An asymptotic formula of a sum function involving gcd and characteristic function of the set of $r$–free numbers

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(Received: 15 August 2020. Received in revised form: 31 August 2020. Accepted: 3 September 2020. Published online: 5 October 2020.)

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Abstract

Let $\text{gcd}(k, j)$ be the greatest common divisor of the positive integers $k$ and $j$. For any real number $x > 1$ and for any fixed positive integers $s$ and $r$, we give an asymptotic formula of the sum function

$$\sum_{k \leq x} \frac{1}{k+s+1} \sum_{j=1}^{k} j^s \mu_r(\text{gcd}(j, k)),$$

where $\mu_r$ is the characteristic function of the set of $r$–free numbers.

Keywords: gcd-sum function; Dirichlet convolution; sum function.

2020 Mathematics Subject Classification: 11A25, 11N37.

1. Introduction

Let $k$ and $j$ be two positive integers. We denote by $\text{gcd}(k, j)$ the greatest common divisor of the integers $k$ and $j$. For any two arithmetical functions $f$ and $g$, let us consider the sum function

$$S_k(j) = S_{f,g}(k, j) := \sum_{d | \text{gcd}(k,j)} f(d) g(k/d).$$

The function given in (1) is a generalization of the following sum function

$$S_{f,g}(k) := \sum_{d | k} f(d) g(k/d) = (f * g)(k),$$

where the symbol "∗" denotes the Dirichlet convolution of arithmetic functions. We remark here that Anderson and Apostol [1] are the first who created this sum function. However, the function $S_k(j)$ has been studied by several researchers, including Johnson [6], Apostol [2]; and Kiuchi, Minamide and Ueda [9]. In particular, Kiuchi [8] proved the following formula

$$\sum_{k \leq x} \frac{1}{k+s+1} \sum_{j=1}^{k} j^s S_k(j) = \frac{1}{2} \sum_{n \leq x} \frac{(f * g)(n)}{n} + \frac{1}{s+1} \sum_{n \leq x} \left( \frac{f}{Id} * g \right)(n) + \frac{1}{s+1} \sum_{m=1}^{\lfloor s/2 \rfloor} \left( \frac{s+1}{2m} \right) B_{2m} \sum_{n \leq x} \left( \frac{f}{Id} * \frac{g}{Id_{2m}} \right)(n)$$

which is valid for any positive integer $k$ and any fixed positive integer $s$, where $B_m$ is Bernoulli’s number, $\lfloor t \rfloor$ is the integer part of $t$ and for any positive integer $n$ the functions $Id, Id_m$ and the unit function $1$ are given as $Id(n) = n, Id_m(n) = n^m$, for any real number $m$, and $1(n) = 1$. We note that the formula (2) has a lot of interesting applications (see [7]). The sums of the form

$$\sum_{n \leq x} \sum_{j=1}^{n} f(\text{gcd}(j, n))$$

have also been studied by many researchers (see [3, 4]). In 2010, O. Bordellès [4] gave an asymptotic formula of (3) under the assumption that $x \geq 1$ is sufficiently large and $f$ is an arithmetic function satisfying certain hypotheses. In this note, an asymptotic formula of the sum function

$$\sum_{k \leq x} \frac{1}{k+s+1} \sum_{j=1}^{k} j^s \mu_r(\text{gcd}(j, k))$$

is given, where $x$ is any real number greater than 1; $s$ and $r$ are any fixed positive integers; and $\mu_r$ is the characteristic function of the set of $r$–free numbers.

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2. Main result

Let $\mu_r$ and $f_r$ be two functions defined by

$$f_r(n) = \begin{cases} 
\mu(m) & \text{if } n = m^r, \\
0 & \text{otherwise,}
\end{cases}$$

and

$$\mu_r(n) = \begin{cases} 
1 & \text{if } n \text{ is an } r\text{-free number,} \\
0 & \text{otherwise,}
\end{cases}$$

where $r \geq 2$ is a fixed integer and $\mu$ is the M"obius function. Denote by $\zeta(s)$ the Riemann zeta-function. The proof of the next lemma is well-known, however, we include it for the sake of completeness.

