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Research Article

Jordan Derivation on Generalized Triangular Matrix Rings of Arbitrary Size

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Abstract

In this paper, we prove that any Jordan derivation on the generalized triangular matrix ring $T_n(R, M)$ is an ordinary derivation for any integer $n \geq 2$. This extends some well-known results of Driss and L'Moufadal [*Int. J. Math. Math. Sci.* **2005** (2005) 2125–2132] and Zhang and Yu [*Linear Algebra Appl.* **419** (2006) 251–255], as well as several related results established in the existing literature.

Keywords: Jordan derivation; (generalized) triangular matrix ring; additive map.

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1. Introduction

Let A be an arbitrary associative algebra. Let K be an arbitrary commutative ring with identity. Moreover, assume that A is a K-bimodule. If an additive map $D : A \longrightarrow A$ satisfies the classical Leibniz rule

$$D(ab) = D(a)b + aD(b) \text{ for all } a, b \in A,$$

then it is referred to as an *ordinary derivation* or simply a *derivation*. A *Jordan derivation* J is an additive map that satisfies the condition

$$J(a^2) = J(a)a + aJ(a) \text{ for every } a \in A.$$

The classical result on Jordan derivations is due to Herstein [13], who showed that every Jordan derivation on a 2-torsion-free prime ring is a derivation. This result was later extended to semiprime rings by Brešar [4]. In other contexts, further generalizations have also been obtained (see, e.g., [2, 5, 14] and the references therein). Likewise, Jordan derivations on upper triangular matrix algebras over commutative rings were studied by Benkovič [2], who established that every such Jordan derivation can be expressed as the sum of a derivation and an anti-derivation.

The class of *triangular algebras*, denoted by

$$\mathcal{T} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{pmatrix},$$

where $\mathcal{A}$ and $\mathcal{B}$ are unital algebras and $\mathcal{M}$ is a unital bimodule that is faithful as both a left $\mathcal{A}$-module and a right $\mathcal{B}$-module, was studied by Zhang and Yu [14], who proved that every Jordan derivation on such an algebra is a derivation. In [5, 12], Jordan derivations on trivial extension algebras have been investigated from both algebraic and analytical perspectives. Further related results on derivations and their generalizations can also be found in [7, 8, 10].

Let $R = (R_i)_{i=1}^n$ be a family of rings with identity $(1_{R_i}, 1 \leq i \leq n)$ and let

$$M := (M_{i,j})_{1 \leq i < j \leq n}$$

be a family of unital modules such that, for all i and j with $1 \leq i < j \leq n$, $M_{i,j}$ is an (R_i, R_j)-bimodule. Also, assume that, for all $1 \leq i < j < k \leq n$, there exists an (R_i, R_k)-bimodule homomorphism

$$M_{i,j} \otimes M_{j,k} \longrightarrow M_{i,k}$$

such that $(m_{i,j}m_{j,k})m_{k,l} = m_{i,j}(m_{j,k}m_{k,l})$ for every $(m_{i,j}, m_{j,k}, m_{k,l}) \in M_{i,j} \times M_{j,k} \times M_{k,l}$, where $n \geq l > k \geq 1$. Note that the elements of the tensor product of two modules are finite sums of elementary tensors.

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Thus, following [1], the set

$$T := \begin{pmatrix} R_1 & M_{12} & M_{13} & \dots & M_{1n} \\ 0 & R_2 & M_{23} & \dots & M_{2n} \\ 0 & 0 & R_3 & \dots & M_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & R_n \end{pmatrix},$$

consisting of all matrices of the form

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ 0 & m_{22} & m_{23} & \cdots & m_{2n} \\ \vdots & \ddots & \ddots & \vdots & \\ 0 & \cdots & & 0 & m_{nn} \end{pmatrix}, \ m_{ii} \in R_i, \ m_{ij} \in M_{ij}, \ 1 \le i < j \le n,$$

forms a ring under the standard matrix addition and multiplication, called a *generalized (or formal) triangular matrix ring*, and will henceforth be denoted by $T_n(R, M)$; for further details, we refer the reader to [1, 3].

Generalized triangular matrix representations provide an important tool for investigating the structure of a wide range of algebras. Their diverse applications arise in operator theory, quasi-triangular Hopf algebras, and various aspects of Lie algebra theory.

The main objective of the present work is to show that if M_{1n} is faithful as both a left R_1-module and a right R_n-module, and each M_{in}, $2 \le i \le n-1$, is a faithful left R_i-module, then every Jordan derivation on $T_n(R, M)$ is an ordinary derivation. This generalizes several well-established results in the existing literature due to Brešar, Driss and L'Moufadal, Ferreira, and other researchers. More precisely, a comparison with the corresponding results obtained so far reveals the following differences. In [9], additional restrictions are imposed: all rings are assumed to be 2-torsion-free, all modules are required to be faithful, and only the 2×2 matrix case is considered. Furthermore, in [11], the rings are also assumed to be 2-torsion-free, and the faithfulness of the modules is again required in Definition 2.1; however, some of the conditions imposed there are weaker than ours, and the rings considered are merely associative, without necessarily possessing an identity element.

2. Results

Throughout the paper, M_{in} is a faithful left R_i-module for each $2 \le i \le n-1$, and M_{1n} is both a faithful left R_1-module and right R_n-module. To simplify the exposition, we set $T := T_n(R, M)$. Also, when we use e_i, we mean a diagonal matrix with 1 in the i-row and i-column, and zeros everywhere else, i.e., $e_i = diag(0, \cdot, 0, 1_{R_i}, 0, \cdot, 0)$. We also put $T_{ij} := e_i T e_j$ for any $1 \le i \le j \le n$. So, every $A \in T$ can uniquely be written as $A = A_{11} + \cdots + A_{1n} + A_{22} + \cdots + A_{nn}$, where $A_{ij} \in T_{ij}$, $1 \le i \le j \le n$.

