regular or each of its components is regular.

Corollary 4.9. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$SO_{avr}(G) \geq \frac{\sqrt{2}}{2}\left(M_1(G) - \frac{4m^2}{n}\right)\,. \tag{28}$$

Equality holds if and only if G is regular.

Inequalities (27) and (28) were proven in [31] (see also [19]).

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