Contributions to Mathematics
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Contrib. Math. **2** (2020) 11–15
DOI: 10.47443/cm.2020.0016

An asymptotic formula of a sum function involving gcd and characteristic function of the set of r–free numbers

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(Received: 15 August 2020. Received in revised form: 31 August 2020. Accepted: 3 September 2020. Published online: 5 October 2020.)

Abstract

Let $\gcd(k,j)$ be the greatest common divisor of the positive integers k and j. For any real number $x>1$ and for any fixed positive integers s and r, we give an asymptotic formula of the sum function

$$\sum_{k\le x}\frac{1}{k^{s+1}}\sum_{j=1}^{k} j^s\mu_r\left(\gcd\left(j,k\right)\right),$$

where μ_r is the characteristic function of the set of r-free numbers.

Keywords: gcd-sum function; Dirichlet convolution; sum function.

2020 Mathematics Subject Classification: 11A25, 11N37.

1. Introduction

Let k and j be two positive integers. We denote by $\gcd(k,j)$ the greatest common divisor of the integers k and j. For any two arithmetical functions f and g, let us consider the sum function

$$S_k\left(j\right)=S_{f,g}\left(k,j\right):=\sum_{d\,|\,\gcd(k,j)} f\left(d\right)g\left(k/d\right). \tag{1}$$

The function given in (1) is a generalization of the following sum function

$$S_{f,g}\left(k\right):=\sum_{d\,|\,k} f\left(d\right)g\left(k/d\right)=\left(f*g\right)\left(k\right),$$

where the symbol "$*$" denotes the Dirichlet convolution of arithmetic functions. We remark here that Anderson and Apostol [1] are the first who created this sum function. However, the function $S_k\left(j\right)$ has been studied by several researchers, including Johnson [6], Apostol [2]; and Kiuchi, Minamide and Ueda [9]. In particular, Kiuchi [8] proved the following formula

$$\sum_{k\le x}\frac{1}{k^{s+1}}\sum_{j=1}^{k} j^s S_k\left(j\right)=\frac{1}{2}\sum_{n\le x}\frac{\left(f*g\right)\left(n\right)}{n}+\frac{1}{s+1}\sum_{n\le x}\left(\frac{f}{Id}*g\right)\left(n\right)+\frac{1}{s+1}\sum_{m=1}^{\lfloor s/2\rfloor}\binom{s+1}{2m}B_{2m}\sum_{n\le x}\left(\frac{f}{Id}*\frac{g}{Id_{2m}}\right)\left(n\right) \tag{2}$$

which is valid for any positive integer k and any fixed positive integer s, where B_m is Bernoulli's number, $\lfloor t\rfloor$ is the integer part of t and for any positive integer n the functions Id, Id_m and the unit function 1 are given as $Id(n)=n$, $Id_m(n)=n^m$, for any real number m, and $1(n)=1$. We note that the formula (2) has a lot of interesting applications (see [7]). The sums of the form

$$\sum_{n\le x}\sum_{j=1}^{n} f\left(\gcd\left(j,n\right)\right) \tag{3}$$

have also been studied by many researchers (see [3, 4]). In 2010, O. Bordellès [4] gave an asymptotic formula of (3) under the assumption that $x\ge 1$ is sufficiently large and f is an arithmetic function satisfying certain hypotheses. In this note, an asymptotic formula of the sum function

$$\sum_{k\le x}\frac{1}{k^{s+1}}\sum_{j=1}^{k} j^s\mu_r\left(\gcd\left(j,k\right)\right)$$

is given, where x is any real number greater than 1; s and r are any fixed positive integers; and μ_r is the characteristic function of the set of r-free numbers.

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2. Main result

Let μ_r and f_r be two functions defined by

$$f_r(n) = \begin{cases} \mu(m) & \text{if } n = m^r, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

and

$$\mu_r(n) = \begin{cases} 1 & \text{if } n \text{ is an } r\text{-free number,} \\ 0 & \text{otherwise,} \end{cases}$$

where $r \geq 2$ is a fixed integer and μ is the Möbius function. Denote by $\zeta(s)$ the Riemann zeta-function. The proof of the next lemma is well-known, however, we include it for the sake of completeness.

