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Research Article

On a Diophantine equation related to the difference of two Pell numbers

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Abstract

In this paper, the Diophantine equation $P_n - P_m = 3^a$ is considered and all solutions for this equation are obtained. In the proof of the main theorem, lower bounds for the absolute value of linear combinations of logarithms and a version of the Baker-Davenport reduction method are used.

Keywords: Pell numbers; Diophantine equation; Baker's theory.

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1. Introduction

In recent years, many researchers investigated the solutions of Diophantine equations of the form

$$u_n \pm u_m = p^a$$

where (u_n) is a fixed linear recurrence sequence and p is a prime. For example, Bravo and Luca [3,4] solved the equation $u_n + u_m = 2^a$ for the cases when (u_n) is the Fibonacci sequence and when (u_n) is the Lucas sequence. Also, Bitim and Keskin [1] found all the solutions of the equation $u_n - u_m = 3^a$ for the case when (u_n) is the Fibonacci sequence. Also, many other researches on this topic, such as [6], have been carried out.

In this paper, we search all the solutions of the Diophantine equation

$$P_n - P_m = 3^a \tag{1}$$

where P_n is the Pell sequence and n, m and a are nonnegative integers such that $n \geq m$. The main argument used for the solution of such problems is Baker's theory (lower bound for the absolute value of linear combinations of logarithms of algebraic numbers) and a version of the Baker-Davenport reduction method.

2. Preliminaries

A linear recurrence sequence of order k is a sequence whose general term is $(a_n) = L(a_{n-1}, a_{n-2}, \dots, a_{n-k})$ where k is a fixed positive integer and L is a linear function. A linear recurrence sequence of order 2 is known as *a binary recurrence sequence*. Pell sequence, one of the most familiar binary recurrence sequence, is defined by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$. Some of the terms of the Pell sequence are given by $0, 1, 2, 5, 12, 29, 70, \dots$. Its characteristic polynomial is of the form $x^2 - 2x - 1 = 0$ whose roots are $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Binet's formula enables us to rewrite the Pell sequence by using the roots α and β as

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}. \tag{2}$$

Also, it is known that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1}. \tag{3}$$

We give the definition of the logarithmic height of an algebraic number and some of its properties.

Definition 2.1. *Let* ξ *be an algebraic number of degree* d *with minimal polynomial*

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \cdot \prod_{i=1}^{d} (x - \xi_i)$$

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where the a_i's are relatively prime integers with $a_0 > 0$ and the ξ_i's are conjugates of ξ. Then

$$h(\xi) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\xi_i|, 1\}\right)\right)$$

is called the logarithmic height of ξ.

Proposition 2.1. *Let $\xi, \xi_1, \xi_2, \ldots, \xi_t$ be elements of an algebraic closure of $\mathbb{Q}$ and $m \in \mathbb{Z}$. Then*

1. $h(\xi_1 \cdots \xi_t) \leq \sum_{i=1}^{t} h(\xi_i)$,
2. $h(\xi_1 + \cdots + \xi_t) \leq (t-1)\log 2 + \sum_{i=1}^{t} h(\xi_i)$,
3. $h(\xi^m) = |m|\, h(\xi)$.

We will use the following theorem (see [8] or Theorem 9.4 in [5]) and lemma (see [2] which is a variation of the result due to [7]) for proving our results.

Theorem 2.1 (Matveev's theorem)**.** *Let $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive elements of a number field $\mathbb{L}$ of degree D, and $b_1, b_2, \ldots, b_t$ be rational integers. Set*

$$B := \max\{|b_1|, \ldots, |b_t|\}$$

and

$$\Lambda := \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1.$$

If Λ is nonzero, then

$$\log|\Lambda| > -3 \cdot 30^{t+4} \cdot (t+1)^{5.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log(tB)) \cdot A_1 \cdots A_t$$

where

$$A_i \geq \max\{D \cdot h(\gamma_i), |\log \gamma_i|, 0.16\}$$

for all $1 \leq i \leq t$. If $\mathbb{L} \subset \mathbb{R}$, then

$$\log|\Lambda| > -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log B) \cdot A_1 \cdots A_t.$$

