Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **2** (2020) 47–54
DOI: 10.47443/cm.2020.0041

Theory of hyper-singular integrals and its application to the Navier-Stokes problem*

Alexander G. Ramm†

Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA

(Received: 19 October 2020. Accepted: 24 December 2020. Published online: 29 December 2020.)

Abstract

In this paper, the convolution integrals $\int_0^t (t-s)^{\lambda-1} b(s)ds$ with hyper-singular kernels are considered, where $\lambda \leq 0$ and either b is a smooth function or b belongs to $L^1(\mathbb{R}_+)$. For such λ, these integrals diverge classically even for smooth b. These convolution integrals are defined in this paper for negative non-integer values of λ. Integral equations and inequalities are considered with the hyper-singular kernels $(t-s)_+^{\lambda-1}$ for $\lambda \leq 0$, where $t_+^\lambda := 0$ for $t < 0$. In particular, one is interested in the value $\lambda = -\frac{1}{4}$ because it is important for the Navier-Stokes problem (NSP). Integral equations of the type $b(t) = b_0(t) + \int_0^t (t-s)^{\lambda-1} b(s)ds$, $\lambda \leq 0$, are also studied. The solution of these equations is investigated, and the existence and uniqueness of the solution is proved for $\lambda = -\frac{1}{4}$. The obtained results are applied to the analysis of the NSP in the space $\mathbb{R}^3$ without boundaries. It is proved that the NSP is contradictory in the following sense: even if one assumes that $v(x,0) > 0$, one proves that the solution $v(x,t)$ to the NSP has the property $v(x,0) = 0$, in general. This paradox shows that the NSP is not a correct description of the fluid mechanics problem and it proves that the NSP does not have a solution, in general.

Keywords: hyper-singular integrals; Navier-Stokes problem.

2020 Mathematics Subject Classification: 45A05, 45H05, 35Q30.

1. Introduction

In this paper, the new definition of the convolution $\int_0^t (t-s)^{\lambda-1} b(s)ds$ with hyper-singular functions is given. We compare this definition with the one, based on the distribution theory [2]. The function $b(s)$ is assumed to be a locally integrable function on $\mathbb{R}_+ := [0,\infty)$. This assumption is satisfied in the Navier-Stokes problem (NSP), see Chapter 5 in [11], where the integral equations of the type $b(t) = b_0(t) + \int_0^t (t-s)^{\lambda-1} b(s)ds$ with $\lambda = -\frac{1}{4}$ are of interest. Classically these integral equations do not make sense because the integrals diverge if $\lambda \leq 0$. In Sections 1–4 of this paper, a new definition of such integrals is given and the solution to the integral equation with hyper-singular kernel is investigated. These results are used in Section 5, where the basic results concerning the NSP are obtained.

We analyze the NSP and prove that the NSP is physically not a correct description of the fluid mechanics problem and that the NSP does not have a solution, in general.

For future use, we define $\Phi_\lambda(t) := \frac{t^{\lambda-1}}{\Gamma(\lambda)}$ and the convolution $\Phi_\lambda \star b := \int_0^t \Phi_\lambda(t-s)b(s)ds$. Also, we define $t := t_+$, that is, $t = 0$ if $t < 0$ and $t = t$ if $t \geq 0$.

Let us give the standard definition of the singular integral used in the distribution theory. Let

$$J := \int_0^\infty t^{\lambda-1}\phi(t)dt, \tag{1}$$

where the test function $\phi(t) \in C_0^\infty(\mathbb{R})$.

Integral (1) diverges classically (that is, in the classical sense) if $\lambda \leq 0$. It is defined in distribution theory (for example, in [2]) as follows:

$$J = \int_0^1 t^{\lambda-1}\phi(t)dt + \int_1^\infty t^{\lambda-1}\phi(t)dt := j_1 + j_2. \tag{2}$$

The integral j_2 converges classically for any complex $\lambda \in \mathbb{C}$ and is analytic with respect to λ. The integral j_1 for $\lambda > 0$ converges classically and can be written as

$$j_1 = \int_0^1 t^{\lambda-1}(\phi(t)-\phi(0))dt + \phi(0)\frac{t^\lambda|_0^1}{\lambda} = \int_0^1 t^{\lambda-1}(\phi(t)-\phi(0))dt + \phi(0)/\lambda. \tag{3}$$

The right side of (3) admits analytic continuation with respect to λ from $\mathrm{Re}\lambda > 0$ to the region $\mathrm{Re}\lambda > -1$. Thus, formulas (2) and (3) together define integral (1) for $\mathrm{Re}\lambda > -1$. The singular integral J has a simple pole at $\lambda = 0$, diverges classically

*A correction notice about this article is available at: https://doi.org/10.47443/cm.2021.n1

†E-mail address: ramm@math.ksu.edu

for $-1 < \lambda < 0$, but is defined in this region by formulas (2) and (3), by analytic continuation with respect to λ. This procedure can be continued and J can be defined for an arbitrary large fixed negative λ, $\lambda \neq 0, -1, -2, \ldots$.

