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Research Article

On the Sombor index of graphs

Tamás Réti[1], Tomislav Došlić[2,3,*], Akbar Ali[4]

[1]*Óbuda University, Bécsiút, 96/B, H–1034 Budapest, Hungary*
[2]*Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia*
[3]*Faculty of Information Studies, Novo Mesto, Slovenia*
[4]*Department of Mathematics, Faculty of Science, University of Haʾil, Haʾil, Saudi Arabia*

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Abstract

This paper is concerned with a recently introduced graph invariant, namely the Sombor index. Some bounds on the Sombor index are derived, and then utilized to establish additional bounds by making use of the existing results. One of the direct consequences of one of the obtained bounds is that the cycle graph C_n attains the minimum Sombor index among all connected unicyclic graphs of a fixed order $n \geq 4$. Graphs having the maximum Sombor index are also characterized from the classes of all connected unicyclic, bicyclic, tricyclic, tetracyclic, and pentacyclic graphs of a fixed order, and a conjecture concerning the maximum Sombor index of graphs of higher cyclicity is stated. A structural result is derived for graphs with integer values of Sombor index. Several possible directions for future work are also indicated.

Keywords: chemical graph theory; topological index; Sombor index; bounds; cyclic graphs.

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1. Introduction

For a graph G, its edge set and vertex set are denoted by $E(G)$ and $V(G)$, respectively. If two vertices of G are adjacent, we write uv. For a vertex $u \in V(G)$, its degree and the set of all vertices adjacent to u are denoted by $d_u(G)$ and $N_G(u)$, respectively. Define $N_G[u] = N_G(u) \cup \{u\}$. A graph containing no cycle is known as an acyclic graph. The cyclomatic number of a graph G is denoted by $\nu(G)$ and is defined as the minimum number of those edges of G whose removal makes G as acyclic. From the notation $d_u(G)$ and $\nu(G)$ we drop "(G)", and from $N_G(u)$ and $N_G[u]$ we drop the subscript "G", when there is no confusion possible. A graph with the cyclomatic number ν is also known as the ν-cyclic graph. For $\nu = 1, 2, 3, 4, 5$, the ν-cyclic graph is also referred to as the unicyclic graph, bicyclic graph, tricyclic graph, tetracyclic graph, pentacyclic graph, respectively. A vertex $u \in V(G)$ of degree 1 is called a pendent vertex. The graph-theoretical terminology and notation used in this paper but not described here, can be found in some standard graph-theoretical books, like [5, 7, 11].

A graph invariant of a graph is a numerical quantity that remains same under graph isomorphism. In chemical graph theory, graph invariants are usually referred to as the topological indices. Recently, Gutman [14] devised a new topological index under the name Sombor index. For a graph G, its Sombor index is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}\,.$$

The problem of finding graph attaining the maximum/minimum Sombor index from the class of all trees/graphs/connected graphs of a fixed order was solved in [14]. In the present paper, we derive some bounds on the Sombor index. These bounds can be utilized to establish additional bounds by making use of the existing results. One of the direct consequences of one of the obtained bounds is that the cycle graph C_n attains the minimum Sombor index among all connected unicyclic graphs of a fixed order $n \geq 4$.

Also, for $\nu = 1, 2, 3, 4, 5$, we prove that $H_{n,\nu}$ is the unique graph having the maximum Sombor index in the class of all connected unicyclic, bicyclic, tricyclic, tetracyclic, pentacyclic, respectively, graphs of a fixed order n, where $n \geq 4 + \lceil (\nu+1)/2 \rceil$ and $H_{n,\nu}$ is the graph (see [3]) obtained from the star graph S_n by adding ν edge(s) between a fixed pendent vertex and ν other pendent vertices. Moreover, it is conjectured that $H_{n,\nu}$ is the unique graph attaining the maximum Sombor index among all connected ν-cyclic graphs of order n, where ν and n are fixed integers satisfying the inequality $6 \leq \nu \leq n-2$. Furthermore, a structural result is derived for graphs with integer values of Sombor index. Several possible directions for future work are indicated in the last section.

*Corresponding author (tomislav.doslic@grad.unizg.hr).

