Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **14** (2026) 40–48
DOI: 10.47443/cm.2026.085

Research Article

Spectral Gaps and Reinforcement Principles for Nonlocal Dynamical Boundary Conditions

Pablo M. Berná [*,†]

Facultad de Empresa Digital, Tecnología y Derecho, Universidad Tecnológica Atlántico Mediterráneo–UTAMED, Málaga, Spain

(Received: 4 October 2026. Received in revised form: 14 November 2026. Accepted: 19 November 2026. Published online: 23 November 2026.)

© 2026 the author. This is an open-access article under the CC BY (International 4.0) license (www.creativecommons.org/licenses/by/4.0/).

Abstract

The linear nonlocal dynamical boundary problem introduced by Berná and Rossi [*Nonlinear Anal.* **195** (2020) #111751] is studied through its reduced boundary energy. Eliminating the stationary interior variable gives a bounded nonnegative self-adjoint Dirichlet-to-Neumann operator on the boundary strip. This model-specific reduction identifies the optimized nonlocal Poincaré constant with the bottom of the spectrum on the mean-zero subspace; the spectral theorem then gives the optimal uniform exponential relaxation rate. Kernel reinforcement is interpreted as an application of the classical superadditivity inequality for shorted operators, and the corresponding comparison of reduced spectral gaps is derived. Direct diffusion within the strip provides another nonnegative contribution to the boundary operator. Under concrete smooth-domain and kernel hypotheses, this full-interaction model has a positive gap even at critical strip width. For the original model, an explicit normalized concentration construction is given, showing that the gap closes when the critical geometry contains suitably weakly coupled strip regions. The equality of strip width and interaction radius alone is not asserted to imply gap closure.

Keywords: nonlocal diffusion; dynamical boundary conditions; Dirichlet-to-Neumann operator; spectral gap; exponential stability.

2020 Mathematics Subject Classification: 47D06, 47A10, 45K05.

1. Introduction

Dynamical boundary conditions provide a natural way of coupling an elliptic equilibrium in the interior of a domain with an evolution taking place on its boundary. In the classical linear setting, the interior harmonic extension can be eliminated and the resulting evolution is generated by the Dirichlet-to-Neumann operator. This elementary observation puts the long-time behaviour of the boundary dynamics into a spectral framework.

A nonlocal counterpart of this problem was introduced by Berná and Rossi [4]. In their model, a bounded domain $\widehat{\Omega} \subset \mathbb{R}^n$ is decomposed into an interior region Ω_r and a boundary strip Γ_r. Particles in Ω_r equilibrate instantaneously, whereas the density on Γ_r evolves through interactions with the interior. Existence, uniqueness, conservation of mass and convergence to equilibrium were established for smooth kernels, nonlinear p-Laplacian-type interactions and singular kernels. In the linear smooth-kernel case, exponential convergence follows from a nonlocal Poincaré inequality when the width of the strip is strictly smaller than the interaction radius.

The purpose of this paper is to isolate the Hilbertian structure hidden in that argument. For the linear problem, the stationary equation in Ω_r defines a nonlocal harmonic extension. The flux generated by this extension gives a nonlocal Dirichlet-to-Neumann operator Λ_J on $L^2(\Gamma_r)$. We prove directly that Λ_J is bounded, self-adjoint and nonnegative, and that

$$\langle \Lambda_J g, g\rangle_{L^2(\Gamma_r)} = \frac{1}{2}\iint_H J(x-y)|\widehat{g}(y)-\widehat{g}(x)|^2\,dy\,dx,$$

where $\widehat{g}$ denotes the nonlocal harmonic extension of g and $H = ((\Omega_r \cup \Gamma_r)\times(\Omega_r\cup\Gamma_r))\setminus(\Gamma_r\times\Gamma_r)$. Thus the dynamical problem is simply $u_t + \Lambda_J u = 0$. For this model, the reduction gives an exact spectral interpretation of the optimized Poincaré constant. On

$$L^2_0(\Gamma_r) := \left\{ g \in L^2(\Gamma_r) : \int_{\Gamma_r} g = 0 \right\},$$

we define

$$\lambda_* := \inf_{0\neq g\in L^2_0(\Gamma_r)} \frac{\langle \Lambda_J g, g\rangle}{\|g\|_2^2}.$$

*E-mail address: pablomanuel.berna@utamed.es
†https://orcid.org/0000-0001-7685-0886

The standard variational characterization of the spectrum gives

$$\lambda_* = \inf \sigma \left(\Lambda_J|_{L^2_0(\Gamma_r)} \right).$$

Under the hypotheses of the nonlocal Poincaré estimate in [4, Theorem 2.5], in particular $r < R$, this number is positive. Once the reduced operator has been shown to be nonnegative and self-adjoint, the spectral theorem immediately yields

$$\|e^{-t\Lambda_J}|_{L^2_0(\Gamma_r)}\| = e^{-t\lambda_*}.$$

We use this standard identity to interpret the best uniform relaxation rate for the present dynamical boundary problem. No compactness or existence of a first positive eigenfunction is required.