Let us define the convolution

$$I(t) := \Phi_\lambda \star b := \int_0^t \Phi_\lambda(t-s)b(s)ds.$$

We assume that $b(t) \in L^1(\mathbb{R}_+)$ and the Laplace transform of b is defined for Re$p \geq 0$ by the formula

$$L(b) := \int_0^\infty e^{-pt}b(t)dt.$$

Let us define $L(t^{\lambda-1})$ not using the distribution theory. For $\lambda > 0$, one has:

$$L(t^{\lambda-1}) = \int_0^\infty e^{-pt}t^{\lambda-1}dt = \int_0^\infty e^{-s}s^{\lambda-1}dsp^{-\lambda} = \frac{\Gamma(\lambda)}{p^\lambda}. \tag{4}$$

It follows from (4) and from the definition of $\Phi_\lambda = \frac{t^{\lambda-1}}{\Gamma(\lambda)}$ that

$$L(\Phi_\lambda) = p^{-\lambda}. \tag{5}$$

The gamma function $\Gamma(\lambda)$ is analytic in $\lambda \in \mathbb{C}$, except for the simple poles at $\lambda = -n$, with the residue at $\lambda = -n$ equal to $\frac{(-1)^n}{n!}$, where $n = 0, 1, 2, \cdots$. It is known that

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \pi^{1/2}\Gamma(2z).$$

The function $\frac{1}{\Gamma(\lambda)}$ is an entire function of λ. These properties of $\Gamma(z)$ can be found, for example, in [4].

The right side of (4) is analytic with respect to $\lambda \in \mathbb{C}$ except for $\lambda = 0, -1, -2, \cdots$, and therefore, defines $L(t^{\lambda-1})$ for all these λ by analytic continuation with respect to λ without using the distribution theory.

Let us define the convolution $I(t)$ using its Laplace transform

$$L(I(t)) = L(\Phi_\lambda)L(b) = \frac{L(b)}{p^\lambda} \tag{6}$$

and its inverse:

$$I(t) = L^{-1}\big(L(b)p^{-\lambda}\big), \tag{7}$$

where L^{-1} is the inverse of the Laplace transform. Since the null-space of L is trivial, that is, the zero element, the inverse L^{-1} is well defined on the range of L.

For $-1 < \lambda < 0$, in particular for $\lambda = -\frac{1}{4}$, formula (7), can be interpreted as a generalized Fourier integral. The value $\lambda = -\frac{1}{4}$ is very important in NSP, see Chapter 5 of the monograph [11] and Section 5 of this paper. We return to this question later, when we discuss the integral equations with hyper-singular integrals.

Let us now prove the following result that will be used later.

Theorem 1.1. *One has*

$$\Phi_\lambda \star \Phi_\mu = \Phi_{\lambda+\mu}. \tag{8}$$

for any $\lambda, \mu \in \mathbb{C}$. If $\lambda + \mu = 0$ then

$$\Phi_0(t) = \delta(t), \tag{9}$$

where $\delta(t)$ is the Dirac distribution.

Proof. By formulas (5) and (6) with $b(t) = \Phi_\mu(t)$, one gets

$$L(\Phi_\lambda \star \Phi_\mu) = \frac{1}{p^{\lambda+\mu}}.$$

By formula (5), one has

$$L^{-1}\Big(\frac{1}{p^{\lambda+\mu}}\Big) = \Phi_{\lambda+\mu}.$$

This proves formula (8). If $\lambda + \mu = 0$ then

$$p^{-(\lambda+\mu)} = 1, \quad L^{-1}1 = \delta(t).$$

This proves formula (9) and completes the proof of Theorem 1.1.