2. Bounds and extremal results

The Randić index, for a graph G, is defined [17, 20] as

$$R(G) = \sum_{uv \in E(G)} (d_u\, d_v)^{-1/2}.$$

For any graph G of size m, it holds (see [12, 15]) that

$$R(G) \cdot RR(G) \geq m^2 \tag{1}$$

with equality if and only if G is regular, where $RR(G)$ is the reciprocal Randić index defined [12, 15] as

$$RR(G) = \sum_{uv \in E(G)} \sqrt{d_u\, d_v}\,.$$

Firstly, we give a lower bound on the Sombor index in terms of the reciprocal Randić index.

Proposition 2.1. *For any graph G, it holds that*

$$SO(G) \geq \sqrt{2} \cdot RR(G)$$

where the equality sign holds if and only if every component of G is regular.

Proof. The desired result follows from the fact that the inequality $(d_u - d_v)^2 \geq 0$ holds for every edge $uv \in E(G)$ with equality if and only if $d_u = d_v$. □

By using Proposition 2.1 and (1), we have the following result.

Corollary 2.1. *If G is a graph of size m then*

$$R(G) \cdot SO(G) \geq \sqrt{2}\, m^2$$

with equality if and only if G is regular.

In literature, there exists many upper bounds on the Randić index (for example, see [17]) and hence one can obtain many lower bounds on the Sombor index by using Corollary 2.1. For instance, let us derive one such bound. It is a well-known fact (for example, see [6]) that if G is a graph without isolated vertices then

$$R(G) \leq \frac{n}{2} \tag{2}$$

with equality if and only if every component of G is regular. The next result follows from Corollary 2.1 and (2).

Corollary 2.2. *If G is a graph of order n, size m and minimum degree at least 1, then*

$$SO(G) \geq \frac{2\sqrt{2}\, m^2}{n}$$

with equality if and only if G is regular.

The next result is an obvious but significant consequence of Corollary 2.2.

Corollary 2.3. *Among all connected unicyclic graphs of a fixed order $n \geq 4$, cycle C_n is the unique graph with the minimum Sombor index and this minimum value is $2\sqrt{2}\, n$.*

For a graph G, its forgotten topological index is defined [4, 13] as

$$F(G) = \sum_{u \in V(G)} d_u^3 = \sum_{uv \in E(G)} (d_u^2 + d_v^2).$$

Now, we give an upper bound on the Sombor index of a graph G in terms of the forgotten topological index and size of G.

Proposition 2.2. *For any graph G of size $m \geq 1$, it holds that*

$$SO(G) \leq \sqrt{m \cdot F(G)}$$

where the equality sign holds if and only if there exist a constant λ such that the equation $d_u^2 + d_v^2 = \lambda$ holds for every pair of adjacent vertices $u, v \in V(G)$.

Proof. By Cauchy-Bunyakovsky-Schwarz's inequality, it holds that

$$\left(\sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2} \right)^2 \leq \sum_{uv \in E(G)} (1) \sum_{uv \in E(G)} (d_u^2 + d_v^2)$$

with equality if and only if there exist a real number λ' such that $\sqrt{d_u^2 + d_v^2} = \lambda'$ for every pair of adjacent vertices $u, v \in V(G)$.

□

The next corollary follows from Proposition 2.2 and from the fact: If T is a tree of order n then $F(T) \leq (n-1)[(n-1)^2+1]$ with equality if and only if T is the star graph S_n (see [18]).

Corollary 2.4. [14] *Among all trees of a fixed order $n \geq 4$, star S_n is the unique graph with the maximum Sombor index and this maximum value is $(n-1)\sqrt{(n-1)^2+1}$.*

Next, in order to characterize graphs attaining the maximum Sombor index in the classes of all connected unicyclic, bicyclic, tricyclic, tetracyclic, and pentacyclic graphs of a fixed order, we need some results first.