This distinction is useful because the bottom of the spectrum of a bounded nonlocal operator need not be an isolated eigenvalue. At critical width $r = R$, the concentration mechanism in [4, Example 2.4] can close the gap when suitable weakly coupled strip regions are present. We specify these geometric requirements and the trial extensions explicitly in Section 4. The equality $r = R$ by itself is not used as a gap-closing criterion.

Nonlocal boundary operators also occur in fractional Neumann and Steklov-type problems [6, 7]. Nonlocal Douglas identities connect extension energies with boundary forms; see [5]. Nonlinear dynamical boundary problems on metric random walk spaces are treated in [10]. Recent related developments include Green identities and regularity for finite-range nonlocal problems with local boundary conditions [9], and scaling and local convergence for nonlocal Neumann operators with integrable kernels [8]. These works concern different boundary formulations but provide relevant context for nonlocal boundary energies and fluxes.

The operator-theoretic background of our comparison argument is the theory of shorted operators developed by Anderson [1] and Anderson–Trapp [2]. In particular, superadditivity under elimination of the interior variable is a classical shorting inequality. The contribution here is its application to the reduced energy and relaxation rate of the Berná–Rossi boundary-strip model, together with a precise treatment of strip diffusion and critical weak coupling. We use these classical operator-theoretic principles to organize the spectral and variational analysis of this model.

The paper is organized as follows. Section 2 introduces the harmonic extension and the reduced boundary operator. Section 3 establishes its energy and spectral properties, including a comparison principle under reinforcement of the kernel. Section 4 applies them to the evolution problem, compares the original dynamics with the variant containing direct strip-to-strip interactions, and discusses the critical geometry. Section 5 presents the concluding remarks.

2. The Nonlocal Dirichlet-to-Neumann Operator

Let $\widehat{\Omega} \subset \mathbb{R}^n$ be a bounded connected smooth domain and let

$$\Gamma_r = \{x \in \widehat{\Omega} : \operatorname{dist}(x, \partial\widehat{\Omega}) \le r\} \quad \text{and} \quad \Omega_r = \widehat{\Omega} \setminus \Gamma_r.$$

We assume that $J : \mathbb{R}^n \to [0, \infty)$ is continuous, radial, nonincreasing in the radial variable and symmetric, that

$$\int_{\mathbb{R}^n} J(z)\, dz = 1, \quad \operatorname{supp} J = \overline{B_R(0)},$$

and $0 < r \le R$, with $|\Omega_r| > 0$ and $|\Gamma_r| > 0$. We make the connectivity assumption precise for the interactions actually retained in the model. Put

$$H = (\widehat{\Omega} \times \widehat{\Omega}) \setminus (\Gamma_r \times \Gamma_r) \quad \text{and} \quad W_J(x, y) = J(x - y)\mathbf{1}_H(x, y).$$

We call this interaction configuration *connected* if, for every measurable $F \subset \widehat{\Omega}$ with $0 < |F| < |\widehat{\Omega}|$,

$$\int_F \int_{\widehat{\Omega}\setminus F} W_J(x, y)\, dy\, dx > 0. \tag{1}$$

This is a condition on J, Ω_r and Γ_r; it excludes a positive-measure component separated from the other retained interactions. Equivalently, the only $L^2(\widehat{\Omega})$ functions with zero W_J-energy are constants: if a zero-energy function were not constant, a level set with both it and its complement of positive measure would contradict (1). We assume (1) throughout for the original kernel J.

The interior Dirichlet estimate, which is distinct from a positive reduced boundary gap, is

$$\frac{1}{2}\iint_H J(x-y)|\widetilde{w}(y)-\widetilde{w}(x)|^2\,dy\,dx \geq c_D\|w\|^2_{L^2(\Omega_r)}, \qquad w\in L^2(\Omega_r), \tag{2}$$

where $\widetilde{w}=w$ on Ω_r and $\widetilde{w}=0$ on Γ_r. Under the stated smooth connected-domain and kernel hypotheses this is the standard zero-Dirichlet nonlocal Poincaré estimate used in [4, Lemma 2.2]; see also [3]. Indeed, J is bounded below by a positive constant on some ball around zero, and the full-domain nonlocal Poincaré inequality, together with the positive-measure zero set Γ_r, gives (2). Strip-to-strip terms vanish for $\widetilde{w}$, so the full and truncated energies agree here.

Given $g\in L^2(\Gamma_r)$, let $Tg\in L^2(\Omega_r)$ be the unique solution of

$$0=\int_{\Omega_r\cup\Gamma_r} J(x-y)\big(\widehat{g}(y)-\widehat{g}(x)\big)\,dy, \qquad x\in\Omega_r, \tag{3}$$

where

$$\widehat{g}(x)=\begin{cases} Tg(x), & x\in\Omega_r,\\ g(x), & x\in\Gamma_r.\end{cases}$$

Existence, uniqueness and the estimate

$$\|Tg\|_{L^2(\Omega_r)}\leq C\|g\|_{L^2(\Gamma_r)} \tag{4}$$

follow from the variational argument for the stationary nonlocal Dirichlet problem; see [4, Lemma 2.2]. Since (3) is linear and its solution is unique, T is linear.