□

Remark 1.1. *Let us give an alternative proof of formula* (8)*. For* $Re\lambda > 0$*,* $Re\mu > 0$ *one has*

$$\Phi_\lambda \star \Phi_\mu = \frac{1}{\Gamma(\lambda)\Gamma(\mu)} \int_0^t (t-s)^{\lambda-1} s^{\mu-1} ds = \frac{t_+^{\lambda+\mu-1}}{\Gamma(\lambda)\Gamma(\mu)} \int_0^1 (1-u)^{\lambda-1} u^{\mu-1} du = \frac{t_+^{\lambda+\mu-1}}{\Gamma(\lambda+\mu)}, \tag{10}$$

where the right side of (10) *is equal to* $\Phi_{\lambda+\mu}$ *and we have used the known formula for the beta function:*

$$B(\lambda, \mu) := \int_0^1 u^{\lambda-1}(1-u)^{\mu-1} du = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}.$$

Analytic properties of beta function follow from those of the gamma function.

Remark 1.2. *Theorem 1.1 is proved in [2], pp.150–151. Our proof differs from the proof in [2]. It is not clear how the proof in [2] is related to the definition of regularized hyper-singular integrals used in [2].*

2. Preparation for investigation of integral equations with hyper-singular kernels

In this section, we start an investigation of equations of the following type

$$b(t) = b_0(t) + c \int_0^t (t-s)^{\lambda-1} b(s) ds, \tag{11}$$

where b_0 is a smooth function rapidly decaying with all its derivatives as $t \to \infty$ and $b_0(t) = 0$ if $t < 0$. We are especially interested in the value $\lambda = -\frac{1}{4}$, because of its importance for the Navier-Stokes theory, see [5, 10] and Chapter 5 in [11].

The integral in (11) *diverges in the classical sense for* $\lambda \le 0$*. Our aim is to define this hyper-singular integral.*

There is a regularization method to define singular integrals $J := \int_{\mathbb{R}} t_+^{\lambda-1}\phi(t)dt$, $\lambda \le 0$, in the distribution theory, see the Introduction section. The integral in (11) is a convolution, which is defined in [2], p.135, as a *direct product of two distributions*. This definition *is not suitable* for our purposes because $t_+^{\lambda-1}$ for any $\lambda \le 0$, $\lambda \ne 0, -1, -2, \cdots$, is a distribution on the space $\mathcal{K} := C_0^\infty(\mathbb{R}_+)$ of the test functions, but *it is not a distribution in the space of the test functions* $K := C_0^\infty(\mathbb{R})$ *used in [2]*. Indeed, one can find $\phi \in K$ such that $\lim_{n\to\infty} \phi_n = \phi$ in K, but $\lim_{n\to\infty} \int_{\mathbb{R}} t_+^{\lambda-1}\phi_n(t)dt = \infty$ for $\lambda \le 0$, so that $t_+^{\lambda-1}$ is not a linear bounded functional in K, i.e., not a distribution. For example, the integral $\int_0^\infty t^{\lambda-1}\phi(t)dt$ is not a bounded linear functional on K: take a ϕ which is vanishing for $t > 1$, positive near $t = 0$ and non-negative on $[0, 1]$. Then this integral diverges at such ϕ and is not a bounded linear functional on K.

On the other hand, one can check that $t_+^{\lambda-1}$ for any $\lambda \in R$ is a distribution (a bounded linear functional) in the space $\mathcal{K} = C_0^\infty(\mathbb{R}_+)$ with the convergence $\phi_n \to \phi$ in $\mathcal{K}$ defined by the following requirements:

a) the supports of all ϕ_n belong to an interval $[a, b]$, $0 < a \le b < \infty$,

b) $\phi_n^{(j)} \to \phi^{(j)}$ in $C([a, b])$ for all $j = 0, 1, 2, \cdots$.

Indeed, the functional $\int_0^\infty t_+^\lambda \phi(t)dt$ is linear and bounded in $\mathcal{K}$:

$$\left| \int_0^\infty t_+^\lambda \phi_n(t)dt \right| \le (a^\lambda + b^\lambda) \int_a^b |\phi_n(t)|\, dt.$$

A similar estimate holds for all the derivatives of ϕ_n.

