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Research Article

On Ostrowski-Type Inequalities for Multiplicative Atangana–Baleanu Fractional Integral Operators

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Abstract

In this paper, we investigate Ostrowski-type inequalities within the framework of multiplicative Atangana–Baleanu fractional calculus. By employing the logarithmic correspondence between ordinary and multiplicative calculus, an Atangana–Baleanu Ostrowski identity is formulated in multiplicative form. Based on this formulation, several estimates are established for positive multiplicatively differentiable functions under convexity assumptions imposed on $|\ln f^*|$ and $|\ln f^*|^q$. Direct convexity arguments, the power-mean inequality, and Young's inequality are employed to derive the corresponding bounds. As consequences of the main results, midpoint-type estimates and limiting cases corresponding to $\alpha = 1$ are also obtained. The results are compared with previously established Atangana–Baleanu and multiplicative Ostrowski inequalities, thereby clarifying the relationship between the additive and multiplicative formulations.

Keywords: multiplicative Atangana–Baleanu fractional integral; Ostrowski-type inequality; multiplicative fractional calculus; convex functions; multiplicative derivative.

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1. Introduction

Fractional calculus has become an active area of research due to its wide range of applications in mathematics, physics, engineering, economics, and biological sciences [3, 12, 21, 23]. In recent years, numerous fractional integral and derivative operators have been introduced in order to model complex phenomena more accurately than their classical counterparts [3, 4, 10, 12, 21, 26, 27]. Among these operators, the Atangana–Baleanu fractional operators have attracted considerable attention because of their non-singular and non-local kernels, which provide a useful framework for describing memory and hereditary effects in various applied problems [3, 4, 13, 25].

On the other hand, multiplicative calculus, originally developed as an alternative to classical Newtonian calculus, provides a natural framework for studying multiplicative growth processes and proportional changes. Unlike ordinary calculus, multiplicative calculus is particularly suited to positive-valued functions and has found applications in economics, finance, information theory, and various branches of mathematical analysis [5, 20]. The notions of multiplicative derivative and multiplicative integral have led to the development of a theory parallel to classical calculus and have recently been employed in the study of generalized convexity and fractional integral inequalities [2, 16, 18, 20].

The interaction between multiplicative calculus and fractional calculus has recently attracted increasing attention [2, 5, 7, 20]. Several authors have introduced multiplicative versions of fractional integral operators and established various inequalities within this setting. In particular, Hermite–Hadamard, Ostrowski, midpoint, trapezoid, and Simpson-type inequalities have been extensively investigated because of their importance in approximation theory, numerical analysis, and error estimation [1, 2, 6, 8, 9, 16–19, 22, 24]. These inequalities provide quantitative estimates for the deviation between function values and suitable fractional integral means.

Motivated by these developments, researchers have recently considered multiplicative fractional integral operators associated with the Atangana–Baleanu fractional calculus. In particular, Du et al. [7] established Hermite–Hadamard-type inequalities via multiplicative Atangana–Baleanu fractional integral operators, while Long and Du [14] investigated multiplicative k-Atangana–Baleanu fractional integrals and their applications to Mercer-type inequalities. Moreover, inequalities involving Atangana–Baleanu fractional integral operators have been investigated for several classes of convex functions [13, 25].

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Closely related Ostrowski-type results have also been obtained in both the additive Atangana–Baleanu and multiplicative fractional settings. Meftah et al. [15] established fractional multiplicative Ostrowski-type inequalities for multiplicatively differentiable convex functions and derived several particular and midpoint-type estimates. More recently, Kalsoom and Khan [11] investigated Ostrowski-type inequalities involving the Atangana–Baleanu fractional integral operator. They established an Atangana–Baleanu Ostrowski identity and used it, together with preinvexity assumptions, to derive estimates based on Hölder's inequality, the power-mean inequality, Young's inequality, and Jensen's inequality. These results provide two closely related frameworks for the present investigation: multiplicative fractional Ostrowski inequalities on the one hand and Atangana–Baleanu Ostrowski inequalities on the other.