Definition 2.1. *For $g\in L^2(\Gamma_r)$ we define the nonlocal Dirichlet-to-Neumann operator by*

$$(\Lambda_J g)(x):=-\int_{\Omega_r} J(x-y)\big(Tg(y)-g(x)\big)\,dy, \qquad x\in\Gamma_r. \tag{5}$$

The sign is chosen so that Λ_J is nonnegative and the evolution takes the form $u_t+\Lambda_J u=0$.

Proposition 2.1. *The operator $\Lambda_J: L^2(\Gamma_r)\to L^2(\Gamma_r)$ is linear and bounded. Moreover, $\Lambda_J\mathbf{1}=0$.*

Proof. Linearity follows from that of T. Writing

$$\Lambda_J g(x)=a(x)g(x)-\int_{\Omega_r} J(x-y)Tg(y)\,dy, \qquad a(x)=\int_{\Omega_r} J(x-y)\,dy,$$

the first term is bounded on $L^2(\Gamma_r)$ because $0\leq a\leq 1$. For the second one, Jensen's inequality, Fubini's theorem and (4) give

$$\int_{\Gamma_r}\left|\int_{\Omega_r} J(x-y)Tg(y)\,dy\right|^2 dx\leq C\|Tg\|^2_{L^2(\Omega_r)}\leq C'\|g\|^2_{L^2(\Gamma_r)}.$$

Finally, the harmonic extension of the constant function 1 is the constant function 1, hence (5) gives $\Lambda_J\mathbf{1}=0$. □

For $v,w\in L^2(\Omega_r\cup\Gamma_r)$ set

$$\mathcal{E}_J(v,w):=\frac{1}{2}\iint_H J(x-y)(v(y)-v(x))(w(y)-w(x))\,dy\,dx, \tag{6}$$

with

$$H=((\Omega_r\cup\Gamma_r)\times(\Omega_r\cup\Gamma_r))\setminus(\Gamma_r\times\Gamma_r).$$

We write $\mathcal{E}_J(v)=\mathcal{E}_J(v,v)$.

Lemma 2.1 (Dirichlet principle)**.** *Let $g\in L^2(\Gamma_r)$ and let $\widehat{g}$ be its J-harmonic extension. For every $v\in L^2(\Omega_r\cup\Gamma_r)$ satisfying $v=g$ on Γ_r,*

$$\mathcal{E}_J(v)=\mathcal{E}_J(\widehat{g})+\mathcal{E}_J(v-\widehat{g}). \tag{7}$$

Consequently,

$$\mathcal{E}_J(\widehat{g})=\min\{\mathcal{E}_J(v): v|_{\Gamma_r}=g\}.$$

The minimizer is unique by (2)*.*

Proof. Put $w = v - \widehat{g}$. Then $w = 0$ on Γ_r. Expanding the quadratic form gives

$$\mathcal{E}_J(v) = \mathcal{E}_J(\widehat{g}) + 2\mathcal{E}_J(\widehat{g}, w) + \mathcal{E}_J(w).$$

Using symmetry of J, the fact that w vanishes on Γ_r, and the same integration-by-parts calculation as in Lemma 2.2 below,

$$\mathcal{E}_J(\widehat{g}, w) = -\int_{\Omega_r} w(x) \int_{\Omega_r \cup \Gamma_r} J(x-y)(\widehat{g}(y) - \widehat{g}(x))\, dy\, dx = 0$$

by the harmonic equation. This proves (7) and hence minimality. If equality holds for another minimizer, then $\mathcal{E}_J(w) = 0$. Since $w = 0$ on Γ_r, (2) implies $w = 0$ almost everywhere. □

Lemma 2.2 (Green identity)**.** *For every $g, h \in L^2(\Gamma_r)$,*

$$\langle \Lambda_J g, h \rangle_{L^2(\Gamma_r)} = \mathcal{E}_J(\widehat{g}, \widehat{h}). \tag{8}$$

In particular, Λ_J is self-adjoint and nonnegative.

Proof. Write $v = \widehat{g}$ and $w = \widehat{h}$. Since J is symmetric, splitting H into $\Omega_r \times \Omega_r$, $\Omega_r \times \Gamma_r$ and $\Gamma_r \times \Omega_r$ gives

$$\mathcal{E}_J(v, w) = \frac{1}{2} \int_{\Omega_r} \int_{\Omega_r} J(x-y)(v(y) - v(x))(w(y) - w(x))\, dy\, dx + \int_{\Omega_r} \int_{\Gamma_r} J(x-y)(v(y) - v(x))(w(y) - w(x))\, dy\, dx.$$

The first term can be symmetrized once more, and after grouping the terms containing $w(x)$ for $x \in \Omega_r$ we obtain

$$\mathcal{E}_J(v, w) = -\int_{\Omega_r} w(x) \int_{\Omega_r \cup \Gamma_r} J(x-y)(v(y) - v(x))\, dy\, dx + \int_{\Gamma_r} w(y) \int_{\Omega_r} J(x-y)(v(y) - v(x))\, dx\, dy.$$