Lemma 2.1. *Let $f(x,y) = \sqrt{x^2+y^2}$ where $x > 0$ and $y > 0$. The functions f and f_x are strictly increasing in x on the interval $[1, \infty)$, where f_x denotes the partial derivative function of f with respect to x.*

Lemma 2.2. [2] *Let ν and n be fixed integers satisfying the inequality $0 \leq \nu \leq n-2$. Among all connected ν-cyclic graphs of order n, let G be a graph having the maximum value of the graph invariant $BID(G) = \sum_{uv \in E(G)} f(d_u, d_v)$, where f is a non-negative real-valued symmetric function defined on the set of positive real numbers. Also, let both the expressions $f(x_0+t, y_0) - f(x_0, y_0) - (f(c, y_0) - f(c-t, y_0))$ and $f(x_0+t, c-t) - f(x_0, c)$ be non-negative, for every choice of the numbers x_0, y_0, c, t satisfying the inequalities $x_0 \geq c > t \geq 1$, $c \geq 2$ and $y_0 \geq 1$. If one of the following two conditions holds:*

- *The function f is increasing in both variables on the interval $[1, \infty)$ and at least one of the expressions $f(x_0+t, y_0) - f(x_0, y_0) - (f(c, y_0) - f(c-t, y_0))$ and $f(x_0+t, c-t) - f(x_0, c)$ is positive for every choice of the numbers x_0, y_0, c, t satisfying the inequalities $x_0 \geq c > t \geq 1$, $c \geq 2$ and $y_0 \geq 1$.*

- *The function f is strictly increasing in both variables on the interval $[1, \infty)$.*

Then, the maximum degree of G is $n-1$.

The next result is a direct consequence of Lemma 2.2.

Corollary 2.5. *Let ν and n be fixed integers satisfying the inequality $0 \leq \nu \leq n-2$. If G is a graph having the maximum Sombor index among all connected ν-cyclic graphs of order n then the maximum degree of G is $n-1$.*

Proof. Define $f(x,y) = \sqrt{x^2+y^2}$ and let x_0, y_0, c, t be any fixed real numbers satisfying the inequalities $x_0 \geq c > t \geq 1$, $c \geq 2$ and $y_0 \geq 1$. Clearly, the function f is strictly increasing in both variables on the interval $[1, \infty)$. Also, there exist numbers c_1, c_2 satisfying $c - t < c_2 < c \leq x_0 < c_1 < x_0 + t$ such that

$$f(x_0+t, y_0) - f(x_0, y_0) - (f(c, y_0) - f(c-t, y_0)) = t[f_x(c_1, y_0) - f_x(c_2, y_0)];$$

the right hand side of this equation is positive because of Lemma 2.1. Moreover, the inequality $f(x_0+t, c-t) - f(x_0, c) > 0$ follows from the assumption $x_0 \geq c > t \geq 1$. Thus, all the conditions mentioned in Lemma 2.2 are satisfied for f and hence the maximum degree of G is $n-1$.

□

As the star S_n is the unique graph having the maximum degree $n-1$ among all trees of a fixed order $n \geq 4$, Corollary 2.4 follows also from Corollary 2.5.

Lemma 2.3. *For the fixed integers ν and n satisfying the inequality $2 \leq \nu \leq n-2$, let G be a graph having the maximum Sombor index among all connected ν-cyclic graphs of order n. If $u, v \in V(G)$ are non-pendent vertices satisfying $d_v \geq d_u$ then $N(u) \subset N[v]$.*

Proof. From the assumption $d_v \geq d_u$, it follows that $N(u) \neq N[v]$. We have to show that the set $N(u)\setminus N[v]$ is empty. Contrarily, suppose that $N(u)\setminus N[v]$ is non-empty. Let $N(u)\setminus N[v] := \{u_1, u_2, \cdots, u_k\}$. Corollary 2.5 guaranties that the maximum degree of G is $n-1$, which implies that the set $N(u) \cap N(v)$ must be non-empty. Let G' be the graph deduced from G by deleting the edges $u_1u, u_2u, \cdots, u_ku$ and inserting the edges $u_1v, u_2v, \cdots, u_kv$. In what follows, we assume $N(w) = N_G(w)$ and $d_w = d_w(G)$ for any vertex $w \in V(G') = V(G)$. Now, one has