In view of these contributions, the purpose of the present paper is to investigate the connection between these two settings within multiplicative Atangana–Baleanu fractional calculus. By using the logarithmic correspondence between ordinary and multiplicative calculus, the Atangana–Baleanu Ostrowski identity is formulated in multiplicative form, and its relation to the additive identity established by Kalsoom and Khan [11] is discussed explicitly. Based on this formulation, Ostrowski-type estimates involving $|\ln f^*|$ and $|\ln f^*|^q$ are studied under suitable convexity assumptions. Direct convexity arguments, the power-mean inequality, and Young's inequality are then employed to obtain the corresponding bounds. In addition, midpoint-type estimates and the limiting case $\alpha = 1$ are considered, and their relationship with previously established multiplicative Ostrowski inequalities is discussed.

The paper is organized as follows. In Section 2, the basic concepts of multiplicative calculus, Riemann–Liouville fractional integrals, and Atangana–Baleanu fractional integral operators required in the sequel are recalled. Section 3 presents the multiplicative formulation of the Atangana–Baleanu Ostrowski identity and develops the corresponding Ostrowski-type estimates under different convexity assumptions. Several particular and limiting cases, together with their connections to related results in the literature, are discussed in Section 4. Finally, concluding remarks are given in Section 5.

2. Preliminaries

In this section, we recall several notions from multiplicative calculus and fractional calculus that will be used throughout the paper. Most of the notions related to multiplicative Atangana–Baleanu fractional integral operators recalled in this section follow the framework introduced by Du et al. [7], whereas the classical Atangana–Baleanu fractional integral is recalled from Atangana and Baleanu [3].

Definition 2.1 (see [7]). *Let $f : (b_1, b_2) \to (0, \infty)$ be a positive function. The multiplicative derivative of f at $x \in (b_1, b_2)$ is defined by*

$$f^*(x) = \lim_{h \to 0} \left(\frac{f(x+h)}{f(x)} \right)^{\frac{1}{h}},$$

provided that the limit exists.

If f is differentiable in the ordinary sense, then

$$f^*(x) = \exp\left(\frac{f'(x)}{f(x)} \right),$$

and consequently

$$\ln f^*(x) = \frac{f'(x)}{f(x)} = \frac{d}{dx} \ln f(x).$$

Definition 2.2 (see [7]). *A function $f : I \to (0, \infty)$ is said to be multiplicatively convex on an interval I if*

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}$$

for all $x, y \in I$ and $t \in [0, 1]$.

It is well known that a positive function f is multiplicatively convex if and only if $\ln f$ is convex in the ordinary sense [16, 20].

Definition 2.3 (see [12, 23]). *Let $f \in L^1[b_1, b_2]$ and let $\alpha > 0$. The left-sided Riemann–Liouville fractional integral of order α is defined by*

$$I^{\alpha}_{b_1+} f(x) = \frac{1}{\Gamma(\alpha)} \int_{b_1}^{x} (x-t)^{\alpha-1} f(t)\, dt, \qquad x \in (b_1, b_2],$$

whereas the right-sided Riemann–Liouville fractional integral is defined by

$$I^{\alpha}_{b_2-} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b_2} (t-x)^{\alpha-1} f(t)\, dt, \qquad x \in [b_1, b_2).$$

For $b_1 < z < b_2$, we shall also use the following notation on the subintervals $[b_1, z]$ and $[z, b_2]$:

$$I^{\alpha}_{z^-} f(b_1) := \frac{1}{\Gamma(\alpha)} \int_{b_1}^{z} (x - b_1)^{\alpha-1} f(x)\, dx,$$

and

$$I^{\alpha}_{z^+} f(b_2) := \frac{1}{\Gamma(\alpha)} \int_{z}^{b_2} (b_2 - x)^{\alpha-1} f(x)\, dx.$$

These notations will be used in the proof of the auxiliary identity in the next section.

Definition 2.4 (see [3, 7]). *Let $0 < \alpha \leq 1$ and let $f \in L^1[b_1, b_2]$. The left-sided Atangana–Baleanu fractional integral is defined by*

$${}^{AB}I^{\alpha}_{b_1} f(x) = \frac{1-\alpha}{B(\alpha)} f(x) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{b_1}^{x} (x - t)^{\alpha-1} f(t)\, dt.$$

Similarly, the right-sided Atangana–Baleanu fractional integral is defined by

$${}^{AB}I^{\alpha}_{b_2} f(x) = \frac{1-\alpha}{B(\alpha)} f(x) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{x}^{b_2} (t - x)^{\alpha-1} f(t)\, dt.$$

Throughout this paper, $B : [0, 1] \to (0, \infty)$ denotes the normalization function associated with the Atangana–Baleanu fractional integral operator, satisfying $B(0) = B(1) = 1$. In particular, $B(\alpha) \neq 0$ for every $\alpha \in [0, 1]$.