The first integral vanishes because $v = \widehat{g}$ is J-harmonic in Ω_r. Using the symmetry of J in the second one yields

$$\mathcal{E}_J(\widehat{g}, \widehat{h}) = \int_{\Gamma_r} h(y) \int_{\Omega_r} J(y-x)(g(y) - Tg(x))\, dx\, dy = \langle \Lambda_J g, h \rangle_{L^2(\Gamma_r)}.$$

Thus (8) holds. Since its right-hand side is symmetric in g and h, Λ_J is self-adjoint. Taking $h = g$ gives

$$\langle \Lambda_J g, g \rangle = \mathcal{E}_J(\widehat{g}) \geq 0.$$

□

Remark 2.1. *The identity above is the nonlocal analogue of $\langle \Lambda g, g \rangle_{\partial\Omega} = \int_\Omega |\nabla Hg|^2$ for the classical Dirichlet-to-Neumann map and the harmonic extension Hg.*

3. The Spectral Gap

Since constants belong to $\ker \Lambda_J$, the relevant space for relaxation is

$$\mathcal{H}_0 := L^2_0(\Gamma_r) = \left\{ g \in L^2(\Gamma_r) : \int_{\Gamma_r} g = 0 \right\}.$$

The space $\mathcal{H}_0$ is invariant under Λ_J. Indeed, by self-adjointness and Proposition 2.1,

$$\int_{\Gamma_r} \Lambda_J g = \langle \Lambda_J g, \mathbf{1} \rangle = \langle g, \Lambda_J \mathbf{1} \rangle = 0.$$

Let A denote the restriction of Λ_J to $\mathcal{H}_0$.

Definition 3.1. *The spectral-gap constant is*

$$\lambda_* := \inf_{0 \neq g \in \mathcal{H}_0} \frac{\mathcal{E}_J(\widehat{g})}{\|g\|^2_{L^2(\Gamma_r)}}. \tag{9}$$

Theorem 3.1 (Variational characterization)**.** *The operator A is bounded, self-adjoint and nonnegative on $\mathcal{H}_0$, and*

$$\lambda_* = \inf \sigma(A). \tag{10}$$

If $r < R$ and the hypotheses of the nonlocal Poincaré inequality of [4, Theorem 2.5] hold, then

$$\lambda_* > 0.$$

Proof. The first assertion follows from Lemma 2.2 and the invariance of $\mathcal{H}_0$. For a bounded self-adjoint operator, the spectral theorem gives

$$\inf \sigma(A) = \inf_{0 \neq g \in \mathcal{H}_0} \frac{\langle Ag, g \rangle}{\|g\|_2^2}.$$

Using Lemma 2.2 proves (10).

When $r < R$, the Poincaré estimate proved in [4, Theorem 2.5], applied to the harmonic extension $\widehat{g}$, yields a constant $c > 0$ such that

$$\mathcal{E}_J(\widehat{g}) \geq c\|g\|^2_{L^2(\Gamma_r)}, \qquad g \in \mathcal{H}_0.$$

Consequently $\lambda_* \geq c > 0$. □

Comparison under Reinforcement of the Kernel

We state the comparison on the fixed decomposition $\widehat{\Omega} = \Omega_r \cup \Gamma_r$. Here normalization of the kernel mass is immaterial. A nonnegative symmetric measurable convolution kernel K is called *admissible* if

$$M_K := \operatorname*{ess\,sup}_{x \in \widehat{\Omega}} \int_{\widehat{\Omega}} K(x - y)\, dy < \infty$$

and the interior coercivity estimate (2) holds with K and some $c_D(K) > 0$ in place of J and c_D. The first condition makes the energy a bounded quadratic form on $L^2(\widehat{\Omega})$; the second gives, by Lax–Milgram, a unique bounded harmonic extension and the Dirichlet principle. Thus the original smooth kernel is admissible. Sums of admissible kernels, and positive scalar multiples, are admissible: the row bounds add and the coercive lower bound is preserved. In particular, $J + \theta K$ is admissible for every $\theta \geq 0$ when J, K are admissible, since its interior energy dominates that of J.

For an admissible K write

$$Q_K(g) = \min_{v|_{\Gamma_r} = g} \mathcal{E}_K(v) = \mathcal{E}_K(\widehat{g}^K) \quad \text{and} \quad \lambda_*(K) = \inf_{0 \neq g \in \mathcal{H}_0} \frac{Q_K(g)}{\|g\|_2^2}.$$

To identify the classical structure, let L_K be the bounded positive operator on $L^2(\Omega_r) \oplus L^2(\Gamma_r)$ whose quadratic form is the uneliminated energy $\mathcal{E}_K$ (with interaction set H). Relative to this decomposition, its interior block $(L_K)_{II}$ is coercive and

$$\Lambda_K = (L_K)_{BB} - (L_K)_{BI}(L_K)_{II}^{-1}(L_K)_{IB}. \tag{11}$$

This follows by minimizing in the interior variable. Consequently, Λ_K is the boundary block of the shorted operator of L_K to $\{0\} \oplus L^2(\Gamma_r)$, in the terminology of [1,2]. The next proposition is the application of classical shorted-operator superadditivity to this nonlocal model; we include its direct variational proof for completeness.