Let $A := \sum_{j=1}^{n} \sum_{i=1}^{j} A_{ij}$ and $B := \sum_{j=1}^{n} \sum_{i=1}^{j} B_{ij}$, where A_{ij} and B_{ij} are defined only for $i \le j$. Then, we deduce

$$AB = \sum_{k=1}^{n} \sum_{t=1}^{k} C_{tk},$$

where $t \le k$ and $C_{tk} = A_{tt}B_{tk} + A_{tt+1}B_{t+1k} + \cdots + A_{tk}B_{kk}$.

To prove our main result, we first establish a series of technical lemmas.

Lemma 2.1. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then*

$$D(e_i) \in T_{1i} + T_{2i} + \cdots + T_{i-1i} + T_{ii+1} + T_{ii+2} + \cdots + T_{in}.$$

Proof. Assume that

$$D(e_i) = \sum_{k=1}^{n} \sum_{s=1}^{n} e_s m_{sk}^i e_k, \text{ where } m_{sk}^i \text{ is defined only for } s \le k.$$

Since, for any $1 \le i \le n$, $D(e_i) = D(e_i)e_i + e_i D(e_i)$ holds, we have

$$\sum_{k=1}^{n} \sum_{s=1}^{n} e_s m_{sk}^i e_k = e_i \sum_{k=1}^{n} \sum_{s=1}^{n} e_s m_{sk}^i e_k + \sum_{k=1}^{n} \sum_{s=1}^{n} e_s m_{sk}^i e_k e_i = \sum_{k=i}^{n} e_i m_{ik}^i e_k + \sum_{s=1}^{i} e_s m_{si}^i e_i$$

$$= 2m_{ii}^i + \sum_{k=i+1}^{n} e_i m_{ik}^i e_k + \sum_{s=1}^{i-1} e_s m_{si}^i e_i.$$

Therefore, it follows that $m^i_{ii} = 2m^i_{ii}$, and so, $m^i_{ii} = 0$. Thus, it holds that

$$D(e_i) = \sum_{k=i+1}^{n} e_i m^i_{ik} e_k + \sum_{s=1}^{i-1} e_s m^i_{si} e_i. \qquad \square$$

Now, for every $1 \leq i \leq n$, we put

$$D(e_i) := e_1 D(e_i) e_i + e_2 D(e_i) e_i + \cdots + e_{i-1} D(e_i) e_i + e_i D(e_i) e_{i+1} + \cdots + e_i D(e_i) e_n.$$

Remark 2.1 (see [6]). *For a ring A and an A-bimodule M, every Jordan derivation $d : A \longrightarrow M$ satisfies*

$$d(ex) = d(e)x + ed(x)$$

whenever $e \in A$ is an idempotent and $x \in A$ commutes with e.

Lemma 2.2. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then*

$$D(R_i) \subseteq T_{1i} + \cdots T_{i-1i} + T_{ii} + T_{ii+1} + \cdots + T_{in}$$

for every $1 \leq i \leq n$.

Proof. Using Remark 2.1, we get $D(e_i r_i) = D(e_i) r_i + e_i D(r_i)$ for every $r_i \in R_i$. Thus, combining Lemma 2.1 and the fact that $e_i D(r_i) \in T_{ii} + T_{ii+1} + \cdots + T_{in}$, we deduce $D(r_i) \in T_{1i} + T_{2i} + \cdots + T_{ii} + T_{ii+1} + T_{ii+2} + \cdots + T_{in}$ for every $r_i \in R_i$, as required. $\square$

Lemma 2.3. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then*

$$D(T_{ij}) \subseteq T_{1j} + \cdots T_{i-1j} + T_{ij} + T_{ij+1} + \cdots + T_{in}$$

for all $1 \leq i < j \leq n$.

Proof. Let $m_{ij} \in T_{ij}$. Then, $m_{ij} = e_i m_{ij} e_j$. Observe that it suffices to show that $e_k D(m_{ij}) e_t = 0$ for all $1 \leq k < t \leq n$ with $k \neq i$ and $t \neq j$. To this end, since $e_k m_{ij} e_t = 0$, we divide the proof into three cases.

Case 1: $k \neq j$ and $t \neq i$.
Since $e_t m_{ij} e_k = 0$, we have $e_k m_{ij} e_t + e_t m_{ij} e_k = 0$. It follows that

$$0 = D(e_k) m_{ij} e_t + e_k D(m_{ij}) e_t + e_k m_{ij} D(e_t) + D(e_t) m_{ij} e_k + e_t D(m_{ij}) e_k + e_t m_{ij} D(e_k) = e_k D(m_{ij}) e_t + e_t D(m_{ij}) e_k.$$

It also follows that $0 = e_k e_k D(m_{ij}) e_t e_t + e_k e_t D(m_{ij}) e_k e_t = e_k D(m_{ij}) e_t$, as desired.

