

Research Article

On truncated and full classical Markov moment problems

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Abstract

Giving necessary and sufficient conditions for the existence of solutions of truncated and full classical Markov moment problems in terms of the given (or measured) moments, in $L_{p,\mu}(S)$ ($1 \leq p < \infty$) spaces setting, is the first aim of this work. Reduced (truncated) moment problems arise in real-world situations, where only a finite number of samples are available. We obtain solutions as nonnegative functions in a $L_{q,\mu}(S)$ space, where $S \subset \mathbb{R}^n$ is a closed subset, μ is a regular Borel probability measure on S and q is the conjugate of $p \in [1, \infty)$. Applying polynomial approximation on Cartesian products of closed unbounded intervals in solving full Markov moment problems on $[0, \infty)^n$, when the uniqueness of the solution follows too, is the second purpose of the paper. A construction of a solution for the truncated one-dimensional moment problem is proposed. Influence of perturbations of the moments on the corresponding solutions in $L_{2,\mu}(S)$ is also briefly discussed; this is the third aim of the paper.

Keywords: truncated moment problem; full moment problem; existence of a solution; uniqueness; construction; perturbations.

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1. Introduction

The truncated moment problem is important in mathematics as well as in mathematical applications, because it involves only a finite number of moments (of limited order), which are assumed to be known (or given, or measurable). This corresponds to real-life problems, where only a finite number of samples are available. Moreover, in the case of one-dimensional truncated moment problem, sometimes the solution can be found as a polynomial function, whose coefficients are determined by solving a Cramer (linear) system. We restrict ourselves to the scalar real classical moment problem on a closed bounded or unbounded subset S of \mathbb{R}^n , where $n \geq 1$ is an integer. The following standard notations will be used: $\mathbb{N} = \{1, 2, \dots\}$; $\mathbb{N}_0 = \{0, 1, 2, \dots\}$; $k = (k_1, \dots, k_n)$ is an arbitrary element of $\mathbb{N}_0^n = \mathbb{N}_0 \times \dots \times \mathbb{N}_0$; k_l is a nonnegative integer for each $l = 1, \dots, n$ and $|k| = k_1 + \dots + k_n$ and $k! = k_1! \dots k_n!$; $p_k(t) = t^k = t_1^{k_1} \dots t_n^{k_n}$ where $t = (t_1, \dots, t_n) \in S$ and $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$. We denote by $\mathbb{R}_d[t_1, \dots, t_n]$ the real vector subspace of all polynomial functions of n real variables, with real coefficients, generated by $t^k = t_1^{k_1} \dots t_n^{k_n}$, $k_i \in \{0, 1, \dots, d\}$, $i = 1, \dots, n$, where $d \geq 1$ is a fixed integer. The dimension of this subspace is clearly equal to $(d+1)^n$. Given a finite set

$$\{m_k : 0 \leq k_i \leq d; i = 1, \dots, n\}$$

of real numbers, and a probability Borel measure μ on S with finite absolute moments of all orders $\leq d$ (i.e., $\int_S |t|^k d\mu < \infty$ for all $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ with $k_i \leq d$ and $i = 1, \dots, n$), existence and eventually construction or approximation of a Lebesgue measurable real nonnegative function $h \in L_{q,\mu}(S)$, satisfying the moment conditions

$$\int_S t^k h(t) d\mu = m_k, \quad k_i \leq d, \quad i = 1, \dots, n, \quad (1)$$

and

$$0 \leq h(t) \text{ for almost all } t \in S, \quad \|h\|_{q,\mu} \leq 1 \quad (2)$$

are under consideration, where q is the conjugate of a given number $p \in [1, \infty)$. Here $\|\cdot\|_{q,\mu}$ is the usual norm on the space $L_{q,\mu}(S)$. By its formulation, the moment problem is an inverse problem, since h is unknown; it is going to be approximated, starting from its given moments m_k , defined by (1). If we require (1) for all $k \in \mathbb{N}_0^n$ (that is, if we have to solve the full moment problem), then sometimes h is unique with the properties (1) and (2), and can be determined as the sum of a Fourier series whose coefficients are known in terms of the moments m_k , $k \in \mathbb{N}_0^n$. The direct problem could be: given