$$\begin{aligned} SO(G) - SO(G') = & \sum_{a \in N(u)\cap N(v)} [f(d_u, d_a) - f(d_u - k, d_a) + f(d_v, d_a) - f(d_v + k, d_a)] \\ & + \sum_{b \in N(v)\setminus N(u)} [f(d_v, d_b) - f(d_v + k, d_b)] + \sum_{i=1}^{k} [f(d_u, d_{u_i}) - f(d_v + k, d_{u_i})] \\ & + \varepsilon_{uv} [f(d_u, d_v) - f(d_u - k, d_v + k)] \end{aligned} \tag{3}$$

where the function f is defined in Lemma 2.1 and

$$\varepsilon_{uv} = \begin{cases} 1 & \text{if } u \text{ and } v \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $f(d_u, d_v) - f(d_u - k, d_v + k) < 0$. Also, note that there exist numbers c_1, c_2 satisfying $d_u - k < c_1 < d_u \leq d_v < c_2 < d_v + k$ such that

$$f(d_u, d_a) - f(d_u - k, d_a) + f(d_v, d_a) - f(d_v + k, d_a) = k[f_x(c_1, d_a) - f_x(c_2, d_a)]. \tag{4}$$

Since $c_2 > c_1$, by Lemma 2.1 the right hand side of Equation (4) is negative, and hence by using Lemma 2.1 in (3), we get $SO(G) - SO(G') < 0$, which contradicts the definition of G. □

Lemma 2.4. *For the fixed integers ν and n satisfying the inequality $2 \leq \nu \leq n-2$, let G be a graph having the maximum Sombor index among all connected ν-cyclic graphs of order n. If $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_{v_1} \geq d_{v_2} \geq \cdots \geq d_{v_n}$, then the vertex v_2 is adjacent to all non-pendent vertices of G.*

Proof. The proof is fully analogous to that of Lemma 6 of the paper [3]. □

For $\nu \geq 1$, denote by $H_{n,\nu}$ the graph deduced from the star S_n by adding ν edge(s) between a fixed pendent vertex and ν other pendent vertices. Now, we are in position to characterize graphs attaining the maximum Sombor index in the classes of all connected unicyclic, bicyclic, tricyclic, and tetracyclic graphs of a fixed order.

Theorem 2.1. *If ν and n are fixed integers such that $\nu \in \{1, 2, 3, 4\}$ and $n \geq 3 + \lceil (\nu+1)/2 \rceil$ then among all connected ν-cyclic graphs of order n, only the graph $H_{n,\nu}$ has the maximum Sombor index and*

$$SO(H_{n,\nu}) = (n-\nu-2)\sqrt{(n-1)^2+1} + \sqrt{(n-1)^2+(\nu+1)^2} + \nu\left[\sqrt{(n-1)^2+4} + \sqrt{(\nu+1)^2+4}\right].$$

Proof. By Corollary 2.5 and Lemma 2.4, a graph having the maximum Sombor index among all connected ν-cyclic graphs of order n must satisfy the following two conditions:

(I). The maximum degree is $n-1$.

(II). If $\{v_1, v_2, \ldots, v_n\}$ is the vertex set such that $d_{v_1} \geq d_{v_2} \geq \cdots \geq d_{v_n}$, then v_2 is adjacent to all non-pendent vertices.

Clearly, if $\nu \in \{1, 2\}$ then $H_{n,\nu}$ is the unique graph satisfying both the conditions (I) and (II). For $\nu = 3$, there are only two graphs satisfying both the conditions (I) and (II): one is $H_{n,3}$ and the other one, say $H'_{n,3}$, is the graph deduced from $H_{n,2}$ by inserting an edge between the two vertices of degree 2. By elementary calculations, one has

$$SO(H_{n,3}) = (n-5)\sqrt{(n-1)^2+1} + 3\sqrt{(n-1)^2+4} + \sqrt{(n-1)^2+16} + 6\sqrt{5}$$

and

$$SO(H'_{n,3}) = (n-4)\sqrt{(n-1)^2+1} + 3\sqrt{(n-1)^2+9} + 9\sqrt{2}.$$

If $n \in \{5, 6, \cdots, 13\}$ then it is directly verified that $SO(H_{n,3}) > SO(H'_{n,3})$. If $n \geq 14$ then

$$\sqrt{(n-1)^2+4} > \sqrt{(n-1)^2+9} - \frac{1}{5}$$

and hence

$$SO(H_{n,3}) - SO(H'_{n,3}) > \left[\sqrt{(n-1)^2+16} - \sqrt{(n-1)^2+1}\right] + \left[6\sqrt{5} - 9\sqrt{2} - \frac{3}{5}\right] > 0.$$