Lemma 2.1. *For any fixed integer $r \geq 2$, we have*

$$\mu_r = 1 * f_r\,. \tag{5}$$

i.e.,

$$\sum_{d^r \mid n} \mu(d) = \mu_r(n)\,.$$

Proof. The function f_r is clearly multiplicative, so the function μ_r, being the convolution product of two multiplicative functions is also a multiplicative function. Therefore, it suffices to show that

$$\mu_r(p^\alpha) = (1 * f_r)(p^\alpha)$$

for all prime powers p^α. Indeed, one has

$$\begin{aligned} (1 * f_r)(p^\alpha) &= \sum_{\alpha=0}^{r} f_r(p^\alpha) = 1 + \sum_{\alpha=1}^{r} f_r(p^\alpha) \\ &= \begin{cases} 1 & \text{if } \alpha < r \\ 0 & \text{if } \alpha \geq r \end{cases} \\ &= \mu_r(p^\alpha)\,. \end{aligned}$$

□

We use the identity (5) and the formula (2) to give an asymptotic formula of the following sum

$$\sum_{k \leq x} \frac{1}{k^{s+1}} \sum_{j=1}^{k} j^s \mu_r(\gcd(j,k))\,.$$

Now, we can state our main result.

Theorem 2.1. *For any positive real number $x > 1$ and any fixed positive integer s, we have*

$$\sum_{k \leq x} \frac{1}{k^{s+1}} \sum_{j=1}^{k} j^s \mu_r(\gcd(k,j)) = \frac{x}{(s+1)\zeta(2r)} + \frac{\log x}{2\zeta(r)} + L(r;s) - \frac{1}{s+1} \sum_{d^r \leq x} \frac{\mu(d)}{d^r} \psi\left(\frac{x}{d^r}\right) + \mathcal{O}\left(x^{-1+\frac{1}{r}} \log x\right)$$

where

$$L(r;s) = \frac{1}{2(s+1)\zeta(r)} \left((s+1)\left(\gamma - \frac{r\zeta'(r)}{\zeta(r)}\right) - 1 + 2 \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} \binom{s+1}{2m} B_{2m} \zeta(2m) \right).$$

In order to prove Theorem 2.1, we firstly need to prove some lemmas.

Lemma 2.2. *For any real number $x > 1$ and any fixed positive integer s, we have*

$$\sum_{k \leq x} \frac{1}{k^{s+1}} \sum_{j=1}^{k} j^s \mu_r(\gcd(k,j)) = \frac{1}{2} \sum_{n \leq x} \frac{\mu_r(n)}{n} + \frac{1}{s+1} \sum_{dl \leq x} \frac{f_r(d)}{d} + \frac{1}{s+1} \sum_{m=1}^{\lfloor s/2 \rfloor} \binom{s+1}{2m} B_{2m} \sum_{dl \leq x} \frac{f_r(d)}{d} \frac{1}{l^{2m}}. \tag{6}$$

Proof. By using the two formulas (1), (5); and by using the definition (4), we get

$$\begin{aligned} S_k(j) &= S_{f_r,1}(k,j) = \sum_{d \mid \gcd(k,j)} f_r(d) \\ &= \sum_{d^r \mid \gcd(k,j)} \mu(d) \\ &= \mu_r(\gcd(k,j)). \end{aligned}$$

Thus,

$$\sum_{k\le x} \frac{1}{k^{s+1}} \sum_{j=1}^{k} j^s S_{f_r,1}(k,j) = \sum_{k\le x} \frac{1}{k^{s+1}} \sum_{j=1}^{k} j^s \mu_r(\gcd(k,j)).$$

On the other hand, the right side of (6) is a direct application of formula (2) when$f = f_r$ and $g = 1$. □