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$h \in L_{q,\mu}(S)$, find its moments defined by (1). The latter problem is a computational one. On the other hand, the condition “ μ is a probability Borel measure” may be weakened assuming that μ is an arbitrary positive Borel measure on S , with finite absolute moments of all orders, accompanied by appropriate modifications in the conclusions of the results. If $n \geq 2$, then the moment problem (1)-(2) is called multidimensional, while when $n = 1$, we have a one-dimensional Markov moment problem. This is not a usual moment problem, (which requires only $h \geq 0$), since it involves also the last inequality (2). The function $h \in L_{q,\mu}(S)$ satisfying (1) and (2) will be called a solution of this moment problem. The solution h appears as the representing function from $L_{q,\mu}(S)$ for a positive linear functional T on $L_{p,\mu}(S)$, $p \in [1, \infty)$, $1/p + 1/q = 1$, $\|T\| \leq 1$. The numbers m_k , $k_i \leq d$, $i = 1, \dots, n$, are called the classical moments with respect to the positive measure $h d\mu$, and this measure is called a representing measure for the extension of the linear functional

$$f : M := \mathbb{R}_d[t_1, \dots, t_n] \rightarrow \mathbb{R}, f \left(\sum_{k_i \leq d} \alpha_k p_k \right) = \sum_{k_i \leq d} \alpha_k m_k, p_k(t) = t^k, \alpha_k \in \mathbb{R}, k_i \leq d, i = 1, \dots, n. \quad (3)$$

The linear functional (3) should be extended to the entire space $L_{p,\mu}(S)$, $1 \leq p < \infty$, preserving linearity, positivity, continuity and eventually controlling the norm of the extension of f (see Section 2). A first step is to point out a necessary and sufficient condition for the existence of such an extension $T : L_{p,\mu}(S) \rightarrow \mathbb{R}$. This is called a truncated moment problem, since the degree of the involved polynomials in each real variable t_i is bounded above. Obviously, a solution T of a truncated moment problem cannot be unique. In the case of full moment problem, (1) must hold for all k in \mathbb{N}_0^n . In this case, the problem of the uniqueness of the solution makes sense and is related to the notion of a M -determinate (moment-determinate) measure μ . Recall that the measure μ is called M -determinate (or simply determinate) if it is uniquely determinate by its moments $m_k = \int_S t^k d\mu$, $k \in \mathbb{N}_0^n$, or, equivalently, by its values on the vector space \mathcal{P} of all polynomials. For results on the moment problem see [1, 11, 24]. See some chapters of [6, 16, 23] for the background on measure theory, analysis and functional analysis. The present work is based on the results published in the papers [9, 12, 17–21, 25] and is indirectly related to the articles [2–5, 7, 8, 10, 13–15, 22, 26]. Some of the references lead to connections of the moment problem with other fields (such as fixed point theory, operator theory, convex functions, complex functions, etc.). The rest of the paper is organized as follows. Section 2 concerns the existence of positive linear solutions for which the norm can be evaluated, formulated in measure theory setting. In some cases, uniqueness and conditions involving quadratic forms are emphasized. For the one-dimensional truncated moment problem, the construction of a solution is sketched. In Section 3, influence of perturbations of the exact moments (determined in the training stage or given), on the corresponding solution is discussed. Section 4 concludes the paper.

2. Existence, uniqueness and construction of some solutions

As we have already seen, under the hypothesis of Section 1, the question on the existence of a solution for the problem defined by (1) and (2) is an extension problem of a linear (bounded) positive functional from the subspace $\mathbb{R}_d[t_1, \dots, t_n]$ to the space $L_{p,\mu}(S)$, $p \in [1, \infty)$. The norm of this linear positive extension will be determinate as well. Usually, such problems are solved via Hahn-Banach theorem. To obtain a positive linear extension, the next result will be partially applied in the sequel.

Theorem 2.1. *Let E be an ordered vector space, F be an order complete vector space, $M \subset E$ be a vector subspace, $T_1 : M \rightarrow F$ be a linear operator, $\Psi : E \rightarrow F$ be a convex operator. The following statements are equivalent:*

- (a) *There exists a positive linear extension $T : E \rightarrow F$ of T_1 such that $T \leq \Psi$ on E .*
- (b) *The inequality $T_1(y) \leq \Psi(x)$ holds for all $(y, x) \in M \times E$ such that $y \leq x$.*

Remark 2.1. *If we apply Theorem 2.1 to the particular case when $E_+ = \{0\}$, we obtain Hahn-Banach theorem.*

Theorem 2.1 was published firstly in [17], being derived from the general constrained extension result ([17], Theorem 1); details and completions can be found in [18]. Then, an equivalent form of Theorem 2.1, formulated in terms of the moment problem, was published in ([19], Theorem 1). All these old statements require a lattice structure on F , which is assumed to be an order complete vector lattice. Recently, a sharp direct proof of Theorem 2.1, which does not require a lattice structure on F , was published in ([20], Theorems 5 and 6). This latter form is exactly Theorem 2.1. The next result is a direct application of Theorem 2.1. Using the above notations, we obtain the next result.