For $b_1 < z < b_2$, consistently with the preceding notation, we write

$${}^{AB}I^{\alpha}_{z^-} f(b_1) = \frac{1-\alpha}{B(\alpha)} f(b_1) + \frac{\alpha}{B(\alpha)} I^{\alpha}_{z^-} f(b_1),$$

and

$${}^{AB}I^{\alpha}_{z^+} f(b_2) = \frac{1-\alpha}{B(\alpha)} f(b_2) + \frac{\alpha}{B(\alpha)} I^{\alpha}_{z^+} f(b_2).$$

Equivalently,

$${}^{AB}I^{\alpha}_{z^-} f(b_1) = \frac{1-\alpha}{B(\alpha)} f(b_1) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{b_1}^{z} (x - b_1)^{\alpha-1} f(x)\, dx,$$

and

$${}^{AB}I^{\alpha}_{z^+} f(b_2) = \frac{1-\alpha}{B(\alpha)} f(b_2) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{z}^{b_2} (b_2 - x)^{\alpha-1} f(x)\, dx.$$

For convenience, we denote by

$$L^*[b_1, b_2] = \left\{ g : [b_1, b_2] \to (0, \infty) : \ln g \in L^1[b_1, b_2] \right\}.$$

Thus, $L^*[b_1, b_2]$ denotes the class of multiplicatively integrable positive functions on $[b_1, b_2]$.

Definition 2.5 (see [7]). *Let $f \in L^*[b_1, b_2]$ and let $0 < \alpha \leq 1$. The left-sided multiplicative Atangana–Baleanu fractional integral is defined by*

$${}^{AB}I^{\alpha,*}_{b_1} f(x) = \exp\left({}^{AB}I^{\alpha}_{b_1} (\ln f)(x) \right),$$

while the right-sided multiplicative Atangana–Baleanu fractional integral is defined by

$${}^{AB}I^{\alpha,*}_{b_2} f(x) = \exp\left({}^{AB}I^{\alpha}_{b_2} (\ln f)(x) \right).$$

Accordingly, for $b_1 < z < b_2$, we use

$${}^{AB}I^{\alpha,*}_{z^-} f(b_1) := \exp\left({}^{AB}I^{\alpha}_{z^-} (\ln f)(b_1) \right),$$

and

$${}^{AB}I^{\alpha,*}_{z^+} f(b_2) := \exp\left({}^{AB}I^{\alpha}_{z^+} (\ln f)(b_2) \right).$$

Finally, for a positive function g, the multiplicative integral is given the following way:

$$\int_a^b g(t)^{\, dt} := \exp\left(\int_a^b \ln g(t)\, dt \right),$$

whenever $\ln g \in L^1[a, b]$.

3. Main Results

Before presenting the main estimates, we recall that Kalsoom and Khan [11] established an Ostrowski-type identity involving the ordinary Atangana–Baleanu fractional integral operator. The identity below is closely related to that result through the logarithmic correspondence between ordinary and multiplicative calculus. In particular, for $G = \ln f$, one has

$$G'(x) = \frac{f'(x)}{f(x)} = \ln f^*(x).$$

Accordingly, Lemma 3.1 is regarded here as the multiplicative formulation of the corresponding Atangana–Baleanu Ostrowski identity rather than as an independent new additive identity.