Lemma 2.1. For any fixed integer $r \geq 2$, we have

$$\mu_r = 1 \ast f_r.$$  \hfill (5)

i.e.,

$$\sum_{d^r | n} \mu(d) = \mu_r(n).$$

Proof. The function $f_r$ is clearly multiplicative, so the function $\mu_r$, being the convolution product of two multiplicative functions is also a multiplicative function. Therefore, it suffices to show that

$$\mu_r(p^\alpha) = (1 \ast f_r)(p^\alpha)$$

for all prime powers $p^\alpha$. Indeed, one has

$$(1 \ast f_r)(p^\alpha) = \sum_{\alpha = 0}^{r} f_r(p^\alpha) = 1 + \sum_{\alpha = 1}^{r} f_r(p^\alpha) = \begin{cases} 
1 & \text{if } \alpha < r \\
0 & \text{if } \alpha \geq r
\end{cases}
= \mu_r(p^\alpha).$$

We use the identity (5) and the formula (2) to give an asymptotic formula of the following sum

$$\sum_{k \leq x} \frac{1}{k^{s+1}} \sum_{j=1}^{k} j^s \mu_r(\gcd(j,k)).$$

Now, we can state our main result.

Theorem 2.1. For any positive real number $x > 1$ and any fixed positive integer $s$, we have

$$\sum_{k \leq x} \frac{1}{k^{s+1}} \sum_{j=1}^{k} j^s \mu_r(\gcd(j,k)) = \frac{x}{(s+1)\zeta(2r)} + \frac{\log x}{2\zeta(r)} + L(r; s) - \frac{1}{s+1} \sum_{d^r \leq x} \mu(d) \psi \left( \frac{x}{d^r} \right) + O \left( x^{-1+\frac{1}{r}} \log x \right)$$

where

$$L(r; s) = \frac{1}{2(s+1)\zeta(r)} \left( (s+1) \left( \gamma - \frac{\zeta'(r)}{\zeta(r)} \right) - 1 + 2 \sum_{m=1}^{\lfloor s/2 \rfloor} \left( \frac{s+1}{2m} \right) B_{2m} \zeta(2m) \right).$$

In order to prove Theorem 2.1, we firstly need to prove some lemmas.

Lemma 2.2. For any real number $x > 1$ and any fixed positive integer $s$, we have

$$\sum_{k \leq x} \frac{1}{k^{s+1}} \sum_{j=1}^{k} j^s \mu_r(\gcd(j,k)) = \frac{1}{2} \sum_{n \leq x} \frac{\mu_r(n)}{n} + \frac{1}{s+1} \sum_{d \leq x} \frac{f_r(d)}{d} + \frac{1}{s+1} \sum_{m=1}^{\lfloor s/2 \rfloor} \left( \frac{s+1}{2m} \right) B_{2m} \sum_{d \leq x} \frac{f_r(d)}{d} \frac{1}{2m}.$$
Proof. By using the two formulas (1), (5); and by using the definition (4), we get

\[ S_k (j) = S_{f_r, 1} (k, j) = \sum_{d \mid \gcd(\kappa, j)} f_r (d) \]

\[ = \sum_{d' \mid \gcd(\kappa, j)} \mu (d) \]

\[ = \mu_r (\gcd(\kappa, j)) . \]

Thus,

\[ \sum_{k \leq x} \frac{1}{k+1} \sum_{j=1}^{k} j^{s} S_{f_r, 1} (k, j) = \sum_{k \leq x} \frac{1}{k+1} \sum_{j=1}^{k} j^{s} \mu_r (\gcd(\kappa, j)) . \]

On the other hand, the right side of (6) is a direct application of formula (2) when \( f = f_r \) and \( g = 1 \).