Lemma 2.1. *Let A, B, μ be some real numbers with $A > 0$ and $B > 1$, and let γ be an irrational number and M be a positive integer. Take p/q as a convergent of the continued fraction of γ such that $q > 6M$. Set $\varepsilon := \|\mu q\| - M\|\gamma q\| > 0$ where $\|\cdot\|$ denotes the distance from the nearest integer. Then there is no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log \frac{Aq}{\varepsilon}}{\log B}.$$

3. Main result

Theorem 3.1. *The only triples of nonnegative integers n, m, a with $n \geq m$ satisfying the Diophantine equation (1) are the following:*

$$(n, m, a) \in \{(1,0,0), (2,1,0), (3,2,1), (5,2,3)\}.$$

Proof. In the case that $n = m$, it is obvious that there exists no solution for the Diophantine equation (1). So we consider the case that $n > m$ in the rest of the paper.

By a simple computation, we observe that all triples (n, m, a) with $0 \leq m < n \leq 200$ satisfying the equation (1) form the set $\{(1,0,0), (2,1,0), (3,2,1), (5,2,3)\}$.

Assume that $n > 200$. From (1) and (3), we get

$$3^a = P_n - P_m \leq P_n \leq \alpha^{n-1} < 3^n$$

and so $a < n$. When we replace P_n in the equation (1) with its closed form, we obtain

$$\frac{\alpha^n}{2\sqrt{2}} - 3^a = \frac{\beta^n}{2\sqrt{2}} + P_m.$$

By taking the absolute value of both sides of the above relation and using the upper bound in relation (3), it is yielded that

$$\left|\frac{\alpha^n}{2\sqrt{2}} - 3^a\right| \leq \frac{|\beta^n|}{2\sqrt{2}} + P_m < \frac{1}{6} + \alpha^{m-1}.$$

When we multiply both sides of the expression above by $\frac{2\sqrt{2}}{\alpha^n}$ to apply Matveev's result in Theorem 2.1, we have

$$\begin{aligned}\left|1 - 3^a \cdot \alpha^{-n} \cdot 2\sqrt{2}\right| &< \frac{2\sqrt{2}}{\alpha^n}\left(\frac{1}{6} + \alpha^{m-1}\right)\\ &= 2\sqrt{2}\alpha^{m-n}\left(\frac{1}{6}\alpha^{-m} + \frac{1}{\alpha}\right)\\ &< 2\sqrt{2}\alpha^{m-n}\left(\frac{1}{6} + \frac{1}{2}\right)\\ &= \frac{4\sqrt{2}}{3}\alpha^{m-n}\\ &< \frac{2}{\alpha^{n-m}}. \end{aligned} \tag{4}$$

Let us take $t := 3$, $(\gamma_1, \gamma_2, \gamma_3) := (3, \alpha, 2\sqrt{2})$ and $(b_1, b_2, b_3) := (a, -n, 1)$. We have $D := 2$ since each γ_i belongs to $\mathbb{Q}(\sqrt{2})$. Note that $1 - 3^a \cdot \alpha^{-n} \cdot 2\sqrt{2}$ is nonzero. Indeed, if it were zero, we could get

$$3^a = \frac{\alpha^n}{2\sqrt{2}} \Rightarrow \alpha^n = 3^a \cdot 2\sqrt{2} \Rightarrow \alpha^{2n} = 8 \cdot 3^{2a},$$

and so $\alpha^{2n} \in \mathbb{Z}$, which is a contradiction.

A_1, A_2, A_3 and B can be chosen as follows:

$$\begin{aligned} &A_1 := 2.2 > 2.1972 \simeq 2 \cdot \log 3 = D \cdot h(\gamma_1),\\ &A_2 := 0.9 > 0.8813 \simeq \log \alpha = D \cdot h(\gamma_2),\\ &A_3 := 2.1 > 2.079 \simeq 2 \cdot \log\left(2\sqrt{2}\right) = D \cdot h(\gamma_3),\\ &B := n. \end{aligned}$$