Although $t_+^{-\frac{5}{4}}$ *is a distribution in* $\mathcal{K}$*, the convolution*

$$h := \int_0^t (t-s)^{-\frac{5}{4}} b(s) ds := t_+^{-\frac{5}{4}} \star b$$

cannot be defined similarly to the definition in the book [2] because the function $\int_0^\infty \phi(u+s)b(s)ds$ *does not, in general, belong to* $\mathcal{K}$ *even if* $\phi \in \mathcal{K}$*.*

Let us define the convolution h using the Laplace transform (4). Laplace transform of distributions is studied in [1]. There one finds a definition of the Laplace transform of distributions, the Laplace transform of convolutions, tables of the Laplace transforms of distributions, in particular, formula (4) and other information. One has

$$L(t_+^{-\frac{5}{4}} \star b) = L(t_+^{-\frac{5}{4}})L(b).$$

To define $L(t^{\lambda-1})$ for $\lambda \le 0$, note that for $\mathrm{Re}\lambda > 0$ the classical definition (4) holds. The right side of (4) admits analytic continuation to the complex plane of λ, $\lambda \ne 0, -1, -2, ...$ This allows one to define integral (4) for any $\lambda \ne 0, -1, -2, ...$ It is known that $\Gamma(z+1) = z\Gamma(z)$, so

$$\Gamma(-\frac{1}{4}) = -4\Gamma(3/4) := -c_1, \quad c_1 > 0.$$

Therefore, we define h by the formula $h = L^{-1}(Lh)$ and define $L(h)$ as follows:

$$L(h) = -c_1 p^{\frac{1}{4}} L(b), \tag{12}$$

where formula (4) with $\lambda = -\frac{1}{4}$ is used and we assume that b is such that $L(b)$ can be defined. That $L(b)$ is well defined in the Navier-Stokes theory, which follows from the priori estimates proved in Chapter 5 of the book [11] and in Section 5 of this paper. From (12), one gets

$$L(b) = -c_1^{-1} p^{-\frac{1}{4}} L(h).$$

3. Integral equation

Equation (11) can be rewritten as

$$b(t) = b_0(t) - cc_1 \Phi_\lambda \star b, \tag{13}$$

where

$$c_1 = \left|\Gamma\left(-\frac{1}{4}\right)\right| \quad \text{and} \quad \lambda = -\frac{1}{4}.$$

Theorem 3.1. *Equation (13) has a unique solution in $C(0,T)$ for any $T > 0$ if b_0 is sufficiently smooth and rapidly decaying as t grows. This solution can be obtained by iterations:*

$$b_{n+1} = -(cc_1)^{-1}\Phi_{1/4} \star b_n + (cc_1)^{-1}\Phi_{1/4} \star b_0, \quad b_{n=0} = (cc_1)^{-1}\Phi_{1/4} \star b_0, \quad b = \lim_{n\to\infty} b_n.$$

Proof. Applying to Equation (13) the operator $\Phi_{1/4}\star$ and using Equation (9), one gets a Volterra-type equation

$$\Phi_{1/4} \star b = \Phi_{1/4} \star b_0 - cc_1 b,$$

or

$$b = -(cc_1)^{-1}\Phi_{1/4} \star b + (cc_1)^{-1}\Phi_{1/4} \star b_0. \tag{14}$$

The operator $\Phi_\lambda\star$ with $\lambda > 0$ is a Volterra-type operator. Therefore, Equation (14) can be solved for b by iterations, see Lemma 3.1 in this section and Lemmas 5.10 and 5.11 in [11].

If $b_0 \geq 0$ and cc_1 is sufficiently large, then the solution to (13) is non-negative and $b \geq 0$; see Remark 3.1 below.

□

For convenience of the reader let us prove the result about solving Equation (14) by iterations, mentioned above.

Lemma 3.1. *The operator $Af := \int_0^t (t-s)^p f(s)ds$ in the space $X := C(0,T)$ for any fixed $T \in [0,\infty)$ and $p > -1$, has spectral radius $r(A)$ equal to zero. The equation $f = Af + g$ is uniquely solvable in X. Its solution can be obtained by iterations*

$$f_{n+1} = Af_n + g, \quad f_0 = g; \quad \lim_{n\to\infty} f_n = f, \tag{15}$$

for any $g \in X$ and the convergence holds in X.

Proof. The spectral radius of a linear operator A is defined by the formula

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

By induction, one proves that

$$|A^n f| \leq t^{n(p+1)} \frac{\Gamma^n(p+1)}{\Gamma(n(p+1)+1)} \|f\|_X, \quad n \geq 1.$$

From this formula and the known asymptotic of the gamma function, the conclusion $r(A) = 0$ follows. The convergence result (15) is analogous to the well known statement for the assumption $\|A\| < 1$. A more detailed argument can be found in [11], p.53.

□

Remark 3.1. *If $c > 0$ is sufficiently large, then the norm of the operator $B := (cc_1)^{-1}\Phi_{1/4}\star$ in $C(0,T)$ is less than one: $\|B\| < 1$. In this case, $(I - B)^{-1} = \sum_{j=0}^{\infty} B^j$ is a positive operator.*

Let us now give another approach to solving integral Equation (13) with $\lambda = -\frac{1}{4}$.