Theorem 2.2. Assume that all absolute moments

$$\int_S |t|^k d\mu < \infty$$

for $k = (k_1, \dots, k_n)$ with $k_i \leq d, i = 1, \dots, n$, are finite. Let $\{m_k : k_i \leq d, i = 1, \dots, n\}$ be a given (finite) sequence of real numbers. The following statements are equivalent:

(a) There exists $h \in L_{\infty, \mu}(S)$ such that $0 \leq h(t) \leq 1$, a.e.,

$$\int_S t^k h(t) d\mu = m_k,$$

where $k_i \leq d$ and $i = 1, \dots, n$.

(b) For any family of scalar $\{a_k : k_i \leq d, i = 1, \dots, n\}$, the inequality

$$\sum_{k_i \leq d; i=1, \dots, n} a_k p_k \leq \varphi \in L_{1, \mu}(S)$$

implies

$$\sum_{k_i \leq d; i=1, \dots, n} a_k m_k \leq \int_S |\varphi(t)| d\mu = \|\varphi\|_{1, \mu}.$$

Proof. The implication (a) implies (b) is obvious. Indeed, according to (a), we have:

$$\sum_{k_i \leq d; i=1, \dots, n} a_k m_k = \int_S \left(\sum_{k_i \leq d; i=1, \dots, n} a_k t^k \right) h(t) d\mu \leq \int_S |\varphi(t)| h(t) d\mu \leq \int_S |\varphi(t)| d\mu.$$

The converse implication is the basic one; it will be proved in the sequel. Let $E = L_{1, \mu}(S)$, M be the subspace of E generated by the monomials $p_k, p_k(t) = t^k, k_i \leq d, i = 1, \dots, n$, and $\Psi : E \rightarrow F = \mathbb{R}$ be defined by

$$\Psi(\varphi) := \int_S |\varphi(t)| d\mu, \quad \varphi \in E.$$

Let $f : M \rightarrow \mathbb{R}$ be the linear functional defined such that the interpolation conditions $f(p_k) = m_k, k_i \leq d, i = 1, \dots, n$ are verified (f is defined by (3)). Then (b) says that

$$\sum_{k_i \leq d; i=1, \dots, n} a_k p_k \leq \varphi \implies f\left(\sum_{k_i \leq d} a_k p_k\right) \leq \Psi(\varphi) = \int_S |\varphi(t)| d\mu$$

for all $y = \sum_{k_i \leq d; i=1, \dots, n} a_k p_k \in M, \varphi \in E$. According to Theorem 2.1 (where T_1 stands for f), there exists a positive linear extension T of f to the entire space $L_{1, \mu}(S)$ such that

$$T(\varphi) \leq \Psi(\varphi) = \int_S |\varphi(t)| d\mu = \|\varphi\|_{1, \mu}$$

for all $\varphi \in L_{1, \mu}(S)$. Writing this for $-\varphi$, we infer that

$$-T(\varphi) = T(-\varphi) \leq \|-\varphi\|_{1, \mu} = \|\varphi\|_{1, \mu}, \quad \varphi \in L_{1, \mu}(S).$$

Thus, it holds that $|T(\varphi)| \leq \|\varphi\|_{1, \mu}$ and $\varphi \in L_{1, \mu}(S)$, so that $\|T\| \leq 1$. According to measure theory reasons [23], there exists $h \in L_{\infty, \mu}(S), 0 \leq h \leq 1$, such that

$$T(\varphi) = \int_S h\varphi d\mu, \quad \varphi \in L_{1, \mu}(S),$$

for all $\varphi \in L_{1, \mu}(S)$. Obviously, $\int_S h p_k d\mu = T(p_k) = f(p_k) = m_k, (k_i \leq d, i = 1, \dots, n)$. This concludes the proof. □

Similar arguments, not involving measure theory appearing in the end of the above proof, lead to the next result, which is not requiring the norm and the order relation are compatible.