From Theorem 2.1, we obtain that

$$\begin{aligned}\left|1 - 3^a \cdot \alpha^{-n} \cdot 2\sqrt{2}\right| &> \exp\left(-C_1 \cdot (1 + \log n) \cdot 2.2 \cdot 0.9 \cdot 2.1\right)\\ \frac{2}{\alpha^{n-m}} &> \exp\left(-C_1 \cdot (1 + \log n) \cdot 2.2 \cdot 0.9 \cdot 2.1\right) \qquad \text{from (4)}\end{aligned}$$

where $C_1 = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)$. Proceeding to appropriate operations, we have

$$\begin{aligned}\frac{2}{\alpha^{n-m}} &> \exp\left(-C_1 \cdot (1 + \log n) \cdot 2.2 \cdot 0.9 \cdot 2.1\right)\\ (n-m)\log\alpha - \log 2 &< C_1 \cdot (1 + \log n) \cdot 2.2 \cdot 0.9 \cdot 2.1.\end{aligned}$$

Since $C_1 < 9.7 \cdot 10^{11}$ and $1 + \log n < 2\log n$ for $n \geq 3$, we get

$$\begin{aligned}(n-m)\log\alpha - \log 2 &< 9.7 \cdot 10^{11} \cdot (1 + \log n) \cdot 2.2 \cdot 0.9 \cdot 2.1\\ (n-m)\log\alpha &< 8.2 \cdot 10^{12} \cdot \log n \end{aligned} \tag{5}$$

To find an upper bound on n, let's rewrite the equation (1) as a second linear form in logarithms and perform some operations as follows:

$$\frac{\alpha^n}{2\sqrt{2}} - \frac{\alpha^m}{2\sqrt{2}} - 3^a = \frac{\beta^n}{2\sqrt{2}} - \frac{\beta^m}{2\sqrt{2}}$$

and taking the absolute value of both sides, we have

$$\left|\frac{\alpha^n}{2\sqrt{2}} - \frac{\alpha^m}{2\sqrt{2}} - 3^a\right| = \left|\frac{\beta^n}{2\sqrt{2}} - \frac{\beta^m}{2\sqrt{2}}\right|.$$

It follows from the triangle inequality that

$$\left|\frac{\alpha^n}{2\sqrt{2}}\left(1 - \alpha^{m-n}\right) - 3^a\right| \leq \frac{|\beta|^n + |\beta|^m}{2\sqrt{2}}.$$

Dividing both sides by $\frac{\alpha^n}{2\sqrt{2}}\left(1-\alpha^{m-n}\right)$, we obtain

$$\left|1-3^a\alpha^{-n}2\sqrt{2}\left(1-\alpha^{m-n}\right)^{-1}\right| \leq \frac{|\beta|^n+|\beta|^m}{\alpha^n(1-\alpha^{m-n})}.$$

It follows from the fact that $\frac{|\beta|^n+|\beta|^m}{(1-\alpha^{m-n})}<0.59$ for $n\geq 3$ and $m\geq 1$, that

$$\left|1-3^a\alpha^{-n}2\sqrt{2}\left(1-\alpha^{m-n}\right)^{-1}\right| < \frac{0.59}{\alpha^n}. \tag{6}$$

Let us apply the result of Matveev once more. We take $t:=3$, $(\gamma_1,\gamma_2,\gamma_3):=\left(3,\alpha,2\sqrt{2}\,(1-\alpha^{m-n})^{-1}\right)$ and $(b_1,b_2,b_3):=(a,-n,1)$. We have $D:=2$ since each γ_i belongs to $\mathbb{Q}\left(\sqrt{2}\right)$. Note that $1-3^a\cdot\alpha^{-n}\cdot 2\sqrt{2}\cdot(1-\alpha^{m-n})^{-1}$ is nonzero. Indeed, if it were zero, we could get

$$\begin{aligned}
3^a\cdot 2\sqrt{2} &= \alpha^n\left(1-\alpha^{m-n}\right)\\
3^a\cdot 2\sqrt{2} &= \alpha^n-\alpha^m\\
-3^a\cdot 2\sqrt{2} &= \beta^n-\beta^m && \text{conjugating both sides in } \mathbb{Q}\left(\sqrt{2}\right)
\end{aligned}$$

and the last two equations would imply that

$$\alpha^n<\alpha^n+\alpha^m=|\beta^n-\beta^m|\leq|\beta|^n+|\beta|^m<1,$$

which contradicts that $\alpha^n>1$ for positive integer n.