Theorem 3.2. *The solution to Equation (13) with $\lambda = -\frac{1}{4}$ does exist, is unique, and belongs to $C(\mathbb{R}_+)$ provided that $b_0(t) \in C(\mathbb{R}_+)$ and $|b_0(t)| + |b'(t)| \leq c(1+t)^{-2}$.*

Proof. Take the Laplace transform of Equation (13) with $\lambda = -\frac{1}{4}$ and use formula (5) to get

$$L(b) = L(b_0) - cc_1 p^{1/4} L(b).$$

Thus,

$$L(b) = \frac{L(b_0)}{1 + cc_1 p^{1/4}}$$

Therefore,

$$b(t) = L^{-1}\Big(\frac{L(b_0)}{1 + cc_1 p^{1/4}}\Big).$$

Let us check that

$$\max_{t\geq 0} |b(t)| \leq c.$$

From our assumptions about $b_0(t)$, it follows that $|L(b_0)| \leq c(1+|p|)^{-1}$, Re$p \geq 0$. Let $p = iw$. Since $b(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{iwt} L(b) dw$, one gets

$$|b(t)| \leq \frac{c}{2\pi} \int_{-\infty}^{\infty} (1+|w|)^{-1} |1 + cc_1 (iw)^{1/4}|^{-1} < c_2,$$

where $c_2 > 0$ is some constant. Here we have used the inequality

$$\inf_{w\in\mathbb{R}} |1 + cc_1 (iw)^{1/4}|^{-1} \leq c. \tag{16}$$

Recall that by $c > 0$ various constants are denoted.

Let us check (16) for $w \geq 0$. For $w < 0$, the argument is similar. One has $(iw)^{1/4} = e^{i\pi/8} w^{1/4}$,

$$J := \frac{1}{|1 + C\cos(\pi/8) + iC\sin(\pi/8)|},$$

where $C := cc_1 w^{1/4} > 0$. Therefore,

$$J^{-2} = [1 + C\cos(\pi/8)]^2 + C^2 \sin^2(\pi/8) = 1 + C^2 + 2C\cos(\pi/8) > c > 0.$$

Consequently, inequality (16) is checked. □

Remark 3.2. *It follows from* (14) *that* $b(0) = 0$ *because* $\lim_{t\to 0} \Phi_{\frac{1}{4}} \star b_0 = 0$ *and* $\lim_{t\to 0} \Phi_{\frac{1}{4}} \star b = 0$ *holds if* b *is a locally integrable bounded function on* $\mathbb{R}_+$.

4. Integral inequality

Consider the following inequality

$$q(t) \leq b_0(t) + ct_+^{\lambda-1} \star q = b_0(t) - cc_1 \Phi_{-\frac{1}{4}} q, \tag{17}$$

where $c_1 = -\Gamma(-\frac{1}{4})$ for $\lambda = -\frac{1}{4}$. Let $f = f(t) \in L^1(\mathbb{R}_+)$ be some function. If $q \leq f$ then $\Phi_{1/4} \star q \leq \Phi_{1/4} \star f$. Therefore, inequality (17) with $\lambda = -\frac{1}{4}$, after applying to both sides the operator $\Phi_{1/4}\star$, implies

$$q \leq -(cc_1)^{-1} \Phi_{1/4} \star q + (cc_1)^{-1} \Phi_{1/4} \star b_0. \tag{18}$$

Inequality (18) for sufficiently large $c > 0$ can be solved by iterations with the initial term $(cc_1)^{-1}\Phi_{1/4} \star b_0$, see Remark 3.1. This yields

$$q(t) \leq b(t), \tag{19}$$

where b solves the integral equation (13). This follows from Theorem 4.1 (given below).

Theorem 4.1. *Assume that* b *solves* (13), $c > 0$ *is sufficiently large,* $b_0(t)$ *satisfies conditions stated in Theorem 3.2 and* $q \geq 0$ *solves inequality* (17). *Then inequality* (19) *holds.*

Proof. Denote $z := b - q$, where

$$b = -(cc_1)^{-1} \Phi_{1/4} \star q + (cc_1)^{-1} \Phi_{1/4} \star b_0.$$

Then

$$0 \leq z + (cc_1)^{-1} \Phi_{\frac{1}{4}} \star z.$$

Solving this inequality by iterations and using Remark 3.1, one obtains (19). If $c > 0$ is arbitrary, then this argument yields (19) for sufficiently small $t > 0$ because the norm of the operator $(cc_1)^{-1}\Phi_{1/4}\star$ tends to zero when $t \to 0$. □

Papers [6, 8, 9] also deal with hyper-singular integrals.