Case 2: $k \neq j$ and $t = i$.
Since $e_t m_{ij} e_k = 0$, we deduce $e_k m_{ij} e_t + e_t m_{ij} e_k = 0$. Thus, it holds that

$$0 = D(e_k) m_{ij} e_t + e_k D(m_{ij}) e_t + e_k m_{ij} D(e_t) + D(e_t) m_{ij} e_k + e_t D(m_{ij}) e_k + e_t m_{ij} D(e_k) = e_k D(m_{ij}) e_t + e_t D(m_{ij}) e_k + e_t m_{ij} D(e_k).$$

It now follows that

$$e_k D(m_{ij}) e_t = e_k e_k D(m_{ij}) e_t e_t + e_k e_t D(m_{ij}) e_k e_t + e_k e_t m_{ij} D(e_k) e_t = 0,$$

as desired.

Case 3: $k = j$ and $t \neq i$.
Since $e_t m_{ij} e_k = 0$, we deduce $e_k m_{ij} e_t + e_t m_{ij} e_k = 0$. So,

$$0 = D(e_k) m_{ij} e_t + e_k D(m_{ij}) e_t + e_k m_{ij} D(e_t) + D(e_t) m_{ij} e_k + e_t D(m_{ij}) e_k + e_t m_{ij} D(e_k) = e_k D(m_{ij}) e_t + D(e_t) m_{ij} e_k + e_t D(m_{ij}) e_k.$$

As $e_k D(m_{ij}) e_t \in T_{kt}$ and $D(e_t) m_{ij} e_k + e_t D(m_{ij}) e_k \in T_{tk}$, we conclude that

$$0 = e_k D(m_{ij}) e_t = D(e_t) m_{ij} e_k + e_t D(m_{ij}) e_k,$$

as required.

In either of the three cases, we establish that $e_k D(m_{ij}) e_t = 0$ with $k \neq i$ and $t \neq j$. Next, we show that $e_s D(m_{ij}) e_s = 0$ for $1 \leq s \leq n$.

If $s \notin \{i,j\}$, then we have either $s<j$ or $j<s$. If $s<j$, then Remark 2.1 implies that $0=D(m_{ij}e_s)=D(m_{ij})e_s+m_{ij}D(e_s)$ since $e_sm_{ij}=m_{ij}e_s$. By Lemma 2.2, it holds that $m_{ij}D(e_s)=0$, and so, $D(m_{ij}e_s)=D(m_{ij})e_s=0$. Consequently, it follows that $e_sD(m_{ij})e_s=0$. If $j<s$, then again from Remark 2.1, it follows that $0=D(e_sm_{ij})=D(e_s)m_{ij}+e_sD(m_{ij})$. Since $D(e_s)m_{ij}=0$, we deduce $D(e_sm_{ij})=e_sD(m_{ij})=0$, and so, $e_sD(m_{ij})e_s=0$.

If $s \in \{i,j\}$, then we have either $s=i$ or $s=j$. If $s=i$, then

$$D(m_{ij})=D(e_im_{ij})=D(e_im_{ij}+m_{ij}e_i)=D(e_i)m_{ij}+e_iD(m_{ij})+D(m_{ij})e_i+m_{ij}D(e_i).$$

By Lemma 2.2, we deduce $m_{ij}D(e_i)=0$, and so, $D(m_{ij})=D(e_i)m_{ij}+e_iD(m_{ij})+D(m_{ij})e_i$. Therefore, it follows that

$$e_iD(m_{ij})e_i=e_iD(e_i)m_{ij}e_i+e_iD(m_{ij})e_i+e_iD(m_{ij})e_i=2e_iD(m_{ij})e_i.$$

Thus, $e_iD(m_{ij})e_i=0$. If $s=j$, then

$$D(m_{ij})=D(m_{ij}e_j)=D(m_{ij}e_j+e_jm_{ij})=D(m_{ij})e_j+m_{ij}D(e_j)+e_jD(m_{ij}).$$

Hence, $e_jD(m_{ij})e_j=2e_jD(m_{ij})e_j$, which implies that $e_jD(m_{ij})e_j=0$. □

Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then motivated by Lemma 2.3, we define $D_{ij}(A):=e_iD(A)e_j$ for all $1 \leq i \leq j \leq n$. We also observe that $m_{ij}=e_im_{ij}e_j \in T_{ij}$, and hence, we deduce

$$D(m_{ij})=\sum_{k=1}^{i}D_{kj}(m_{ij})+\sum_{t=j+1}^{n}D_{it}(m_{ij}),\ 1 \leq i \leq j \leq n.$$

Lemma 2.4. *Let $D : T \longrightarrow T$ be a Jordan derivation. Then,*

$$D\left(\sum_{j=1}^{n}\sum_{i=1}^{j}m_{ij}\right)=\sum_{j=1}^{n}\sum_{i=1}^{j}d_{ij},$$

where $d_{ii}=D_{ii}(m_{ii})$, m_{ij},d_{ij} are defined if $i \leq j$ and

$$d_{ij}=D_{ij}(m_{ii})+D_{ij}(m_{ii+1})+\cdots+D_{ij}(m_{ij})+D_{ij}(m_{i+1j})+\cdots+D_{ij}(m_{jj});$$

$1 \leq i \leq j \leq n$.