Lemma 2.3. *For any $x > 1$ and $r \ge 2$, we have*

$$\sum_{n\le x} \frac{\mu_r(n)}{n} = \frac{\log x}{\zeta(r)} + \frac{\gamma}{\zeta(r)} - r\frac{\zeta'(r)}{\zeta^2(r)} + \mathcal{O}\left(x^{-1+\frac{1}{r}} \log x\right). \tag{7}$$

Proof. Using the identity (3) , and the known formula

$$\sum_{n\le x} \frac{1}{n} = \log x + \gamma + \mathcal{O}\left(x^{-1}\right),$$

we have

$$\begin{aligned} \sum_{n\le x} \frac{\mu_r(n)}{n} &= \sum_{n\le x} \left(\frac{f_r}{Id} * \frac{1}{Id}\right)(n) \\ &= \sum_{d\le x} \frac{f_r(d)}{d} \sum_{m\le x/d} \frac{1}{m} \\ &= \sum_{d^r\le x} \frac{\mu(d)}{d^r} \sum_{m\le x/d^r} \frac{1}{m} \\ &= \log x \sum_{d=1}^{\infty} \frac{\mu(d)}{d^r} - r\sum_{d=1}^{\infty} \frac{\mu(d)\log d}{d^r} + \gamma \sum_{d=1}^{\infty} \frac{\mu(d)}{d^r} + A(x), \end{aligned}$$

where

$$A(x) = \log x \sum_{d>x^{1/r}} \frac{\mu(d)}{d^r} - r\sum_{d>x^{1/r}} \frac{\mu(d)\log d}{d^r} + \gamma \sum_{d>x^{1/r}} \frac{\mu(d)}{d^r} + \mathcal{O}\left(\frac{d^r}{x} \sum_{d^r\le x} \frac{\mu(d)}{d^r}\right).$$

By the known identity

$$\frac{1}{\zeta(r)} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^r},$$

where $r > 1$, we have

$$\frac{\zeta'(r)}{\zeta^2(r)} = \sum_{d=1}^{\infty} \frac{\mu(d)\log d}{d^r},$$

and by using the estimate

$$\sum_{n>x} \frac{1}{n^r} = \mathcal{O}\left(x^{-1+r}\right),$$

where $r > 1$, we get

$$\sum_{n\le x} \frac{\mu_r(n)}{n} = \frac{\log x}{\zeta(r)} + \frac{\gamma}{\zeta(r)} - r\frac{\zeta'(r)}{\zeta^2(r)} + \mathcal{O}\left(x^{-1+\frac{1}{r}} \log x\right).$$

□

Lemma 2.4. *For any $x > 1$ and $r \ge 2$, we have*

$$\sum_{dl\le x} \frac{f_r(d)}{d} = \frac{x}{\zeta(2r)} - \sum_{d^r\le x} \frac{\mu(d)}{d^r} \psi\left(\frac{x}{d^r}\right) - \frac{1}{2\zeta(r)} + \mathcal{O}\left(x^{\frac{1}{r}-1}\right). \tag{8}$$

Proof. For $x > 1$ and $r \geq 2$, we have

$$\begin{aligned} \sum_{dl \leq x} \frac{f_r(d)}{d} &= \sum_{d^r l \leq x} \frac{\mu(d)}{d^r} \\ &= \sum_{d^r \leq x} \frac{\mu(d)}{d^r} \sum_{l \leq x/d^r} 1 \\ &= \sum_{d^r \leq x} \frac{\mu(d)}{d^r} \left\lfloor \frac{x}{d^r} \right\rfloor, \end{aligned}$$