5. Application to the Navier-Stokes problem

In this section, we apply the results of Sections 1–4 to the Navier-Stokes problem. Especially the results of Sections 3 and 4 will be used.

The Navier-Stokes problem (NSP) in $\mathbb{R}^3$ is discussed in many books and papers (see Chapter 5 in [11] and references therein). The uniqueness of a solution in $\mathbb{R}^3$ is proved in [3, 5, 11] in different norms. The existence of the solution to the NSP is discussed in [11].

The goal of this section is to prove that the statement of the NSP is contradictory. Therefore, the NSP is not a physically correct statement of the problem of fluid mechanics. We prove that the solution to the NSP does not exist, in general.

What is a physically correct statement of problems of fluid mechanics remains an open problem.

We prove *the paradox in the NSP*. This paradox can be described as follows:

One can have initial velocity $v(x,0) > 0$ in the NSP and, nevertheless, the solution $v(x,t)$ to this NSP must have the zero initial velocity: $v(x,0) = 0$.

This paradox proves that the statement of the NSP is contradictory, that the NSP is not a physically correct statement of the fluid mechanics problem and the solution to the NSP does not exist, in general.

The NSP in $\mathbb{R}^3$ consists of solving the equations

$$v' + (v,\nabla)v = -\nabla p + \nu\Delta v + f, \quad x \in \mathbb{R}^3,\ t \ge 0, \quad \nabla\cdot v = 0, \quad v(x,0) = v_0(x), \tag{20}$$

see, for example, books [3, 11]. Vector-functions velocity $v = v(x,t)$ and the exterior force $f = f(x,t)$ and the scalar function $p = p(x,t)$, the pressure, are assumed to decay as $|x| \to \infty$ and $t \in \mathbb{R}_+ := [0,\infty)$. The derivative with respect to time is denoted $v' := v_t$, $\nu = const > 0$ is the viscosity coefficient, the velocity $v = v(x,t)$ and the pressure $p = p(x,t)$ are unknown, $v_0 = v(x,0)$ and $f(x,t)$ are known. It is assumed that $\nabla\cdot v_0 = 0$. Equations (20) describe viscous incompressible fluid with density $\rho = 1$. Let us assume for simplicity that $f = 0$. This do not change our arguments and our logic. The solution to NSP (20) solves the integral equation:

$$v(x,t) = F - \int_0^t ds \int_{\mathbb{R}^3} G(x-y,t-s)(v,\nabla)v dy, \tag{21}$$

where $(v,\nabla)v = v_j v_{p,j}$, over the repeated indices summation is assumed and $v_{p,j} := \frac{\partial v_p}{\partial x_j}$. Equation (21) implies an integral inequality of the type studied in Sections 3 and 4 (see also [11], Chapter 5). Formula for the tensor $G = G(x,t) = G_{pm}(x,t)$ is derived in [11], p.41:

$$G(x,t) = (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i\xi\cdot x}\Big(\delta_{pm} - \frac{\xi_p\xi_m}{\xi^2}\Big)e^{-\nu\xi^2 t}d\xi. \tag{22}$$

The term $F = F(x,t)$, in our case when $f = 0$, depends only on the data v_0 (see formula (5.42) in [11]):

$$F(x,t) := \int_{\mathbb{R}^3} g(x-y,t)v_0(y)dy,$$

where

$$g(x,t) = \frac{e^{-\frac{|x|^2}{4\nu t}}}{4\nu\pi t}, \quad t > 0; \quad g(x,t) = 0, \quad t \le 0.$$

We assume throughout that $v_0 = v(x,0) > 0$ is such that F is bounded in all the norms we use. Let us use the Fourier transform:

$$\tilde{v} := \tilde{v}(\xi,t) := (2\pi)^{-3}\int_{\mathbb{R}^3} v(x,t)e^{-i\xi\cdot x}dx.$$

Fourier transform equation (21) and get the integral equation:

$$\tilde{v}(\xi,t) = \tilde{F}(\xi,t) - \int_0^t ds\tilde{G}(\xi,t-s)\tilde{v}\bigstar(i\xi\tilde{v}), \tag{23}$$

where $\bigstar$ denotes the convolution in $\mathbb{R}^3$. For brevity we omitted the tensorial indices: instead of $\tilde{G}_{mp}\tilde{v}_j\bigstar(i\xi_j)\tilde{v}_p$, where one sums up over the repeated indices, we wrote $\tilde{G}\tilde{v}\bigstar(i\xi\tilde{v})$. From formula (5.9) in [11], see formula (22), one gets:

$$\tilde{G}(\xi,t) = (2\pi)^{-3}\Big(\delta_{pm} - \frac{\xi_p\xi_m}{\xi^2}\Big)e^{-\nu\xi^2 t}.$$