Proposition 3.1 (Kernel reinforcement as a shorting inequality)**.** *Let J_1, J_2 be admissible nonnegative symmetric kernels on the same decomposition. Then $J_1 + J_2$ is admissible, and*

$$Q_{J_1+J_2}(g) \geq Q_{J_1}(g) + Q_{J_2}(g), \qquad g \in L^2(\Gamma_r). \tag{12}$$

Consequently,

$$\lambda_*(J_1 + J_2) \geq \lambda_*(J_1) + \lambda_*(J_2). \tag{13}$$

If J, K are admissible, then, for every $\theta \geq 0$, $J + \theta K$ is admissible and

$$\lambda_*(J + \theta K) \geq \lambda_*(J) + \theta\lambda_*(K) \geq \lambda_*(J).$$

These comparisons concern conductance addition without subsequent mass renormalization.

Proof. The admissibility assertions follow from the row bounds and interior coercivity above. For every extension v of g,

$$\mathcal{E}_{J_1+J_2}(v) = \mathcal{E}_{J_1}(v) + \mathcal{E}_{J_2}(v) \geq Q_{J_1}(g) + Q_{J_2}(g).$$

Minimization gives (12), equivalently the boundary-block form of

$$(L_{J_1} + L_{J_2})_{\Gamma_r} \geq (L_{J_1})_{\Gamma_r} + (L_{J_2})_{\Gamma_r},$$

where the subscript denotes shorting. Dividing by $\|g\|_2^2$ and taking the infimum over nonzero $g \in \mathcal{H}_0$ proves (13). For $\theta > 0$, the identity $Q_{\theta K} = \theta Q_K$ proves the last assertion by the same argument. At $\theta = 0$ it is an equality; no harmonic reduction for the zero kernel is required. □

Remark 3.1. *The reduced operators need not be additive because their minimizing extensions depend on the kernel. Superadditivity after eliminating the interior is the classical phenomenon expressed by shorting. Here it permits a direct comparison of boundary-strip energies and their uniform relaxation rates.*

Remark 3.2. *Formula* (10) *does not assert that λ_* is an eigenvalue. This is intentional. The operator Λ_J is bounded and self-adjoint, but the present assumptions provide no compactness mechanism ensuring that λ_* is an eigenvalue. We therefore formulate the results in terms of the bottom of the spectrum, which is sufficient for the dynamical conclusions below.*

We also record the characterization of the zero mode.

Proposition 3.2. *Assume that the interaction configuration satisfies* (1)*. Then*

$$\ker \Lambda_J = \operatorname{span}\{\mathbf{1}\}.$$

Proof. If $\Lambda_J g = 0$, Lemma 2.2 gives $\mathcal{E}_J(\widehat{g}) = 0$. Hence

$$J(x-y)|\widehat{g}(y) - \widehat{g}(x)|^2 = 0$$

for almost every interacting pair in H. If $\widehat{g}$ were not almost everywhere constant, a level set $F = \{\widehat{g} > s\}$ with $0 < |F| < |\widehat{\Omega}|$ would have zero interaction with its complement, contradicting (1). Thus $\widehat{g}$ is almost everywhere constant on $\Omega_r \cup \Gamma_r$. Thus g is constant on Γ_r. The converse follows from Proposition 2.1. □

4. Optimal Relaxation and the Critical Geometry

Consider the linear dynamical boundary problem

$$\begin{cases} 0 = \displaystyle\int_{\Omega_r \cup \Gamma_r} J(x-y)(u(y,t) - u(x,t))\, dy, & x \in \Omega_r, \\ u_t(x,t) = \displaystyle\int_{\Omega_r} J(x-y)(u(y,t) - u(x,t))\, dy, & x \in \Gamma_r, \\ u(x,0) = u_0(x), & x \in \Gamma_r. \end{cases} \tag{14}$$

The first line means that, at every time, the interior value is the harmonic extension of the trace on Γ_r. Hence (14) is equivalent on $L^2(\Gamma_r)$ to

$$u_t + \Lambda_J u = 0, \qquad u(0) = u_0. \tag{15}$$

Since Λ_J is bounded and nonnegative self-adjoint, $u(t) = e^{-t\Lambda_J} u_0$. Set

$$\bar{u}_0 := \frac{1}{|\Gamma_r|} \int_{\Gamma_r} u_0(x)\, dx.$$

Since $\Lambda_J \mathbf{1} = 0$ and $\mathcal{H}_0$ is invariant, the mean is conserved and $u(t) - \bar{u}_0 = e^{-tA}(u_0 - \bar{u}_0)$.

The following statement is a standard spectral-theorem consequence of the self-adjoint reduction. Its role is to express the exact uniform relaxation rate of (14) in terms of the model's reduced energy.