A_1, A_2 and B can be chosen as follows:

$$\begin{aligned}
A_1 &:= 2.2>2.1972\simeq 2\cdot\log 3=D\cdot h\left(\gamma_1\right),\\
A_2 &:= 0.9>0.8813\simeq\log\alpha=D\cdot h\left(\gamma_2\right),\\
B &:= n.
\end{aligned}$$

Now, let's find an appropriate value for A_3:

$$\begin{aligned}
h\left(\gamma_3\right) &= h\left(\frac{2\sqrt{2}}{1-\alpha^{m-n}}\right)\\
&\leq h\left(2\sqrt{2}\right)+h\left(1-\alpha^{m-n}\right) && \text{from Proposition 2.1(1)}\\
&\leq \log\left(2\sqrt{2}\right)+h\left(1\right)+h\left(\alpha^{m-n}\right)+\log 2 && \text{from Proposition 2.1(2)}\\
&= \log\left(4\sqrt{2}\right)+|m-n|\cdot h\left(\alpha\right) && \text{from Proposition 2.1(3)}\\
&= \log\left(4\sqrt{2}\right)+(n-m)\,\frac{\log\alpha}{2}
\end{aligned}$$

and so,

$$A_3:=3.47+(n-m)\cdot\log\alpha>\log 32+(n-m)\cdot\log\alpha=\max\left\{2h\left(\gamma_3\right),\left|\log\gamma_3\right|,0.16\right\}.$$

Now Theorem 2.1 implies that

$$\begin{aligned}
\frac{0.59}{\alpha^n} &> \left|1-3^a\alpha^{-n}2\sqrt{2}\left(1-\alpha^{m-n}\right)^{-1}\right|\\
&> \exp\left(-C_2\cdot(1+\log n)\cdot 2.2\cdot 0.9\cdot(3.47+(n-m)\log\alpha)\right)\\
&= \exp\left(-C_2\cdot(1+\log n)\cdot 1.98\cdot(3.47+(n-m)\log\alpha)\right)
\end{aligned}$$

where $C_2:=1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2)<9.7\cdot 10^{11}$. Taking the logarithm of both sides in the last inequality, considering that $1+\log n<2\log n$ for $n\geq 3$ and using the inequality (5), one can see that

$$\begin{aligned}
\log 0.59-n\log\alpha &> -C_2\cdot(1+\log n)\cdot 1.98\cdot(3.47+(n-m)\log\alpha)\\
n\log\alpha &< \log 0.59+C_2\cdot(1+\log n)\cdot 1.98\cdot(3.47+(n-m)\log\alpha)\\
n\log\alpha &< 3.85\cdot 10^{12}\cdot\log n\cdot(3.47+(n-m)\log\alpha)\\
n\log\alpha &< 3.85\cdot 10^{12}\cdot\log n\cdot\left(3.47+8.2\cdot 10^{12}\log n\right).
\end{aligned} \tag{7}$$

Thus, we obtain

$$n < 3.59 \cdot 10^{25} \log^2 n$$

and so,

$$n < 1.63 \cdot 10^{29}. \tag{8}$$

Now let's improve the upper bound on n a little bit more. Set

$$z_1 := a \log 3 - n \log \alpha + \log\left(2\sqrt{2}\right).$$

The inequality (4) can be also written as

$$|1 - e^{z_1}| < \frac{2}{\alpha^{n-m}}.$$

By using (1) and (2), we get

$$\frac{\alpha^n}{2\sqrt{2}} = P_n + \frac{\beta^n}{2\sqrt{2}} > P_n > P_n - P_m = 3^a.$$

Therefore, we have

$$z_1 = \log\left(\frac{3^a 2\sqrt{2}}{\alpha^n}\right) < 0.$$

It is easy to see that $\frac{2}{\alpha^{n-m}} < 0.829$ for all $n - m \geq 1$. Therefore we have $e^{|z_1|} < 5.85$. Then we get

$$0 < |z_1| < e^{|z_1|} - 1 \leq e^{|z_1|}\,|1 - e^{z_1}| < \frac{12}{\alpha^{n-m}}$$

and so

$$0 < \left|a \log 3 - n \log \alpha + \log\left(2\sqrt{2}\right)\right| < \frac{12}{\alpha^{n-m}}.$$