Proof. Set $A:=\sum_{j=1}^{n}\sum_{i=1}^{j}m_{ij}$. Then, we have

$$D(A)=D(e_1m_{11}e_1)+D(e_1m_{12}e_2)+\cdots+D(e_1m_{1n}e_n)+\cdots+D(e_{n-1}m_{n-1n}e_n)+D(e_nm_{nn}e_n).$$

Now, we apply Lemmas 2.2 and 2.3 to obtain the stated claim. □

Lemma 2.5. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then $D_{is}(m_{ij})=m_{ij}D_{js}(e_j)$ for all $j+1 \leq s \leq n$ and $m_{ij} \in T_{ij}$ with $i<j$.*

Proof. As $i \neq j$, we derive $m_{ij}=m_{ij}e_j+e_jm_{ij}$. Then, we have $D(m_{ij})=D(m_{ij})e_j+m_{ij}D(e_j)+D(e_j)m_{ij}+e_jD(m_{ij})$. Furthermore,

$$\begin{aligned}D(m_{ij})=&\left(\sum_{k=1}^{i}D_{kj}(m_{ij})+\sum_{t=j+1}^{n}D_{it}(m_{ij})\right)e_j+m_{ij}\left(\sum_{k=1}^{j-1}D_{kj}(e_j)\right)+\sum_{t=j+1}^{n}D_{jt}(e_j))\\&+\left(\sum_{k=1}^{j-1}D_{kj}(e_j)+\sum_{t=j+1}^{n}D_{jt}(e_j)\right)m_{ij}+e_j\left(\sum_{k=1}^{i}D_{kj}(m_{ij})+\sum_{t=j+1}^{n}D_{it}(m_{ij})\right).\end{aligned}$$

However, since $\sum_{t=j+1}^{n}D_{it}(m_{ij}) \in T_{ij+1}+\cdots+T_{in}$ and $e_j \in T_{jj} \cong R_j$, we obtain $\left(\sum_{t=j+1}^{n}D_{it}(m_{ij})\right)e_j=0$. Since $m_{ij}=m_{ij}e_j$ and $i<j$, we deduce

$$m_{ij}\left(\sum_{k=1}^{j-1}D_{kj}(e_j)+\sum_{t=j+1}^{n}D_{jt}(e_j)\right)=m_{ij}e_j\left(\sum_{k=1}^{i-1}D_{kj}(e_j)\right)=m_{ij}e_j\left(\sum_{k=1}^{i-1}D_{kj}(e_j)\right)=0.$$

Analogously, we obtain

$$\left(\sum_{k=1}^{j-1} D_{kj}(e_j) + \sum_{t=j+1}^{n} D_{jt}(e_j)\right) m_{ij} + e_j \left(\sum_{k=1}^{i} D_{kj}(m_{ij}) + \sum_{t=j+1}^{n} D_{it}(m_{ij})\right) = 0,$$

and hence

$$D(m_{ij}) = \sum_{k=1}^{i} D_{kj}(m_{ij})e_j + m_{ij} \sum_{t=j+1}^{n} D_{jt}(e_j).$$

Therefore, it follows that $D_{is}(m_{ij}) = m_{ij}D_{js}(e_j)$ for all $j+1 \leq s \leq n$ and $m_{ij} \in T_{ij}$ with $i < j$. □

Lemma 2.6. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then $D_{ij}(e_i) + D_{ij}(e_j) = 0$ for any $i \leq j$.*

Proof. If $i = j$, then by Lemma 2.1, $D_{ii}(e_i) = 0$, as desired. Without loss of generality, we may assume that $i < j$. As $e_ie_j + e_je_i = 0$, we deduce

$$D(e_i)e_j + e_iD(e_j) + e_jD(e_i) + D(e_j)e_i = 0.$$

Now, by Lemma 2.1, we deduce

$$\left[\sum_{k=1}^{i-1} D_{ki}(e_i) + \sum_{t=i+1}^{n} D_{it}(e_i)\right] e_j + e_i \left[\sum_{k=1}^{j-1} D_{kj}(e_j) + \sum_{t=j+1}^{n} D_{jt}(e_j)\right] + e_j \left[\sum_{k=1}^{i-1} D_{ki}(e_i) + \sum_{t=i+1}^{n} D_{it}(e_i)\right]$$

$$+ \left[\sum_{k=1}^{j-1} D_{kj}(e_j) + \sum_{t=j+1}^{n} D_{jt}(e_j)\right] e_i = 0.$$

Since $i < j$, we deduce

$$\sum_{k=1}^{i-1} D_{ki}(e_i)e_j = 0, \quad e_i \sum_{t=j+1}^{n} D_{jt}(e_j) = 0, \quad e_j \left[\sum_{k=1}^{i-1} D_{ki}(e_i) + \sum_{t=i+1}^{n} D_{it}(e_i)\right] = 0, \quad \text{and} \quad \left[\sum_{k=1}^{j-1} D_{kj}(e_j) + \sum_{t=j+1}^{n} D_{jt}(e_j)\right] e_i = 0.$$

Consequently, it follows that

$$\sum_{t=i+1}^{n} D_{it}(e_i)e_j + e_i \sum_{k=1}^{j-1} D_{kj}(e_j) = 0,$$

which implies that $D_{ij}(e_i) + D_{ij}(e_j) = 0$. □

Lemma 2.7. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then $D_{sj}(m_{ij}) = D_{si}(e_i)m_{ij}$ for all $s \leq i-1$ and $m_{ij} \in T_{ij}$ with $i < j$.*

Proof. As $i \neq j$, we obtain $m_{ij} = e_im_{ij} + m_{ij}e_i$. So, $D(m_{ij}) = D(e_i)m_{ij} + e_iD(m_{ij}) + D(m_{ij})e_i + m_{ij}D(e_i)$. Also, we have

$$\begin{aligned} D(m_{ij}) &= \left(\sum_{k=1}^{i-1} D_{ki}(e_i) + \sum_{t=i+1}^{n} D_{it}(e_i)\right) m_{ij} + e_i \left(\sum_{k=1}^{i} D_{kj}(m_{ij}) + \sum_{t=j+1}^{n} D_{it}(m_{ij})\right) \\ &\quad + \left(\sum_{k=1}^{i} D_{kj}(m_{ij}) + \sum_{t=j+1}^{n} D_{it}(m_{ij})\right) e_i + m_{ij} \left(\sum_{k=1}^{i-1} D_{ki}(e_i) + \sum_{t=i+1}^{n} D_{it}(e_i)\right) \\ &= \left(\sum_{k=1}^{i-1} D_{ki}(e_i)\right) m_{ij} + D_{ij}(m_{ij}) + \sum_{t=j+1}^{n} D_{it}(m_{ij}). \end{aligned}$$

Thus, $D_{sj}(m_{ij}) = D_{si}(e_i)m_{ij}$ for all $s \leq i-1$ and $m_{ij} \in T_{ij}$ with $i < j$. □

As a consequence of Lemma 2.7, we have the following result.