Theorem 4.1. *For every $t \geq 0$,*

$$\|e^{-tA}\|_{\mathcal{L}(\mathcal{H}_0)} = e^{-t\lambda_*}. \tag{16}$$

Consequently, every solution of (14) *satisfies*

$$\|u(t) - \bar{u}_0\|_{L^2(\Gamma_r)} \leq e^{-t\lambda_*} \|u_0 - \bar{u}_0\|_{L^2(\Gamma_r)}. \tag{17}$$

Moreover, λ_* is the largest number $\lambda \geq 0$ for which

$$\|e^{-tA}\| \leq e^{-\lambda t} \qquad \textit{for every } t \geq 0.$$

In particular, under the Poincaré hypotheses in Theorem 3.1, including $r < R$, the equilibrium is uniformly exponentially stable on $\mathcal{H}_0$.

Proof. By the continuous functional calculus for bounded self-adjoint operators,

$$\sigma(e^{-tA}) = e^{-t\sigma(A)}.$$

Since e^{-tA} is self-adjoint and positive,

$$\|e^{-tA}\| = \sup_{\mu\in\sigma(A)} e^{-t\mu} = e^{-t\inf\sigma(A)} = e^{-t\lambda_*},$$

which proves (16) and (17). If an estimate with $\lambda > \lambda_*$ held for every $t \geq 0$, then (16) would imply $e^{-t\lambda_*} \leq e^{-t\lambda}$ for all $t > 0$, a contradiction. The final assertion follows from Theorem 3.1. □

Thus the constant controlling the Poincaré inequality is not only sufficient for exponential convergence. After optimization, its Rayleigh quotient gives exactly the best uniform rate permitted by the dynamics.

A Second Reinforcement Mechanism: Strip Diffusion

The preceding comparison concerns reinforcement of the conductance kernel. A different mechanism appears in the modified problem considered in [4], where particles in Γ_r interact not only with Ω_r but also directly with other points of Γ_r. The stationary equation in Ω_r is unchanged, so the same harmonic extension Tg is used. The reduced operator is now

$$(\Lambda_J^* g)(x) := -\int_{\Omega_r\cup\Gamma_r} J(x-y)\big(\widehat{g}(y) - g(x)\big)\,dy, \qquad x \in \Gamma_r. \tag{18}$$

Introduce the boundary-strip diffusion operator

$$(B_\Gamma g)(x) := \int_{\Gamma_r} J(x-y)(g(x) - g(y))\,dy. \tag{19}$$

Then

$$\Lambda_J^* = \Lambda_J + B_\Gamma. \tag{20}$$

Proposition 4.1. *The operator B_Γ is bounded, self-adjoint and nonnegative on $L^2(\Gamma_r)$, and*

$$\langle B_\Gamma g, g\rangle = \frac{1}{2}\int_{\Gamma_r}\int_{\Gamma_r} J(x-y)|g(y) - g(x)|^2\,dy\,dx.$$

Consequently Λ_J^* is bounded, self-adjoint and nonnegative. If

$$\lambda_*^* := \inf_{0\neq g\in\mathcal{H}_0} \frac{\langle \Lambda_J^* g, g\rangle}{\|g\|_2^2},$$

then

$$\lambda_*^* \geq \lambda_*. \tag{21}$$

Moreover,

$$\|e^{-t\Lambda_J^*}|_{\mathcal{H}_0}\| = e^{-t\lambda_*^*} \leq e^{-t\lambda_*}, \qquad t \geq 0.$$

Proof. Boundedness of B_Γ follows from the same Schur–Young estimate used for the integral term in Proposition 2.1. Symmetry of J gives

$$\langle B_\Gamma g, h\rangle = \frac{1}{2}\int_{\Gamma_r}\int_{\Gamma_r} J(x-y)(g(y) - g(x))(h(y) - h(x))\,dy\,dx,$$

and the asserted quadratic identity follows by taking $h = g$. Hence $B_\Gamma \geq 0$. By (20),

$$\langle \Lambda_J^* g, g\rangle = \langle \Lambda_J g, g\rangle + \langle B_\Gamma g, g\rangle \geq \langle \Lambda_J g, g\rangle,$$

so taking the infimum over nonzero $g \in \mathcal{H}_0$ proves (21). The semigroup norm formula is the same spectral-calculus argument as in Theorem 4.1. □

Corollary 4.1. *Suppose $\widehat{\Omega}$ is a bounded connected smooth domain, $|\Gamma_r| > 0$, and J is a continuous nonnegative symmetric integrable kernel with $J(z) \geq j_0 > 0$ for $|z| \leq \rho$ for some $\rho > 0$. Suppose also that the interior reduction is well defined, as above. Then there exists $c_* > 0$ such that*

$$\frac{1}{2}\int_{\widehat{\Omega}}\int_{\widehat{\Omega}} J(x-y)|\widehat{g}(y) - \widehat{g}(x)|^2\,dy\,dx \geq c_*\|g\|^2_{L^2(\Gamma_r)}, \qquad g \in \mathcal{H}_0. \tag{22}$$

Consequently $\lambda_*^* \geq c_* > 0$. This includes $r = R$ for the smooth kernels of Section 2, provided $|\Omega_r| > 0$. The constant may depend on the kernel, domain and strip; no uniformity over varying geometries is claimed. Direct strip interactions therefore restore a positive gap in any such configuration in which the original gap vanishes.