Finally, we consider the case $\nu = 4$. In this case, there are exactly 11 non-isomorphic connected ν-cyclic graphs of order $n \geq 6$ and maximum degree $n-1$ (these 11 graphs are given in Figure 3 of [2]). However, from these 11 graphs, only two graphs obey the condition (II); one is $H_{n,4}$ and the other one, say $H'_{n,4}$, is the graph deduced from $H_{n,3}$ by inserting an edge between two vertices of degree 2. Here, one has

$$SO(H_{n,4}) = (n-6)\sqrt{(n-1)^2+1} + 4\sqrt{(n-1)^2+4} + \sqrt{(n-1)^2+25} + 4\sqrt{29}$$

and

$$SO(H'_{n,4}) = (n-5)\sqrt{(n-1)^2+1} + \sqrt{(n-1)^2+4} + 2\sqrt{(n-1)^2+9} + \sqrt{(n-1)^2+16} + 2\sqrt{5} + 3\sqrt{2} + 10,$$

and hence

$$SO(H_{n,4}) - SO(H'_{n,4}) > \left[\sqrt{(n-1)^2+9} - \sqrt{(n-1)^2+1}\right] + \left[\sqrt{(n-1)^2+25} - \sqrt{(n-1)^2+16}\right] + \left[4\sqrt{29} - \frac{3}{2} - 2\sqrt{5} - 3\sqrt{2} - 10\right], \tag{5}$$

because $\sqrt{(n-1)^2+4} > \sqrt{(n-1)^2+9} - \frac{1}{2}$ for every $n \geq 6$. On the right hand side of (5), note that the whole expression within every pair of square brackets "[...]" is positive for each $n \geq 6$ and thereby $SO(H_{n,4}) > SO(H'_{n,4})$. □

We end this section with the following natural conjecture arising from Theorem 2.1.

Conjecture 2.1. *If ν and n are fixed integers satisfying the inequality $5 \leq \nu \leq n-2$ then among all connected ν-cyclic graphs of order n, only the graph $H_{n,\nu}$ has the maximum Sombor index.*

Conjecture 2.1 can be easily verified for $\nu = 5$ by using the way adopted in the proof of Theorem 2.1. In [1], all the possible 26 non-isomorphic connected 5-cyclic (pentacyclic) graphs of a fixed order $n \geq 7$ and having maximum degree $n-1$ are given. From these 26 graphs, only three graphs obey the condition (II) mentioned in the proof of Theorem 2.1. Among these three graphs, one is $H_{n,5}$; one is the graph, say $H'_{n,5}$, deduced from $H_{n,4}$ by inserting an edge between two vertices of degree 2; and the remaining one is the graph, say $H''_{n,5}$, deduced from $H'_{n,4}$ by inserting an edge between a vertex of degree 2 and a vertex of degree 3; where the graph $H'_{n,4}$ is defined in the proof of Theorem 2.1. We have verified that $SO(H_{n,5}) > SO(H'_{n,5})$ and $SO(H_{n,5}) > SO(H''_{n,5})$ for $n \geq 7$.

3. Graphs with integer-valued Sombor indices

This section is motivated by remarks in [14] about the equality of degree-radii of two degree-points. Recall that the degree-radius is, in fact, equal to the contribution $\varphi(e) = \sqrt{d_u^2 + d_v^2}$ of an edge $e = uv$ between the vertices u and v with degrees d_u and d_v, respectively. For chemical graphs, i.e., for graphs whose maximum degree Δ does not exceed 4, it immediately leads to the following result.