Corollary 2.1. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then*

$$D(m_{ij}) = \left(\sum_{k=1}^{i-1} D_{ki}(e_i)m_{ij}\right) + D_{ij}(m_{ij}) + \left(m_{ij} \sum_{t=j+1}^{n} D_{jt}(e_j)\right) \text{ for all } i < j.$$

Lemma 2.8. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then $D(m_{ij}n_{jt}) = D(m_{ij})n_{jt} + m_{ij}D(n_{jt})$ for all $1 \leq i \leq j < t \leq n$ with $m_{ij} \in T_{ij}$ and $n_{jt} \in T_{jt}$.*

Proof. Since $i \leq j < t$, from the definition of a generalized triangular matrix ring, it follows that $T_{ti} = 0$, which gives $n_{jt}m_{ij} = 0$. Since D is a Jordan derivation, we deduce

$$D(m_{ij}n_{jt}) = D(m_{ij}n_{jt} + n_{jt}m_{ij}) = D(m_{ij})n_{jt} + m_{ij}D(n_{jt}) + D(n_{jt})m_{ij} + n_{jt}D(m_{ij}).$$

However, as $D(n_{jt}) \in T_{1t} + \cdots + T_{jt} + T_{jt+1} + \cdots + T_{jn}$ and $i \leq j < t$, we obtain $D(n_{jt})m_{ij} = 0$. In the same way, we have $n_{jt}D(m_{ij}) = 0$. Finally, we obtain $D(m_{ij}n_{jt}) = D(m_{ij})n_{jt} + m_{ij}D(n_{jt})$. □

The next three results are consequences of Lemma 2.8.

Corollary 2.2. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then*

$$D_{it}(m_{ij}n_{jt}) = D_{ij}(m_{ij})n_{jt} + m_{ij}D_{jt}(n_{jt})$$

for all $1 \leq i < j < t \leq n$ with $m_{ij} \in T_{ij}$ and $n_{jt} \in T_{jt}$.

Proof. Using Lemma 2.8, we write $D(m_{ij}n_{jt}) = D(m_{ij})n_{jt} + m_{ij}D(n_{jt})$ for all $1 \leq i < j < t \leq n$ with $m_{ij} \in T_{ij}$ and $n_{jt} \in T_{jt}$. As $m_{ij} = e_i m_{ij} e_j$ and $n_{jt} = e_j n_{jt} e_t$, we deduce

$$D(m_{ij}n_{jt}) = D(m_{ij})e_j n_{jt}e_t + e_i m_{ij}e_j D(n_{jt}) = D(m_{ij})e_j n_{jt}e_t + e_i m_{ij}e_j D(n_{jt}).$$

Therefore,

$$D_{it}(m_{ij}n_{jt}) = e_i D(m_{ij}n_{jt})e_t = e_i D(m_{ij})e_j n_{jt}e_t + e_i m_{ij}e_j D(n_{jt})e_t = D_{ij}(m_{ij})n_{jt} + m_{ij}D_{jt}(n_{jt}).$$ □

Corollary 2.3. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then $D_{st}(m_{ij}n_{jt}) = D_{sj}(m_{ij})n_{jt}$ for all $1 \leq s < i < j < t \leq n$ with $m_{ij} \in T_{ij}$ and $n_{jt} \in T_{jt}$.*

Corollary 2.4. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then $D_{ik}(m_{ij}n_{jt}) = m_{ij}D_{jk}(n_{jt})$ for all $1 \leq i < j < t < k \leq n$ with $m_{ij} \in T_{ij}$ and $n_{jt} \in T_{jt}$.*

We now proceed with the following useful results.

Lemma 2.9. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then*

$$D_{ii+k}(m_{i+ti+k})m_{i+kj} = D_{ij}(m_{i+ti+k}m_{i+kj})$$

for all $1 \leq i < i + t \leq i + k < j \leq n$ with $m_{i+ti+k} \in T_{i+ti+k}$ and $m_{i+kj} \in T_{i+kj}$.