Proof. The concrete domain and positivity assumptions imply the standard full-domain nonlocal Poincaré inequality (see [3] and the full-interaction discussion in [4, Section 2.3]): for some $c_* > 0$,

$$\frac{1}{2}\iint_{\widehat{\Omega}\times\widehat{\Omega}} J(x-y)|v(y)-v(x)|^2\,dy\,dx \geq c_*\|v-m\|^2_{L^2(\widehat{\Omega})}, \quad m = \frac{1}{|\widehat{\Omega}|}\int_{\widehat{\Omega}} v.$$

Indeed, positivity on a ball gives control on sufficiently small overlapping subsets, and a finite connected covering of the smooth connected domain propagates that control to the whole domain. Apply this inequality to $v = \widehat{g}$. Since $\int_{\Gamma_r} g = 0$,

$$\|\widehat{g}-m\|^2_{L^2(\widehat{\Omega})} \geq \int_{\Gamma_r} |g-m|^2 = \|g\|^2_{L^2(\Gamma_r)} + |\Gamma_r|\,|m|^2 \geq \|g\|^2_{L^2(\Gamma_r)}.$$

This proves (22). By the Green identity and Proposition 4.1, its left-hand side is $\langle \Lambda_J^* g, g\rangle$, so taking Rayleigh quotients proves the assertion. For the radial nonincreasing continuous kernels of Section 2, normalization implies $J(0) > 0$, hence the required lower bound on a small ball. □

The Critical Width

The concentration mechanism of [4, Example 2.4] requires weakly coupled regions of the strip, not merely the numerical equality $r = R$. Here is an explicit sufficient geometric condition and the associated extensions. Set

$$a_J(x) = \int_{\Omega_r} J(x-y)\,dy, \qquad x \in \mathbb{R}^n.$$

Since J is continuous with compact support, it is uniformly continuous, and

$$|a_J(x) - a_J(x')| \leq |\Omega_r| \sup_{z\in\mathbb{R}^n} |J(z+x-x') - J(z)|;$$

hence a_J is continuous on $\mathbb{R}^n$. In particular, a_J restricted to Γ_r extends continuously to $\overline{\Gamma_r}$, including the points of $\partial\widehat{\Omega}$, which do not belong to Γ_r because $\widehat{\Omega}$ is open. Suppose $r = R$ and there are distinct $x_0, x_1 \in \overline{\Gamma_r}$ (not necessarily in Γ_r) such that

$$a_J(x_i) = 0, \qquad |\Gamma_r \cap B_\varepsilon(x_i)| > 0 \quad (i = 0, 1) \quad \text{for every sufficiently small } \varepsilon > 0. \tag{23}$$

The sets $\Gamma_r \cap B_\varepsilon(x_i)$ in (23) are subsets of the strip itself; only the base points x_i are allowed to lie on $\partial\widehat{\Omega}$. A checkable sufficient condition for its vanishing is $\operatorname{dist}(x_i, \Omega_r) \geq R$, because the continuous kernel is zero for $|z| \geq R$. The distance here is to the interior region Ω_r. We do not deduce (23) from a distance-to-$\partial\widehat{\Omega}$ condition alone.

Choose $\varepsilon_k \downarrow 0$ so that $E_{i,k} = \Gamma_r \cap B_{\varepsilon_k}(x_i)$ are disjoint, and put $m_{i,k} = |E_{i,k}| > 0$. By the continuity of a_J at x_i and (23), $\sup_{E_{i,k}} a_J \to a_J(x_i) = 0$ as $k \to \infty$. Define

$$h_k = \frac{\mathbf{1}_{E_{0,k}}}{m_{0,k}} - \frac{\mathbf{1}_{E_{1,k}}}{m_{1,k}}, \qquad d_k = (m_{0,k}^{-1} + m_{1,k}^{-1})^{1/2}, \qquad g_k = h_k/d_k. \tag{24}$$

Then

$$\int_{\Gamma_r} h_k = 1 - 1 = 0$$

and $\|h_k\|_2^2 = d_k^2$, so $\int_{\Gamma_r} g_k = 0$ and $\|g_k\|_2 = 1$. Extend these traces by zero into the interior:

$$v_k(x) = \begin{cases} 0, & x \in \Omega_r, \\ g_k(x), & x \in \Gamma_r. \end{cases} \tag{25}$$

There are no interior–interior contributions and strip–strip interactions are excluded from H. The two cross terms cancel the factor $1/2$, giving

$$\mathcal{E}_J(v_k) = \int_{\Gamma_r} a_J(x)|g_k(x)|^2\,dx \leq \delta_k, \qquad \delta_k := \sup_{x\in E_{0,k}\cup E_{1,k}} a_J(x) \longrightarrow 0.$$

The Dirichlet principle therefore yields

$$\mathcal{E}_J(\widehat{g}_k) \leq \mathcal{E}_J(v_k) \leq \delta_k \longrightarrow 0. \tag{26}$$

For instance, these geometric conditions hold in one dimension for $\widehat{\Omega} = (0, L)$, $L > 2R$, $r = R$, with $x_0 = 0$ and $x_1 = L$: $\Omega_r = (R, L - R)$ and both endpoints have distance R from Ω_r. Here $\Gamma_r = (0, R] \cup [L - R, L)$, so $x_0, x_1 \notin \Gamma_r$ but

$$x_0, x_1 \in \overline{\Gamma_r} = [0, R] \cup [L - R, L].$$

Moreover $a_J(0) = a_J(L) = 0$, since J vanishes for $|z| \geq R$, and $E_{0,k} = (0, \varepsilon_k)$, $E_{1,k} = (L - \varepsilon_k, L)$ have positive measure. This provides a concrete configuration to which the conditional conclusion below applies.