Theorem 2.3. Let E be a normed vector space endowed with a linear order relation, $M \subset E$ be a vector subspace, $T_1 : M \rightarrow \mathbb{R}$ be a linear functional. The following assertions are equivalent:

(a) *There exists a positive linear extension $T : E \rightarrow \mathbb{R}$ of T_1 such that $\|T\| \leq 1$.*

(b) *The inequality $T_1(y) \leq \|x\|$ holds for all $(y, x) \in M \times E$ such that $y \leq x$.*

Proof. Apply Theorem 2.1 to $F = \mathbb{R}$, $\Psi(\varphi) = \|\varphi\|$, $\varphi \in E$. □

Most of concrete function spaces are Banach lattices. However, there exist important ordered Banach spaces which are not lattices (see [15]). Many concrete Banach spaces are ordered Banach spaces (the order relation is compatible with the norm). On the other hand, applying Theorem 2.1 once more, and using the notations and hypothesis on the finite absolute moments of the measure μ , the following result holds:

Theorem 2.4. *Let $p \in (1, \infty)$, $E = L_{p,\mu}(S)$, $\{m_k : k_i \leq d, i = 1, \dots, n\}$ be a given (finite) sequence of real numbers, and q be the conjugate of p . The following statements are equivalent:*

(a) *There exists $h \in L_{q,\mu}(S)$ such that $0 \leq h(t)$, a.e., $\|h\|_{q,\mu} \leq 1$ and*

$$\int_S t^k h(t) d\mu = m_k$$

where $k_i \leq d$ and $i = 1, \dots, n$.

(b) *For any family of scalar $\{a_k : k_i \leq d, i = 1, \dots, n\}$, the inequality*

$$\sum_{k_i \leq d, i=1, \dots, n} a_k p_k \leq \varphi \in L_{p,\mu}(S)$$

implies

$$\sum_{k_i \leq d, i=1, \dots, n} a_k m_k \leq \|\varphi\|_{p,\mu}.$$

Proof. (a) \implies (b). By using Hölder’s inequality, we obtain

$$\sum_{k_i \leq d, i=1, \dots, n} a_k m_k = \int_S \left(\sum_{k_i \leq d, i=1, \dots, n} a_k t^k \right) h(t) d\mu \leq \int_S \varphi(t) h(t) d\mu \leq \|\varphi\|_{p,\mu} \|h\|_{q,\mu} \leq \|\varphi\|_{p,\mu}.$$

The implication (b) \implies (a) follows from the corresponding implication of Theorem 2.1, applied to M , f being defined by (3), $\Psi : E \rightarrow \mathbb{R}$, $\Psi(\varphi) = \|\varphi\|_{p,\mu}$ (here M is the subspace defined in the proof of Theorem 2.2). The conclusion follows via measure theory arguments. □

Remark 2.2. *For any positive measure μ on $S \subseteq \mathbb{R}^n$, a positive linear functional $T : L_{1,\mu}(S) \rightarrow \mathbb{R}$ verifies the condition*

$$T(\varphi) \leq \int_S \varphi d\mu, \quad \forall \varphi \in (L_{1,\mu}(S))_+$$

if and only if $\|T\| \leq 1$. Indeed, if T is dominated by the functional defined by the integral on the positive cone of $L_{1,\mu}(S)$, let $\psi \in L_{1,\mu}(S)$. The following inequalities hold:

$$|T(\psi)| \leq T(\psi^+) + T(\psi^-) = T(|\psi|) \leq \int_S |\psi| d\mu = \|\psi\|_{1,\mu}.$$

Hence $\|T\| \leq 1$. The converse implication is obvious:

$$\varphi \in (L_{1,\mu}(S))_+ \implies T(\varphi) \leq |T(\varphi)| \leq \|T\| \int_S |\varphi| d\mu \leq \int_S |\varphi| d\mu = \int_S \varphi d\mu = T_2(\varphi).$$

On the other hand, it is straightforward that the norm of the functional $T_2 : L_{1,\mu}(S) \rightarrow \mathbb{R}$, $T_2(\varphi) = \int_S \varphi d\mu$ is equal to 1, if μ is a probability measure.

Remark 2.2 allows application of the results from [21]; namely, the following theorem holds.