Proof. Since $j > i + t$, we deduce $m_{i+kj}m_{i+ti+k} = 0$, and thus,

$$\begin{aligned} D(m_{i+ti+k}m_{i+kj}) &= D(m_{i+ti+k}m_{i+kj} + m_{i+kj}m_{i+ti+k}) \\ &= D(m_{i+ti+k})m_{i+kj} + m_{i+ti+k}D(m_{i+kj}) + D(m_{i+kj})m_{i+ti+k} + m_{i+kj}D(m_{i+ti+k}). \end{aligned}$$

It follows that

$$e_i D(m_{i+ti+k}m_{i+kj})e_j = e_i D(m_{i+ti+k})m_{i+kj}e_j + e_i m_{i+ti+k}D(m_{i+kj})e_j + e_i D(m_{i+kj})m_{i+ti+k}e_j + e_i m_{i+kj}D(m_{i+ti+k})e_j.$$

As $e_i m_{i+ti+k} = m_{i+ti+k}e_j = e_i m_{i+kj} = 0$, we deduce

$$D_{ij}(m_{i+ti+k}m_{i+kj}) = e_i D(m_{i+ti+k}m_{i+kj})e_j = e_i D(m_{i+ti+k})m_{i+kj}e_j = e_i D(m_{i+ti+k})m_{i+kj}.$$

On the other hand, it is easy to see that $e_i D_{st}(m_{i+ti+k})m_{i+kj} = 0$ for all $s \neq i$ and $t \neq i + k$. Consequently, we have

$$D_{ij}(m_{i+ti+k}m_{i+kj}) = D_{ii+k}(m_{i+ti+k})m_{i+kj}.$$ □

Lemma 2.10. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then $D_{ij}(m_{kj})m_{jj} = D_{ij}(m_{kj}m_{jj})$ for all $1 \leq i \leq k < j \leq n$ with $m_{kj} \in T_{kj}$ and $m_{jj} \in T_{jj}$.*

Proof. Since $j > k$, it holds that $m_{jj}m_{kj} = 0$, and hence,

$$D(m_{kj}m_{jj}) = D(m_{kj}m_{jj} + m_{jj}m_{kj}) = D(m_{kj})m_{jj} + m_{kj}D(m_{jj}) + D(m_{jj})m_{kj} + m_{jj}D(m_{kj}).$$

It follows that

$$e_iD(m_{kj}m_{jj})e_j = e_iD(m_{kj})m_{jj}e_j + e_im_{kj}D(m_{jj})e_j + e_iD(m_{jj})m_{kj}e_j + e_im_{jj}D(m_{kj})e_j.$$

As $e_im_{kj}D(m_{jj})e_j = e_im_{jj}D(m_{kj})e_j = 0$, we have

$$D_{ij}(m_{kj}m_{jj}) = e_iD(m_{kj})m_{jj}e_j + e_iD(m_{jj})m_{kj}e_j.$$

Since $e_iD(m_{jj}) \in T_{ij}$ and $m_{kj}e_j \in T_{kj}$, we obtain $e_iD(m_{jj})m_{kj}e_j = 0$. Therefore, we deduce

$$D_{ij}(m_{kj}m_{jj}) = e_iD(m_{kj})m_{jj}e_j = D_{ij}(m_{kj})m_{jj}$$ □

Lemma 2.11. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then $D_{ij}(m_{ik}m_{kt}) = m_{ik}D_{kj}(m_{kt})$ for all $1 \leq i \leq k < t < j \leq n$ or $1 \leq i < k \leq t < j \leq n$ and $m_{ik} \in T_{ik}, m_{kt} \in T_{kt}$.*

Proof. Since $t > i$, we deduce $m_{kt}m_{ik} = 0$, and thus,

$$D(m_{ik}m_{kt}) = D(m_{ik}m_{kt} + m_{kt}m_{ik}) = D(m_{ik})m_{kt} + m_{ik}D(m_{kt}) + D(m_{kt})m_{ik} + m_{kt}D(m_{ik}).$$

It follows that

$$e_iD(m_{ik}m_{kt})e_j = e_iD(m_{ik})m_{kt}e_j + e_im_{ik}D(m_{kt})e_j + e_iD(m_{kt})m_{ik}e_j + e_im_{kt}D(m_{ik})e_j = e_im_{ik}D(m_{kt})e_j.$$

Therefore, $D_{ij}(m_{ik}m_{kt}) = m_{ik}D_{kj}(m_{kt})$ for all $1 \leq i \leq k < t < j \leq n$ or $1 \leq i < k \leq t < j \leq n$ and $m_{ik} \in T_{ik}, m_{kt} \in T_{kt}$, which completes the proof. □

Lemma 2.12. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then*

$$D(m_{ii}m_{in}) = D(m_{ii})m_{in} + m_{ii}D(m_{in})$$

for all $m_{ii} \in R_i, m_{in} \in M_{in}$, and $1 \leq i \leq n-1$.

Proof. Since $m_{in}m_{ii} = 0$ for $1 \leq i \leq n-1$, we write

$$D(m_{ii}m_{in}) = D(m_{ii}m_{in} + m_{in}m_{ii}) = D(m_{ii})m_{in} + m_{ii}D(m_{in}) + D(m_{in})m_{ii} + m_{in}D(m_{ii}),$$

$1 \leq i \leq n-1$. Furthermore, as $D(m_{in}) \in T_{1n} + \cdots + T_{in}$, we obtain $D(m_{in})m_{ii} = 0$. Analogously, $m_{in}D(m_{ii}) = 0$. Therefore, we deduce

$$D(m_{ii}m_{in}) = D(m_{ii})m_{in} + m_{ii}D(m_{in}),\ 1 \leq i \leq n-1.$$ □

Lemma 2.13. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then*

$$D(m_{1n}m_{nn}) = D(m_{1n})m_{nn} + m_{1n}D(m_{nn})$$

for all $m_{nn} \in R_n$ and $m_{1n} \in M_{1n}$.