One has $|\delta_{pm} - \frac{\xi_p\xi_m}{\xi^2}| \le c$. Therefore,

$$|\tilde{G}(\xi,t-s)| \le ce^{-\nu(t-s)\xi^2}.$$

We denote by $c > 0$ *various constants* independent of t and ξ, by $\|\tilde{v}\|$ the norm in $L^2(\mathbb{R}^3)$ and by (v, w) the inner product in $L^2(\mathbb{R}^3)$. Let us introduce the norm

$$\|v\|_1 := \|v\| + \|\nabla v\|.$$

One has

$$(2\pi)^{3/2}\|\tilde{v}\| = \|v\|, \quad (2\pi)^3\||\xi|\tilde{v}\|^2 = \|\nabla v\|^2,$$

by the Parceval equality.

Assumption A. *Assume that $F(x,t)$ is a smooth function rapidly decaying together with all its derivatives. In particular,*

$$\sup_{t\geq 0}\Big((1+t^m)\|F(x,t)\|_1\Big) + \sup_{t\geq 0, \xi\in\mathbb{R}^3}\Big((1+t^m+|\xi|^m)|\tilde{F}(\xi,t)|\Big) < c, \quad m = 1,2,3.$$

Assumption A holds throughout this section and is not repeated. It is known that

$$\sup_{t\geq 0}\left(\|v\| + \int_0^t \|\nabla v\|^2 ds\right) < c, \quad \sup_{t\geq 0}\int_0^t \|\tilde{v}|\xi|\|^2 ds < c, \tag{24}$$

$$\sup_{t\geq 0}(|\xi||\tilde{v}(\xi,t)|) < \infty, \quad |v(\tilde{\xi},t)| \leq c(1+t^{1/2}), \quad \sup_{t\geq 0}\|\nabla v\| < \infty, \tag{25}$$

see [11], p.52.

Theorem 5.1. *Inequalities* (24)–(25) *hold.*

Proof. Proof of this theorem can be found in [11]. Because of the importance of the third inequality in (25) and of its novelty, we give its proof in detail. Let $|\tilde{v}(\xi,t)| := u$, $|\tilde{F}| := \mu(\xi,t) := \mu$. From Equation (23) one gets:

$$u \leq \mu + c\int_0^t e^{-\nu(t-s)\xi^2}\|u\|\||\xi|u\|ds \leq \mu + c\int_0^t e^{-\nu(t-s)\xi^2} b(s)ds, \quad b(s) := \||\xi|u\|, \tag{26}$$

where the Parceval formula

$$(2\pi)^{3/2}\|\tilde{v}\| = \|v\| < c$$

was used. By direct calculation, one derives the following inequality:

$$\|e^{-\nu(t-s)\xi^2}|\xi|\| \leq c(t-s)^{-\frac{5}{4}}.$$

It follows from this inequality and from (26) by multiplying by $|\xi|$ and taking the norm $\|\cdot\|$ of the resulting inequality that the following integral inequality holds:

$$b(t) \leq b_0(t) + c\int_0^t (t-s)^{-\frac{5}{4}} b(s)ds, \tag{27}$$

where

$$b_0(t) := \||\xi|\mu(\xi,t)\|, \quad b(s) := \||\xi|u\|.$$

The function $b_0(t)$ is smooth and rapidly decaying due to Assumption A. Let β solve the following equation:

$$\beta(t) = b_0(t) + c\int_0^t (t-s)^{-\frac{5}{4}}\beta(s)ds. \tag{28}$$

Equation (28) can be written as

$$\beta(t) = b_0(t) - cc_1\Phi_{-\frac{1}{4}} \star \beta, \tag{29}$$

where $\star$ denotes the convolution of two functions on $\mathbb{R}_+$ and $c_1 = |\Gamma(-\frac{1}{4})|$. The convolution on $\mathbb{R}_+$ was defined in the Introduction. In Section 4, the relation between the solutions to integral equation (29) and integral inequality (27) was studied and the inequality $b(t) \leq \beta(t)$ was proved. Taking the Laplace transform of Equation (28) and using Equation (4) with $\lambda = -\frac{1}{4}$, we get

$$L(\Phi_{-\frac{1}{4}} \star \beta) = L(\Phi_{-\frac{1}{4}})L(\beta) = p^{1/4}L(\beta),$$

so

$$L(\beta) = L(b_0) - cc_1 p^{1/4}L(\beta).$$

Therefore,

$$L(\beta) = \frac{L(b_0)}{1+cc_1p^{1/4}}, \quad 0 \leq b(t) \leq \beta(t). \tag{30}$$