Theorem 2.5. *Let $S \subseteq \mathbb{R}^n$ be a closed unbounded subset, μ be a Borel regular M -determinate probability measure on S with finite absolute moments of all orders, $(m_k)_{k \in \mathbb{N}_0^n}$ be a given sequence of real numbers. The following statements are equivalent:*

(a) *There exists a unique $h \in L_{\infty,\mu}(S)$ such that $0 \leq h \leq 1$ and*

$$\int_S t^k h(t) d\mu = m_k$$

for all $k \in \mathbb{N}_0^n$.

(b) *For any $d \in \mathbb{N}_0$ and for any family of scalar $\{a_k : k_i \leq d, i = 1, \dots, n\}$, the inequality*

$$\sum_{k_i \leq d, i=1, \dots, n} a_k p_k \leq \varphi \in L_{1,\mu}(S)$$

implies

$$\sum_{k_i \leq d, i=1, \dots, n} a_k m_k \leq \int_S |\varphi(t)| d\mu = \|\varphi\|_{1,\mu}.$$

(c) *For any finite subset $J_0 \subset \mathbb{N}_0^n$, and for any $\{a_k\}_{k \in J_0} \subset \mathbb{R}$, the following implication holds*

$$\sum_{k \in J_0} a_k p_k \geq 0 \text{ on } S \implies 0 \leq \sum_{k \in J_0} a_k m_k \leq \sum_{k \in J_0} a_k \int_S t^k d\mu.$$

Proof. If there exists a function $h \in L_{\infty,\mu}(S)$ such that the moment conditions $\int_S t^k h(t) d\mu = m_k$ for all $k \in \mathbb{N}_0^n$, then h is unique in $L_{\infty,\mu}(S)$ with this property. Indeed, according to Lemma 3 of [21], the subspace \mathcal{P} of all polynomial functions on S is dense in $L_{1,\mu}(S)$, since μ is M -determinate by hypothesis. It remains to prove the implications (b) \implies (a), (c) \implies (a), their converses (a) \implies (b), (a) \implies (c) being obvious. To prove (b) \implies (a), the corresponding implication of Theorem 2.1 is applied, where E stands for $L_{1,\mu}(S)$, $M = \mathcal{P}$, $T_1\left(\sum_{k_i \leq d} \alpha_k p_k\right) = \sum_{k_i \leq d} \alpha_k m_k$, $\Psi = \|\cdot\|_{1,\mu}$, $F = \mathbb{R}$. The implication (c) \implies (a) is the implication (b) \implies (a) from Theorem 2 of [21], where y_j stands for m_j , $j \in \mathbb{N}_0^n$, $Y = \mathbb{R}$, $T_2(\varphi) = \int_S \varphi d\mu$, $\varphi \in L_{1,\mu}(S)$. Then one uses Remark 2.2. This concludes the proof. □

Next, we prove a variant of Theorem 2.5 for a special set S , endowed with a natural probability measure μ . Namely, when S is a Cartesian product of closed unbounded intervals, a part of the necessary and sufficient conditions for the existence of the solution can be expressed in terms of quadratic forms. In the one-dimensional case, if $S = [0, \infty)$ or $S = \mathbb{R}$, then the equivalent checkable condition for the existence of the (unique) solution is completely expressed in terms of quadratic forms.

Theorem 2.6. *Let $S = [0, \infty)^n$, $n \in \mathbb{N}$, $n \geq 2$, be endowed with the measure*

$$d\mu = \exp\left(-\sum_{k=1}^n t_k\right) dt_1 \cdots dt_n,$$

and $(m_k)_{k \in \mathbb{N}_0^n}$ a given sequence of real numbers. The following statements are equivalent:

(a) *There exists a unique $h \in L_{\infty,\mu}(S)$ such that $0 \leq h \leq 1$ and*

$$\int_S t^k h(t) d\mu = m_k$$

for all $k \in \mathbb{N}_0^n$.