Proof. Since $m_{nn}m_{1n} = 0$, we obtain

$$D(m_{1n}m_{nn}) = D(m_{1n}m_{nn} + m_{nn}m_{1n}) = D(m_{1n})m_{nn} + m_{1n}D(m_{nn}) + D(m_{nn})m_{1n} + m_{nn}D(m_{1n}).$$

But, $D(m_{nn}) \in T_{1n} + \cdots + T_{nn}$, and hence, $D(m_{nn})m_{1n} = 0$. Similarly, we have $m_{nn}D(m_{1n}) = 0$. Consequently,

$$D(m_{1n}m_{nn}) = D(m_{1n})m_{nn} + m_{1n}D(m_{nn}).$$ □

Lemma 2.14. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then*

$$D_{ii}(m_{ii}k_{ii}) = D_{ii}(m_{ii})k_{ii} + m_{ii}D_{ii}(k_{ii})$$

for all $m_{ii}, k_{ii} \in R_i$ with $1 \leq i \leq n$.

Proof. By Lemma 2.12, we deduce $D_{in}(m_{ii}k_{ii}m_{in}) = D_{ii}(m_{ii}k_{ii})m_{in} + m_{ii}k_{ii}D_{in}(m_{in})$ for any $1 \leq i \leq n-1$, where $m_{in} \in M_{in}$. On the other side, again with the help of Lemma 2.12, we write

$$D_{in}(m_{ii}k_{ii}m_{in}) = D_{ii}(m_{ii})k_{ii}m_{in} + m_{ii}D_{in}(k_{ii}m_{in}) = D_{ii}(m_{ii})k_{ii}m_{in} + m_{ii}D_{ii}(k_{ii})m_{in} + m_{ii}k_{ii}D_{in}(m_{in})$$

for any $1 \leq i \leq n-1$, where $m_{in} \in M_{in}$. So, we deduce

$$D_{ii}(m_{ii}k_{ii})m_{in} = D_{ii}(m_{ii})k_{ii}m_{in} + m_{ii}D_{ii}(k_{ii})m_{in},$$

where $1 \leq i \leq n-1$. However, as M_{in} is a faithful left R_i-module, it holds that $D_{ii}(m_{ii}k_{ii}) = D_{ii}(m_{ii})k_{ii} + m_{ii}D_{ii}(k_{ii})$ for all $m_{ii}, k_{ii} \in R_i$ with $1 \leq i \leq n-1$.

Next, assume that $m_{nn}, k_{nn} \in R_n$ and fix arbitrary $m_{1n} \in M_{1n}$. Employing Lemma 2.13, we deduce

$$D_{1n}(m_{1n}m_{nn}k_{nn}) = D_{1n}(m_{1n})m_{nn}k_{nn} + m_{1n}D_{nn}(m_{nn}k_{nn}).$$

Furthermore, again Lemma 2.13 is applicable to extract that

$$D_{1n}(m_{1n}m_{nn}k_{nn}) = D_{1n}(m_{1n}m_{nn})k_{nn} + m_{1n}m_{nn}D_{nn}(k_{nn}) = D_{1n}(m_{1n})m_{nn}k_{nn} + m_{1n}D_{nn}(m_{nn})k_{nn} + m_{1n}m_{nn}D_{nn}(k_{nn}),$$

and hence,

$$m_{1n}D_{nn}(m_{nn}k_{nn}) = m_{1n}D_{nn}(m_{nn})k_{nn} + m_{1n}m_{nn}D_{nn}(k_{nn}).$$

Since M_{1n} is a faithful right R_n-module, we deduce $D_{nn}(m_{nn}k_{nn}) = D_{nn}(m_{nn})k_{nn} + m_{nn}D_{nn}(k_{nn})$, as claimed. □

We now present the key lemma that establishes the validity of our main result.

Lemma 2.15. *Let $D : T \longrightarrow T$ be an additive map. If D is a Jordan derivation, then $(D(A)B)_{ij} + (AD(B))_{ij} = D_{ij}(AB)$ for all $1 \leq i < j \leq n$ and $A, B \in T$.*

Proof. Assume that $A = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$ and $B = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}$, where $a_{ij} = e_i a_{ij} e_j$ and $b_{ij} = e_i b_{ij} e_j$, $1 \leq i, j \leq n$. Then,

$$AB = \sum_{t=1}^{n}\sum_{k=1}^{n}(ab)_{tk},$$

where $t \leq k$ and $(ab)_{tk} = a_{tt}b_{tk} + a_{tt+1}b_{t+1k} + \cdots + a_{tk}b_{kk}$. Note that

$$(D(A)B)_{ij} = e_i D(A)Be_j = e_i D(A)(b_{1j} + b_{2j} + \cdots + b_{jj})e_j.$$

As $e_i D_{ks}(A) = 0$ for every $k \neq i$, we deduce

$$(D(A)B)_{ij} = e_i(D_{ii}(A) + D_{ii+1}(A) + \cdots + D_{in}(A))(b_{1j} + b_{2j} + \cdots + b_{jj})e_j.$$

Since $D_{ij}(X) = e_i D_{ij}(X) e_j$, $1 \leq i, j \leq n$, for $X \in T$, we write

$$(D(A)B)_{ij} = \sum_{t=i}^{j} D_{it}(A)b_{tj}e_j = \sum_{t=i}^{j} D_{it}\left(\sum_{s=i}^{t} a_{is} + \sum_{k=i+1}^{t} a_{kt}\right)b_{tj}.$$

By Lemmas 2.9 and 2.10, we deduce

$$(D(A)B)_{ij} = \sum_{t=i}^{j} D_{it}\left(\sum_{s=i}^{t} a_{is}\right)b_{tj} + \sum_{t=i}^{j} D_{ij}\left(\sum_{k=i+1}^{t} a_{kt}b_{tj}\right).$$