Lemma 2.3. For any \( x > 1 \) and \( r \geq 2 \), we have

\[ \sum_{n \leq x} \frac{\mu_r (n)}{n} = \log x + \gamma - \frac{\zeta' (1)}{\zeta^2 (1)} + O \left( x^{-1 + \frac{1}{r}} \log x \right) . \]  

(7)

Proof. Using the identity (3), and the known formula

\[ \sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O \left( x^{-1} \right) , \]

we have

\[ \sum_{n \leq x} \frac{\mu_r (n)}{n} = \sum_{n \leq x} \left( f_r \frac{1}{d} * \frac{1}{d} \right) (n) \]

\[ = \sum_{d \leq x} \frac{f_r (d)}{d} \sum_{m \leq \frac{x}{d}} \frac{1}{m} \]

\[ = \sum_{d' \leq x} \frac{\mu (d)}{d'} \sum_{m \leq \frac{x}{d'}} \frac{1}{m} \]

\[ = \log x \sum_{d=1}^{\infty} \frac{\mu (d)}{d^r} - r \sum_{d=1}^{\infty} \frac{\mu (d) \log d}{d^r} + \gamma \sum_{d=1}^{\infty} \frac{\mu (d)}{d^r} + O (x) , \]

where

\[ A (x) = \log x \sum_{d > x^{1/r}} \frac{\mu (d)}{d^r} - r \sum_{d > x^{1/r}} \frac{\mu (d) \log d}{d^r} + \gamma \sum_{d=1}^{\infty} \frac{\mu (d)}{d^r} + O \left( \frac{d^r}{x} \sum_{d' \leq x} \frac{\mu (d)}{d^r} \right) . \]

By the known identity

\[ \frac{1}{\zeta (r)} = \sum_{d=1}^{\infty} \frac{\mu (d)}{d^r} , \]

where \( r > 1 \), we have

\[ \frac{\zeta' (r)}{\zeta^2 (r)} = \sum_{d=1}^{\infty} \frac{\mu (d) \log d}{d^r} , \]

and by using the estimate

\[ \sum_{n>x} \frac{1}{n^r} = O \left( x^{-1+r} \right) , \]

where \( r > 1 \), we get

\[ \sum_{n \leq x} \frac{\mu_r (n)}{n} = \log x \frac{\zeta (1)}{\zeta (2)} + \gamma \frac{\zeta' (1)}{\zeta^2 (1)} - r \frac{\zeta' (1)}{\zeta^2 (1)} + O \left( x^{-1+\frac{1}{r}} \log x \right) . \]

Lemma 2.4. For any \( x > 1 \) and \( r \geq 2 \), we have

\[ \sum_{d \leq x} \frac{f_r (d)}{d} = \frac{x}{\zeta (2r)} - \sum_{d' \leq x} \frac{\mu (d)}{d^r} \psi \left( \frac{x}{d^r} \right) - \frac{1}{2 \zeta (r)} + O \left( x^{\frac{1}{r}-1} \right) . \]

(8)
Proof. For \( x > 1 \) and \( r \geq 2 \), we have
\[
\sum_{d \leq x} f_r(d) = \sum_{d' \leq x} \left( \sum_{d \mid d'} \frac{\mu(d)}{d^r} \right) \sum_{(s \leq x/d^r)} \frac{\mu(d)}{d^r} \sum_{t \leq x} 1
\]
Using the fact that
\[
\psi(x) = x - \lfloor x \rfloor - \frac{1}{2},
\]
we get
\[
\sum_{d \leq x} f_r(d) = x \sum_{d' \leq x} \frac{\mu(d)}{d^r} - x \sum_{d' \leq x} \frac{\mu(d)}{d^r} \psi \left( \frac{x}{d^r} \right) - \frac{1}{2} \sum_{d' \leq x} \frac{\mu(d)}{d^r} = \frac{x}{\zeta(2r)} - \sum_{d' \leq x} \frac{\mu(d)}{d^r} \psi \left( \frac{x}{d^r} \right) - \frac{1}{2} \zeta(2r) \psi \left( \frac{x}{d^r} \right) + O \left( \frac{x^{1-\varepsilon}}{d^r} \right).
\]