(b) *For any finite subset $J_0 \subset \mathbb{N}_0^n$, and for any $\{a_k\}_{k \in J_0} \subset \mathbb{R}$, the following implication holds:*

$\sum_{k \in J_0} a_k p_k \geq 0$ on $S \implies \sum_{k \in J_0} a_k m_k \geq 0$; *for any finite subsets $J_k \subset \mathbb{N}_0$, $k = 1, \dots, n$, and for any $\{\lambda_{j_k}\}_{j_k \in J_k}$, $k = 1, \dots, n$, the following relations hold*

$$\sum_{i_1, j_1 \in J_1} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \cdots \lambda_{i_n} \lambda_{j_n} m_{i_1+j_1+l_1, \dots, i_n+j_n+l_n} \right) \cdots \right) \leq \sum_{i_1, j_1 \in J_1} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \cdots \lambda_{i_n} \lambda_{j_n} (i_1 + j_1 + l_1)! \cdots (i_n + j_n + l_n)! \right) \cdots \right), \quad (l_1, \dots, l_n) \in \{0, 1\}^n.$$

Proof. The measure $d\mu_1 = \exp(-t) dt$ is moment determinate on $[0, \infty)$ (see Theorem 2 in [25]). According to Lemma 4 of [21] and Theorem 5 in [21], where Y stands for \mathbb{R} and

$$F_2(\varphi) = \int_{[0, \infty)^n} \varphi d\mu, \varphi \in L_{1, \mu}([0, \infty)^n), \mu = \mu_1 \times \dots \times \mu_n, d\mu_j = \exp(-t_j) dt_j, j = 1, \dots, n,$$

the conclusion follows via standard measure theory arguments. □

The case $n = 1, S = [0, \infty)$, is of special interest (it leads to the one-dimensional Stieltjes moment problem). Namely, the following result holds, where only quadratic forms are involved. This formulation is possible due to the explicit form of nonnegative polynomials on $[0, \infty)$ in terms of sums of squares.

Theorem 2.7. *Let $(m_k)_{k \in \mathbb{N}_0}$ be a sequence of real numbers. The following statements are equivalent:*

(a) *There exists a unique $h \in L_{\infty, \mu}(S)$ such that $0 \leq h \leq 1$ and*

$$\int_S t^k h(t) \exp(-t) dt = m_k$$

for all $k \in \mathbb{N}_0^n$.

(b) *For any finite subset $J_0 \subset \mathbb{N}_0$ and any $\{\lambda_k\}_{k \in J_0} \subset \mathbb{R}$, the following inequalities hold:*

$$0 \leq \sum_{i, j \in J_0} \lambda_i \lambda_j m_{i+j+l} \leq \sum_{i, j \in J_0} \lambda_i \lambda_j (i+j+l)!, \quad l \in \{0, 1\}.$$

Proof. By using Theorem 2.6 and the explicit form of nonnegative polynomials p on $[0, \infty)$, we have: $p(t) \geq 0$ for all $t \in [0, \infty)$ if and only if $p(t) = q^2(t) + tr^2(t), t \geq 0$, for some $q, r \in \mathbb{R}[t]$ (see [1]); the conclusion follows. □

In other words, conditions (b) of Theorem 2.7 say that

$$0 \preceq (m_{i+j})_{i, j=0}^d \preceq ((i+j)!)_{i, j=0}^d \quad \text{and} \quad 0 \preceq (m_{i+j+1})_{i, j=0}^d \preceq ((i+j+1)!)_{i, j=0}^d \tag{4}$$

for any natural number $d \in \mathbb{N}_0$. We recall that the natural order relation on the space of symmetric $(d+1) \times (d+1)$ matrices with real coefficients, appearing in (4) is: $A \preceq B$ if and only if $\langle Ax, x \rangle \leq \langle Bx, x \rangle$ for all $x \in \mathbb{R}^{d+1}$ (see also [15] for details and related results).

We next consider the construction of the solution h for the truncated moment problem related to the Theorem 2.7, in the space $L_{2, \mu}([0, \infty))$, $d\mu = \exp(-t) dt$. The problem: find a function $h \in L_{2, \mu}([0, \infty))$ such that

$$\int_0^\infty t^j h(t) \exp(-t) dt = m_j, \quad j \in \{0, 1, \dots, d\},$$

seems to admit the simplest solution a polynomial function of degree d . We start by looking for a polynomial

$$h(t) = \sum_{j=0}^d \lambda_j t^j, \quad \lambda_j \in \mathbb{R}, \quad \sum_{j=0}^d \lambda_j^2 > 0,$$

such that

$$m_l = \int_0^\infty t^l h(t) \exp(-t) dt = \sum_{j=0}^d \lambda_j (l+j)!, \quad l \in \{0, 1, \dots, d\}. \tag{5}$$