Thus we have

$$0 < \left|a \frac{\log 3}{\log \alpha} - n + \frac{\log\left(2\sqrt{2}\right)}{\log \alpha}\right| < \frac{12}{\log \alpha} \cdot \alpha^{-(n-m)} \tag{9}$$

by dividing both sides of the inequality above by $\log \alpha$. From Lemma 2.1, we have the irrational number $\gamma = \frac{\log 3}{\log \alpha}$ with

$$\mu = \frac{\log\left(2\sqrt{2}\right)}{\log \alpha}, A = \frac{12}{\log \alpha}, B = \alpha, w = n - m.$$

On the other hand, we recall that $a < n < 1.63 \cdot 10^{29}$. From Lemma 2.1, we can set $M := 1.63 \cdot 10^{29}$ and if we take the denominator of the 58th convergent of γ, then we get $q = 15.50 \cdot 10^{29} > 6M$. By using Mathematica Script Language, we obtain $\varepsilon = \|\mu q\| - M\,\|\gamma q\| = 0.184766 > 0$.

Applying Lemma 2.1 to the above parameters, we conclude that there is no solution to the inequality (9) for the values $n - m$ with

$$n - m \geq \frac{\log\left(Aq/\varepsilon\right)}{\log B} = 81.788.$$

Therefore, for the inequality (9) to be solvable, our upper limit for $n - m$ must be at most 81. By substituting the upper bound value for $n - m$ in the inequality (7), we get $n < 1.211 \cdot 10^{16}$. Let us improve this upper bound value on n a little more. Put

$$z_2 := a \log 3 - n \log \alpha + \log\left(2\sqrt{2}\left(1 - \alpha^{m-n}\right)^{-1}\right).$$

Therefore, (6) implies that

$$|1 - e^{z_2}| < \frac{0.59}{\alpha^n}.$$

It is easy to see that $\frac{0.59}{\alpha^n} < \frac{1}{2}$. Suppose that $z_2 > 0$. Then $0 < z_2 < e^{z_2} - 1 < \frac{0.59}{\alpha^n}$. If $z_2 < 0$, then $1 - e^{z_2} < \frac{0.59}{\alpha^n} < \frac{1}{2}$ and we obtain $\frac{1}{2} < e^{z_2}$ so that again $e^{|z_2|} < 2$. Therefore, we have

$$0 < |z_2| < e^{|z_2|} - 1 \leq e^{|z_2|} \cdot |1 - e^{z_2}| < 2 \cdot \frac{0.59}{\alpha^n}$$

and

$$0 < \left|a \frac{\log 3}{\log \alpha} - n + \frac{\log\left(2\sqrt{2}\left(1 - \alpha^{m-n}\right)^{-1}\right)}{\log \alpha}\right| < \frac{1.18}{\log \alpha} \cdot \alpha^{-n} \tag{10}$$

by dividing both sides of the inequality above by $\log\alpha$. Now to apply Lemma 2.1 again, set

$$\gamma=\frac{\log 3}{\log\alpha},\quad \mu=\frac{\log\left(2\sqrt{2}\,(1-\alpha^{m-n})^{-1}\right)}{\log\alpha},\quad A=\frac{1.18}{\log\alpha},\quad B=\alpha,\quad w=n.$$

Firstly, we can choose $M=1.211\cdot 10^{16}$. Since $6M=7.266\cdot 10^{16}$, in order to apply Lemma 2.1, we must choose $q=8.27\cdot 10^{18}$ which is the 33rd denominator of the continued fraction of γ. Therefore, with the aid of Mathematica, we get $\varepsilon\leq 0.49473$ for $n-m\in\{1,\ldots,81\}$. From Lemma 2.1, there is no solution to the inequality (10) for

$$n\geq\frac{\log\left(Aq/\varepsilon\right)}{\log B}=50.551.$$

Thus, n must be less than or equal to 50 for a solution which contradicts our assumption. This completes the proof. □

4. Conclusion

We obtain all solutions of the Diophantine equation $P_n-P_m=3^a$. Linear forms in logarithms and Baker's theory are the main tools used in our proofs. The method used in this paper may be applied to other Diophantine equations.

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