On the other hand, we obtain

$$(AD(B))_{ij} = e_i AD(B)e_j = \left(\sum_{t=i}^{n} a_{it}\right)D(B)e_j.$$

Since $D_{st}(B)e_j = 0$ for every $t \neq j$, we deduce that

$$(AD(B))_{ij} = \left(\sum_{t=i}^{n} a_{it}\right)\left(\sum_{s=1}^{j} D_{sj}(B)\right)e_j = \sum_{k=i}^{j} a_{ik}D_{kj}(B).$$

We note that $D_{ij}(b_{ts})$ can be nonzero only if $t = i$ or $s = j$, for all $1 \leq i \leq j \leq n$ and $1 \leq t \leq s \leq n$. Thus, it follows that

$$(AD(B))_{ij} = \sum_{t=i}^{j} a_{it}D_{tj}\left(\sum_{k=t}^{j} b_{tk} + \sum_{s=t+1}^{j} b_{sj}\right) = \sum_{s=i}^{j}\sum_{t=i}^{s} a_{it}D_{tj}(b_{sj}) + \sum_{s=i}^{j-1}\left(a_{is}D_{sj}\left(\sum_{m=s}^{j-1} b_{sm}\right)\right).$$

By Lemmas 2.11 and 2.14, we deduce

$$(AD(B))_{ij} = \sum_{s=i}^{j}\sum_{t=i}^{s} a_{it}D_{tj}(b_{sj}) + \sum_{s=i}^{j-1} D_{sj}\left(a_{is}\sum_{m=s}^{j-1} b_{sm}\right).$$

So, it holds that

$$(AD(B))_{ij} = \sum_{s=i}^{j}\sum_{t=i}^{s} a_{it}D_{tj}(b_{sj}) + \sum_{k=i}^{j-1}\sum_{t=i}^{k} D_{ij}(a_{it}b_{tk}).$$

Thus, we deduce

$$(D(A)B)_{ij} + (AD(B))_{ij} = \sum_{t=i}^{j} D_{it}\left(\sum_{s=i}^{t} a_{is}\right)b_{tj} + \sum_{t=i}^{j} D_{ij}\left(\sum_{k=i+1}^{t} a_{kt}b_{tj}\right) + \sum_{s=i}^{j}\sum_{t=i}^{s} a_{it}D_{tj}(b_{sj}) + \sum_{k=i}^{j-1}\sum_{t=i}^{k} D_{ij}(a_{it}b_{tk}).$$

Furthermore, using Lemmas 2.8 and 2.14, we obtain

$$(D(A)B)_{ij} + (AD(B))_{ij} = \sum_{t=i}^{j} D_{ij}(a_{it}b_{tj}) + \sum_{t=i}^{j} D_{ij}\left(\sum_{k=i+1}^{t} a_{kt}b_{tj}\right) + \sum_{k=i}^{j-1}\sum_{t=i}^{k} D_{ij}(a_{it}b_{tk}).$$

Consequently, we have

$$\begin{aligned}(D(A)B)_{ij} + (AD(B))_{ij} = {} & D_{ij}((AB)_{ij}) + D_{ij}((AB)_{i+1j}) + \cdots + D_{ij}((AB)_{jj}) + D_{ij}((AB)_{ii}) \\ & + D_{ij}((AB)_{ii+1}) + \cdots + D_{ij}((AB)_{ij-1}).\end{aligned}$$

Since $D_{ij}((AB)_{st}) = 0$ whenever $s \neq i$ and $t \neq j$, we conclude that

$$(D(A)B)_{ij} + (AD(B))_{ij} = D_{ij}(AB). \qquad \square$$

Lemma 2.15 immediately yields the following main result, which is the primary motivation for the present work.

Theorem 2.1. *Let M_{in} be a faithful left R_i-module for each $2 \leq i \leq n-1$, and M_{1n} be both a faithful left R_1-module and a faithful right R_n-module, where $n \geq 2$. If $D : T \longrightarrow T$ is an additive map, then D is a Jordan derivation if and only if it is a derivation.*

Assume now that K is a commutative 2-torsion-free ring, that both $\mathcal{A}$ and $\mathcal{B}$ are algebras over K, and that $\mathcal{M}$ is a unital $(\mathcal{A},\mathcal{B})$-bimodule. Then, the following three assertions hold.

Corollary 2.5. [9, Theorem 3.1] *If M is faithful as a left A-module and as a right B-module, then any Jordan derivation of the generalized upper triangular matrix ring*

$$\begin{pmatrix}\mathcal{A} & \mathcal{M}\\ 0 & \mathcal{B}\end{pmatrix}$$

is an ordinary derivation.

Corollary 2.6. *If $S \neq 4S$ is a ring, then any Jordan derivation of the generalized upper triangular matrix ring*

$$\begin{pmatrix}S/4S & S/4S\\ 0 & S/4S\end{pmatrix}$$

is an ordinary derivation.

We observe that the ring $T_2(S/4S)$ is not 2-torsion free, and hence, Corollary 2.6 is not deducible from [9, Theorem 3.1].

Corollary 2.7. *Let S be a ring. Let*

$$T_n(S) := \left\{\begin{pmatrix}a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ 0 & a_{22} & a_{23} & \cdots & a_{2n}\\ \vdots & \ddots & \ddots & \vdots & \\ 0 & \cdots & & 0 & a_{nn}\end{pmatrix},\ a_{ij} \in S,\ 1 \leq i \leq j \leq n\right\}$$

be the upper triangular matrix ring over S. Then, any Jordan derivation on $T_n(S)$ is an ordinary derivation whenever $n \geq 2$.

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