Proposition 3.1. *Let G be a (connected) chemical graphs without isolated vertices. Then $SO(G)$ is an integer if and only if G is a biregular bipartite graph with degrees $\delta = 3$ and $\Delta = 4$. In that case, $SO(G) = 5|E(G)|$.*

Proof. As all degrees are nonnegative integers, all contributions must be either integers or irrational numbers. If a contribution $\varphi(e)$ is an integer for $e = uv$, then $(d_u, d_v, \varphi(e))$ must form a Pythagorean triple, and $(3, 4, 5)$ is the only such triple whose two smaller entries can be degrees of a chemical graph. □

Let $G = G(\Delta, \delta)$ be a biregular bipartite graph. Graph G is called *semiregular* if no edge connects vertices of the same degree. Hence, a biregular bipartite graph is not necessarily semiregular. Because the complete bipartite graphs are semiregular, it follows that $K_{3,4}$ is the smallest chemical graph with integer Sombor index. Another familiar example is RD, the graph of rhombic dodecahedron - semiregular bipartite graph with 14 vertices (8 of degree 3 and 6 of degree 4), 24 edges and 12 faces, all of them congruent rhombi. It follows immediately from Proposition 3.1 that $SO(K_{3,4}) = 60$ and $SO(RD) = 120$. Another nice example of a semiregular bipartite graph is the rhombic triacontahedron with 32 vertices, 60 edges and 30 rhombic faces. However, its Sombor index is not integer, since its degrees, $\Delta = 5$ and $\delta = 3$, do not form two smaller elements in a Pythagorean triple. Yet, there are infinitely many $(3,4)$-semiregular graphs for which the Sombor index is integer and equal to $5|E(G)|$. Construction of such semiregular graphs is outlined in ref. [22].

We observe that, being semiregular, all such graphs satisfy Proposition 2.2 with equality.

Proposition 3.2. *Let G be a (3,4)-semiregular graph with m edges. Then*

$$SO(G) = \sqrt{mF(G)}.$$

Some of these results can be extended to the case of unrestricted vertex degrees. We start with an auxiliary statement and then formulate the structural result.

Lemma 3.1. *Let G be a connected graph whose Sombor index has an integer value. Then the vertices of the same degree form independent sets in G and the minimum degree of G is at least three.*

Proof. Let $e = uv$ be an edge between two vertices of the same degree $d_u = d_v = d$. Then the contribution $\varphi(e)$ is an irrational number, $\varphi(e) = \sqrt{2}d$, contradiction with the assumed integrality of $SO(G)$. The second claim follows from the fact that 1 and 2 cannot appear in a Pythagorean triple. □

Theorem 3.1. *Let G be a connected graph and let $SO(G)$ be an integer. Then G is a multipartite graph with the smallest degree at least three.*

We are inclined to believe that in that case G must be a bipartite graph, but proving it would bring us beyond the scope of this contribution.

4. Further remarks

In this section we offer some concluding remarks and indicate some possible directions for further work. The most obvious thing to do would be to go beyond the Euclidean norm used by Gutman in definition of $SO(G)$, i.e., to consider the $p-$variant $SO_p(G)$ defined as

$$SO_p(G) = \sum_{uv \in E(G)} (d_u^p + d_u^p)^{1/p}.$$

We call SO_p as the p-Sombor index, where $p \neq 0$. Clearly, SO_1 is the first Zagreb index (see [13]) and SO_2 is the original Sombor index.

For a positive p, the edge contributions $\varphi_p(e) = (d_u^p + d_u^p)^{1/p}$ to $SO_p(G)$ are well known: they appear as the sums $\mathfrak{S}_p(a)$ related to p-means in the classical monograph [16] on inequalities. (Here a stands for the pair (d_u, d_v) of degrees of the end-vertices of $e = uv$.) The next result follows immediately from the monotonicity of $\mathfrak{S}_p(a)$ (see [16], p. 28).

Proposition 4.1. *Let G be a simple graph and $0 < p < q$. Then*

$$SO_q(G) < SO_p(G).$$

The following results is an easy consequence of the Fermat Last Theorem.