It follows from (30) that

$$b(t) \leq \beta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t}\frac{L(b_0)}{1+cc_1(i\tau)^{1/4}}d\tau \leq \frac{1}{\pi}\int_0^{\infty}\frac{|L(b_0)|}{|1+cc_1e^{i\pi/8}\tau^{1/4}|}d\tau \leq c, \tag{31}$$

where the argument p of the function $L(b_0)$ is equal to $i\tau$, $p = i\tau$, and we have used the decay $O(|\tau|^{-1})$ of $|L(b_0)|$ as a function of $p = i\tau$ as $|\tau| \to \infty$. This decay follows from Assumption A and implies that the integrand in (31) belongs to $L^1(\mathbb{R})$ because of the following inequality, proved at the end of Section 3:

$$\inf_{\tau\in[0,\infty)} |1 + cc_1e^{i\pi/8}\tau^{1/4}| > 0, \quad cc_1 > 0.$$

Thus, the third estimate of (25) is proved. □

Theorem 5.2. *The NSP* (20) *does not have a solution, in general.*

Proof. If $v_0(x) = v(x, 0) > 0$, then $b_0(0) > 0$. Apply to Equation (28) the operator $\Phi_{1/4}\star$ and use Theorem 1.1. This yields

$$\Phi_{1/4} \star \beta = \Phi_{1/4} \star b_0 - cc_1\beta(t), \tag{32}$$

where formula (4) was used, $c_1 = -\Gamma(-\frac{1}{4}) > 0$ and $\Phi_{\frac{1}{4}} \star \Phi_{-\frac{1}{4}} = \delta$, where δ is the delta-function, see formulas (8)–(9). We assume that $b_0(t)$ satisfies Assumption A, so it is smooth and rapidly decaying. Then Equation (32) can be solved by iterations by Theorem 3.1 and the solution β is also smooth. Therefore,

$$\lim_{t\to 0}\Phi_{1/4} \star \beta = 0, \quad \lim_{t\to 0}\Phi_{1/4} \star b_0 = 0.$$

Consequently, it follows from (32) that $\beta(0) = 0$. Since $0 \leq b(t) \leq \beta(t)$, one concludes that

$$b(0) = 0.$$

This proves that the NSP problem does not have a solution, in general. Indeed, starting with an initial data which is positive we prove that the corresponding solution to the NSP must have the initial data equal to zero. This is the *NSP paradox*, see [7]. Of course, if the data $v_0(x) = v(x, 0) = 0$ then the solution exists and is equal to zero by the uniqueness theorem, see, for example [5, 11]. Other paradoxes of the theory of fluid mechanics are mentioned in [3].

□

References

[1] Y. Brychkov, A. Prudnikov, *Integral Tranforms of Generalized Functions*, Nauka, Moskow, 1977 (in Russian).
[2] I. Gel'fand, G. Shilov, *Generalized Functions*, Vol. 1, GIFML Press, Moscow, 1959 (in Russian).
[3] O. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Fluid*, Gordon and Breach, New York, 1969.
[4] N. Lebedev, *Special Functions and Their Applications*, Dover, New York, 1972.
[5] A. G. Ramm, Concerning the Navier-Stokes problem, *Open J. Math. Anal.* **4** (2020) 89–92.
[6] A. G. Ramm, Existence of the solutions to convolution equations with distributional kernels, *Global J. Math. Anal.* **6** (2018) 1–2.
[7] A. G. Ramm, Navier-Stokes equations paradox, *Rep. Math. Phys.*, To appear.
[8] A. G. Ramm, On a hyper-singular equation, *Open J. Math. Anal.* **4** (2020) 8–10.
[9] A. G. Ramm, On hyper-singular integrals, *Open J. Math. Anal.* **4** (2020) 101–103.
[10] A. G. Ramm, Solution of the Navier-Stokes problem, *Appl. Math. Lett.* **87** (2019) 160–164.
[11] A. G. Ramm, *Symmetry Problems. The Navier-Stokes Problem*, Morgan & Claypool Publishers, San Rafael, 2019.