The linear system (5) in the unknowns $\lambda_j, j = 0, 1, \dots, d$ has the square symmetric matrix $A = ((l+j)!)_{l, j=0}^d$. The latter matrix is positive definite because

$$\sum_{l, j=0}^d \lambda_l \lambda_j (l+j)! = \int_0^\infty \left(\sum_{j=0}^d \lambda_j t^j \right)^2 \exp(-t) dt > 0$$

for all $(\lambda_0, \dots, \lambda_d) \in \mathbb{R}^{d+1} \setminus \{0\}$. In particular, A is invertible (and its inverse is positive definite); so the system (5) is a Cramer system and from (5) it results its unique solution

$$(\lambda_0, \dots, \lambda_d)^t = A^{-1}(m_0, \dots, m_d)^t. \tag{6}$$

This method seems to work for any closed subset $S \subseteq \mathbb{R}$, having nonempty interior, endowed with a measure $d\mu = wdt$, where w is positive and continuous on S and

$$\int_S |t|^k w(t) dt < \infty, \quad \forall k \in \mathbb{N}_0.$$

Then, the system (5) becomes

$$m_l = \int_0^\infty t^l h(t) w(t) dt = \sum_{j=0}^d \lambda_j \int_S t^{j+l} w(t) dt, \quad l \in \{0, 1, \dots, d\}.$$

The matrix $A = (\int_S t^{j+l} w(t) dt)_{j,l=0}^d$ is positive definite since

$$\sum_{l,j=0}^d \lambda_l \lambda_j \int_S t^{l+j} w(t) dt = \int_S \left(\sum_{j=0}^d \lambda_j t^j \right)^2 w(t) dt > 0$$

for all $(\lambda_0, \dots, \lambda_d) \neq 0$. In particular, A is invertible and (6) holds.

3. Perturbations of moments and the corresponding perturbations of solutions

Assume now that the moments m_k , where $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$, $0 \leq k_i \leq d, i \in \{1, 2, \dots, n\}$, are exact and determined in a training stage when no external influence can occur. Also, assume that the moments in real time stage, denoted by v_k , where $k_i \leq d$ and $i = 1, \dots, n$, can be measured, but errors may occur due to external influences. The vector subspace of polynomials involved this way has the dimension $(d + 1)^n$. Assume now that there exists h, w in $L_{2,\mu}(S)$ such that

$$\int_S t^k h(t) d\mu = m_k \quad \text{and} \quad \int_S t^k w(t) d\mu = v_k$$

for all multi-index $k \in \mathbb{N}_0^n$ having all components $0 \leq k_i \leq d, i = 1, \dots, n$. Let $\{e_k\}_{0 \leq k_i \leq d}$ be the orthogonal system of polynomials having unit norms, obtained from the system $\{p_k\}_{0 \leq k_i \leq d}$ via Gram-Schmidt process, in the space $L_{2,\mu}(S)$. We denote by

$$P_d : L_{2,\mu}(S) \rightarrow \text{Span} \{e_k; k_i \in \{0, \dots, d\}, i = 1, \dots, n\}$$

the orthogonal projection. The following evaluations hold:

$$e_k = \sum_{l \leq k} c_{l,k} p_l,$$

where $l \leq k$ means $l_i \leq k_i, i = 1, \dots, n$, and the coefficients $c_{l,k}$ are known from Gram-Schmidt process. This yield:

$$\langle P_d(h) - P_d(w), e_k \rangle = \sum_{l \leq k} c_{l,k} \langle P_d(h) - P_d(w), p_l \rangle = \sum_{l \leq k} c_{l,k} (m_l - v_l).$$

Hence,

$$\|P_d(w) - P_d(h)\|_{2,\mu}^2 = \sum_{k_i \leq d} \langle P_d(h) - P_d(w), e_k \rangle^2 = \sum_{k_i \leq d} \left(\sum_{l \leq k} c_{l,k} (m_l - v_l) \right)^2. \tag{7}$$

Thus, one determines the integral mean of the square $(P_d(w) - P_d(h))^2$ in the left hand side of (7), in terms of the squares of the errors $|m_k - v_k|, (k_i \leq d \text{ for all } i \in \{1, \dots, n\})$.

4. Conclusion

Existence (and uniqueness) of the solutions for the classical (full) Markov moment problem are emphasized. Truncated moment problems are also under attention. For the one-dimensional case, a polynomial solution of truncated moment problem is sketched. Improving and completing this last result in the context of the multidimensional or/and full moment problem could be a subject for future work. In the last part of the paper, perturbation of the solution in terms of perturbations of the moments is discussed.

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