Proposition 4.2. *Let G be a simple graph and $p > 2$ an integer. Then $SO_p(G)$ is not an integer.*

Proof. According to the Fermat Last Theorem, no contribution $\varphi_p(e)$ can be an integer for integer values of $p > 2$. Moreover, all contributions must be irrational, since a p-th root of an integer is either integer, or irrational. As the number of contributions is an integer, the claim follows. □

An interesting thing, however, is that the p-Sombor index of any graph is an integer for $p = \infty$. This follows from the fact that $\lim_{p\to\infty} \mathfrak{S}_p(a) = \max(a)$, hence an integer, and the integrality of the number of contributions. $SO_\infty(G)$ can be neatly expressed in terms of the so-called M-polynomial of G.

Let G be a graph. Its M-polynomial is a bivariate polynomial defined as

$$M(G; x, y) = \sum_{i \leq j} m_{i,j}(G) x^i y^j,$$

where $m_{i,j}(G)$ is the number of edges of G whose end-vertices have degrees i and j. We refer the reader to [10] for more information on M-polynomials.

Proposition 4.3. *Let G be a graph. Then*

$$SO_\infty(G) = \frac{\partial}{\partial y} M(G; x, y) \left.\right|_{(1,1)} .$$

For positive values of p between 0 and 1, the following interesting result is obtained for $p = 1/2$; it can be easily verified by direct computation.

Proposition 4.4. *If G is any graph then*

$$SO_{1/2}(G) = M_1(G) + 2RR(G).$$

The limiting behavior of $SO_p(G)$ for $p \to 0$ follows, again, directly from the behavior of $\mathfrak{S}_p(a)$ at the lower end of the range.

Proposition 4.5. *If G is a graph without isolated vertices then*

$$\lim_{p \to 0} SO_p(G) = \infty.$$

The p-Sombor indices are of interest also for negative values of p. For example, for $p = -1$, one obtains a known quantity, the inverse sum indeg index:

$$SO_{-1}(G) = \sum_{e=uv \in E(G)} \frac{d_u d_v}{d_u + d_v} = ISI(G).$$

(See [23] for more information on the inverse sum indeg index.) We leave the systematic exploration of p-Sombor indices for negative values of p to the interested reader.

Another direction could be pursued by getting rid of the adjacency condition and by considering sums of contributions $\sqrt{d_u^2 + d_v^2}$ (or $(d_u^p + d_u^p)^{1/p}$) for all pairs of vertices u, v, whether adjacent or not. This would result in a quantity that could be called the global or total Sombor index. Its comparison with the classical case would indicate how much information on $\sqrt{d_u^2 + d_v^2}$ is captured by the adjacency structure. The difference between the total and the classical Sombor index would correspond to the Sombor coindex, another potentially interesting object of study.

The last possibility we mention here is going out of plane, i.e., extending the basic idea to higher dimensions. For an n-vertex graph G we could characterize each of its vertices by $N \geq 2$ parameters and study the distribution of the corresponding radii $r(u)$ on the surface of one or more k-dimensional spheres. We outline here two possible approaches, and leave further study to the interested reader.

Let $f_k(u), k = 1, 2, \ldots, N$, be topological parameters characterizing quantitatively vertex u of an n-vertex graph G. The N-dimensional edge-based Sombor-like index is defined by

$$SO_E(G) = \sum_{uv \in E(G)} r(u, v),$$

where

$$r(u, v) = \sqrt{[f_1(u) - f_1(v)]^2 + \ldots + [f_N(u) - f_N(v)]^2}.$$

A vertex-based Sombor-like index could be constructed starting from the same $f_k(u), k = 1, 2, \ldots, N$, in the following way. Denote by $\overline{f}_k$ the average value of $f_k(u)$ over all vertices of G, defined as

$$\overline{f}_k = \frac{1}{n} \sum_{u \in V(G)} f_k(u).$$

Now define

$$SO_V(G) = \sum_{u \in V(G)} r(u),$$

where

$$r(u) = \sqrt{\left[f_1(u) - \overline{f}_1\right]^2 + \ldots + \left[f_N(u) - \overline{f}_N\right]^2}.$$

By using different parameters for $f_k(u)$ one can construct various vertex- and edge-based Sombor-like indices quantifying various aspects of relationships between the chosen parameters.

We end this paper by pointing out the recent articles [8, 9, 19, 21] on the Sombor index, which came to our attention, and most of which were published, after the submission of the present paper.

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