Using the fact that

$$\psi(x) = x - \lfloor x \rfloor - \frac{1}{2},$$

we get

$$\begin{aligned} \sum_{dl \leq x} \frac{f_r(d)}{d} &= x \sum_{d^r \leq x} \frac{\mu(d)}{d^{2r}} - \sum_{d^r \leq x} \frac{\mu(d)}{d^r} \psi\left(\frac{x}{d^r}\right) - \frac{1}{2} \sum_{d^r \leq x} \frac{\mu(d)}{d^r} \\ &= \frac{x}{\zeta(2r)} - \sum_{d^r \leq x} \frac{\mu(d)}{d^r} \psi\left(\frac{x}{d^r}\right) - \frac{1}{2\zeta(r)} + \mathcal{O}\left(x^{\frac{1-r}{r}}\right). \end{aligned}$$

□

Lemma 2.5. *For any $x > 1$ and for the two fixed integers $r \geq 2$, $m \geq 0$, we have*

$$\sum_{dl \leq x} \frac{f_r(d)}{d} \frac{1}{l^{2m}} = \frac{\zeta(2m)}{\zeta(r)} + \mathcal{O}\left(x^{-1+1/r}\right) + \mathcal{O}\left(x^{-2m/r+1/r}\right). \tag{9}$$

Proof. For $x > 1$ and $r \geq 2$, we have

$$\begin{aligned} \sum_{dl \leq x} \frac{f_r(d)}{d} \frac{1}{l^{2m}} &= \sum_{d^r l \leq x} \frac{\mu(d)}{d^r} \frac{1}{l^{2m}} \\ &= \sum_{d \leq x^{1/r}} \frac{\mu(d)}{d^r} \sum_{l \leq x^{1/r}/d} \frac{1}{l^{2m}} \\ &= \sum_{d \leq x^{1/r}} \frac{\mu(d)}{d^r} \left(\zeta(2m) + \mathcal{O}\left(\left(\frac{x^{1/r}}{d}\right)^{1-2m}\right)\right) \\ &= \frac{\zeta(2m)}{\zeta(r)} + \mathcal{O}\left(x^{-1+1/r}\right) + \mathcal{O}\left(x^{-2m/r+1/r}\right). \end{aligned}$$

□

Now, we are ready to prove the main result of this note.

Proof of Theorem 2.1. By substituting formulas (7) , (8) , (9) in (6), we obtain

$$\sum_{k \leq x} \frac{1}{k^{s+1}} \sum_{j=1}^{k} j^s \mu_r(\gcd(k,j)) = \frac{x}{(s+1)\zeta(2r)} + \frac{\log x}{2\zeta(r)} + L(r;s) - \frac{1}{s+1} \sum_{d^r \leq x} \frac{\mu(d)}{d^r} \psi\left(\frac{x}{d^r}\right) + \mathcal{O}\left(x^{-1+\frac{1}{r}} \log x\right)$$

where

$$L(r;s) = \frac{1}{2(s+1)\zeta(r)} \left((s+1)\left(\gamma - \frac{r\zeta'(r)}{\zeta(r)}\right) - 1 + 2\sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} \binom{s+1}{2m} B_{2m} \zeta(2m)\right).$$

Since for all $t \in \mathbb{R}$, it holds that $|\psi(t)| \leq \frac{1}{2}$, we see that the absolute value of the ψ-sum is not greater than $\frac{\zeta(r)}{4\zeta(2r)}$ and

$$\frac{\zeta(r)}{4\zeta(2r)} \leq \frac{5}{4\pi^2} < \frac{2}{5}.$$

Also, using Euler's formula

$$\zeta(2m) = (-1)^{m+1} 2^{2m-1} \frac{\pi^{2m}}{(2m)!} B_{2m}$$

we write

$$2\sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} \binom{s+1}{2m} B_{2m} \zeta(2m) = \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} \binom{s+1}{2m} (-1)^{m+1} 2^{2m} \frac{\pi^{2m}}{(2m)!} B_{2m}^2.$$

□

Acknowledgments

We thank the anonymous referees for reading this paper carefully and for their valuable suggestions and corrections, which led to a number of improvements in the paper. We also express our gratitude to Olivier Bordellès for his helpful comments on the earlier version of this paper.

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