Proposition 4.2. *Suppose $r = R$ and there exist $g_k \in \mathcal{H}_0$ with $\|g_k\|_2 = 1$ satisfying (26); in particular, the geometric condition (23) is sufficient. Then*

$$\lambda_* = 0 \quad \textit{and} \quad \|e^{-tA}\|_{\mathcal{L}(\mathcal{H}_0)} = 1 \quad \textit{for every } t \geq 0.$$

In particular, there is no estimate

$$\|e^{-tA} g\|_2 \leq e^{-ct} \|g\|_2, \qquad g \in \mathcal{H}_0,$$

valid for all $t \geq 0$ with a constant $c > 0$ independent of g.

Proof. By (9) and (26), $0 \leq \lambda_* \leq \mathcal{E}_J(\widehat{g}_k) \to 0$. Hence $\lambda_* = 0$. The norm identity follows from Theorem 4.1. □

Remark 4.1. *The conclusion concerns* uniform *exponential stability. The equality $\lambda_* = 0$ alone does not rule out convergence to equilibrium for each fixed initial datum, nor does it determine a universal slower rate. It says precisely that the semigroup has no positive spectral gap on the mean-zero space.*

5. Final Comments

The linear boundary-strip model admits a useful variational reduction: eliminating the stationary interior variable produces a bounded nonnegative self-adjoint Dirichlet-to-Neumann operator whose quadratic form is the minimized interaction energy. The optimized boundary Poincaré constant is therefore the bottom of its spectrum on the mean-zero subspace. The standard spectral theorem transfers this identification directly to the exact uniform exponential relaxation rate of the model.

Likewise, kernel reinforcement is an application of the classical superadditivity of shorted operators [1, 2]. The application identifies how this order inequality controls the reduced energy and relaxation rate in the present model. Direct strip diffusion adds a nonnegative boundary form, and Corollary 4.1 supplies concrete connected-domain and near-origin kernel-positivity hypotheses under which the full-interaction dynamics has a positive gap, including at $r = R$.

For the original dynamics, $r < R$ gives a positive gap under the hypotheses of [4, Theorem 2.5]. At $r = R$, gap closure is established only for weak-coupling configurations such as (23); it is not inferred from critical width alone. The trial traces and their zero-interior extensions make this mechanism explicit.

Additional assumptions may ensure that the spectral bottom is an eigenvalue. Dependence on the strip width is subtler because the boundary state space varies with r; the critical concentration example does not by itself imply continuity of $\lambda_*(r)$ as $r \uparrow R$. Quantitative width dependence and local limits under kernel rescaling remain questions for further work. Recent studies of nonlocal Green identities and Neumann scaling [8, 9] provide related context, but their boundary formulations differ from the present strip dynamics.

References

[1] W. N. Anderson, Shorted operators, *SIAM J. Appl. Math.* **20** (1971) 520–525.

[2] W. N. Anderson, G. E. Trapp, Shorted operators. II, *SIAM J. Appl. Math.* **28** (1975) 60–71.

[3] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, J. J. Toledo-Melero, *Nonlocal Diffusion Problems*, American Mathematical Society, Providence, 2010.

[4] P. M. Berná, J. D. Rossi, Nonlocal diffusion equations with dynamical boundary conditions, *Nonlinear Anal.* **195** (2020) #111751.

[5] K. Bogdan, T. Grzywny, K. Pietruska-Pałuba, A. Rutkowski, Nonlinear nonlocal Douglas identity, *Calc. Var. Partial Differential Equations* **62** (2023) #151.

[6] L. M. Del Pezzo, J. D. Rossi, A. M. Salort, Fractional eigenvalue problems that approximate Steklov eigenvalue problems, *Proc. Roy. Soc. Edinburgh Sect. A* **148** (2018) 499–516.

[7] S. Dipierro, X. Ros-Oton, E. Valdinoci, Nonlocal problems with Neumann boundary conditions, *Rev. Mat. Iberoam.* **33** (2017) 377–416.

[8] M. L. Parks, P. Radu, On nonlocal problems with Neumann boundary conditions: scaling and convergence for nonlocal operators and solutions, *Adv. Contin. Discrete Models* **2025** (2025) #66.

[9] J. M. Scott, Q. Du, Nonlocal problems with local boundary conditions II: Green's identities and regularity of solutions, *SIAM J. Math. Anal.* **57** (2025) 404–451.

[10] M. Solera, J. Toledo, Nonlocal doubly nonlinear diffusion problems with nonlinear boundary conditions, *J. Evol. Equ.* **23** (2023) #24.