Lemma 2.5. For any \( x > 1 \) and for the two fixed integers \( r \geq 2, m \geq 0 \), we have
\[
\sum_{d \leq x} f_r(d) \frac{1}{d^{2m}} = \zeta(2m) \zeta(r) + O \left( x^{-1+1/r} \right) + O \left( x^{-2m/r+1/r} \right).
\]

Proof. For \( x > 1 \) and \( r \geq 2 \), we have
\[
\sum_{d \leq x} f_r(d) \frac{1}{d^{2m}} = \sum_{d' \leq x} \frac{\mu(d)}{d^r} \frac{1}{d^{2m}}
\]
Using the fact that
\[
\psi(x) = x - \lfloor x \rfloor - \frac{1}{2},
\]
we get
\[
\sum_{d \leq x} f_r(d) \frac{1}{d^{2m}} = x \sum_{d' \leq x} \frac{\mu(d)}{d^r} \sum_{1 \leq t \leq x} \frac{1}{d^{2m}} = \frac{x}{\zeta(2r)} \sum_{d' \leq x} \frac{\mu(d)}{d^r} \left( \zeta(2m) + O \left( \frac{x^{1/r}}{d^{1-2m}} \right) \right) = \frac{\zeta(2m)}{\zeta(r)} + O \left( x^{-1+1/r} \right) + O \left( x^{-2m/r+1/r} \right).
\]

Now, we are ready to prove the main result of this note.

Proof of Theorem 2.1. By substituting formulas (7), (8), (9) in (6), we obtain
\[
\sum_{k \leq x} \frac{1}{k^{s+1}} \sum_{j=1}^{k} j^s \mu_r(\gcd(k, j)) = \frac{x}{(s+1) \zeta(2r)} + \frac{\log x}{2 \zeta(r)} + L(r; s) - \frac{1}{s+1} \sum_{d' \leq x} \frac{\mu(d)}{d^r} \psi \left( \frac{x}{d^r} \right) + O \left( x^{-1+\frac{1}{2} \log x} \right)
\]
where
\[
L(r; s) = \frac{1}{2(s+1) \zeta(r)} \left( (s+1) \left( \gamma - \frac{r \zeta'(r)}{\zeta(r)} \right) - 1 + \sum_{m=1}^{\lfloor s \rfloor} \left( \frac{s+1}{2m} \right) B_{2m} \zeta(2m) \right).
\]
Since for all \( t \in \mathbb{R} \), it holds that \( |\psi(t)| \leq \frac{1}{2} \), we see that the absolute value of the \( \psi \)-sum is not greater than \( \frac{\zeta(r)}{4 \zeta(2r)} \) and
\[
\frac{\zeta(r)}{4 \zeta(2r)} \leq \frac{5}{4 \pi^2} < \frac{2}{5}.
\]
Also, using Euler’s formula
\[
\zeta(2m) = (-1)^{m+1} 2^{2m-1} \frac{\pi^{2m}}{(2m)!} B_{2m}
\]
we write
\[
2 \sum_{m=1}^{\lfloor s \rfloor} \left( \frac{s+1}{2m} \right) B_{2m} \zeta(2m) = \sum_{m=1}^{\lfloor s \rfloor} \left( \frac{s+1}{2m} \right) (-1)^{m+1} 2^{2m-1} \frac{\pi^{2m}}{(2m)!} B_{2m}.
\]
Acknowledgments

We thank the anonymous referees for reading this paper carefully and for their valuable suggestions and corrections, which led to a number of improvements in the paper. We also express our gratitude to Olivier Bordellès for his helpful comments on the earlier version of this paper.

References