Lemma 3.1. *Let $f : [b_1, b_2] \to (0, \infty)$ be a function such that $\ln f$ is absolutely continuous on $[b_1, b_2]$ and $\ln f^* \in L^1[b_1, b_2]$. Let $0 < \alpha \leq 1$ and $z \in (b_1, b_2)$. Then, the following identity holds:*

$$\frac{[f(z)]^{\frac{(z-b_1)^\alpha + (b_2-z)^\alpha}{b_2-b_1}} [f(b_1)f(b_2)]^{\frac{(1-\alpha)\Gamma(\alpha)}{b_2-b_1}}}{\left[{}^{AB}I^{\alpha,*}_{z^-} f(b_1)\, {}^{AB}I^{\alpha,*}_{z^+} f(b_2)\right]^{\frac{B(\alpha)\Gamma(\alpha)}{b_2-b_1}}} = \frac{\left[\exp\left(\int_0^1 t^\alpha \ln f^*\big(tz + (1-t)b_1\big)\, dt\right)\right]^{\frac{(z-b_1)^{\alpha+1}}{b_2-b_1}}}{\left[\exp\left(\int_0^1 t^\alpha \ln f^*\big(tz + (1-t)b_2\big)\, dt\right)\right]^{\frac{(b_2-z)^{\alpha+1}}{b_2-b_1}}}.$$

Proof. Set

$$I_1 = \left[\exp\left(\int_0^1 t^\alpha \ln f^*\big(tz + (1-t)b_1\big)\, dt\right)\right]^{\frac{(z-b_1)^{\alpha+1}}{b_2-b_1}}.$$

Since

$$\ln f^*(x) = \frac{f'(x)}{f(x)} = \frac{d}{dx} \ln f(x)$$

for almost every $x \in [b_1, b_2]$, we have

$$\frac{d}{dt} \ln f\big(tz + (1-t)b_1\big) = (z - b_1) \ln f^*\big(tz + (1-t)b_1\big).$$

Therefore,

$$\int_0^1 t^\alpha \ln f^*\big(tz + (1-t)b_1\big)\, dt = \frac{1}{z-b_1} \int_0^1 t^\alpha \frac{d}{dt} \ln f\big(tz + (1-t)b_1\big)\, dt.$$

Applying integration by parts gives

$$\int_0^1 t^\alpha \ln f^*\big(tz + (1-t)b_1\big)\, dt = \frac{\ln f(z)}{z-b_1} - \frac{\alpha}{z-b_1} \int_0^1 t^{\alpha-1} \ln f\big(tz + (1-t)b_1\big)\, dt.$$

Consequently,

$$\ln I_1 = \frac{(z-b_1)^\alpha}{b_2-b_1} \ln f(z) - \frac{\alpha(z-b_1)^\alpha}{b_2-b_1} \int_0^1 t^{\alpha-1} \ln f\big(tz + (1-t)b_1\big)\, dt.$$

Using the change of variable

$$x = tz + (1-t)b_1, \qquad dt = \frac{dx}{z-b_1},$$

we obtain

$$\int_0^1 t^{\alpha-1} \ln f\big(tz + (1-t)b_1\big)\, dt = \frac{1}{(z-b_1)^\alpha} \int_{b_1}^z (x-b_1)^{\alpha-1} \ln f(x)\, dx$$

$$= \frac{\Gamma(\alpha)}{(z-b_1)^\alpha} I^\alpha_{z^-}(\ln f)(b_1).$$

It follows that

$$\ln I_1 = \frac{(z-b_1)^\alpha}{b_2-b_1} \ln f(z) - \frac{\Gamma(\alpha+1)}{b_2-b_1} I^\alpha_{z^-}(\ln f)(b_1).$$

From

$${}^{AB}I^\alpha_{z^-} g(b_1) = \frac{1-\alpha}{B(\alpha)} g(b_1) + \frac{\alpha}{B(\alpha)} I^\alpha_{z^-} g(b_1),$$

we have

$$\alpha I^\alpha_{z^-} g(b_1) = B(\alpha)\, {}^{AB}I^\alpha_{z^-} g(b_1) - (1-\alpha) g(b_1).$$

Hence,

$$\ln I_1 = \frac{(z-b_1)^\alpha}{b_2-b_1} \ln f(z) + \frac{(1-\alpha)\Gamma(\alpha)}{b_2-b_1} \ln f(b_1) - \frac{B(\alpha)\Gamma(\alpha)}{b_2-b_1}\, {}^{AB}I^\alpha_{z^-}(\ln f)(b_1).$$

By the definition of the multiplicative Atangana–Baleanu fractional integral,

$$ {}^{AB}I^{\alpha,*}_{z^-} f(b_1) = \exp\left({}^{AB}I^{\alpha}_{z^-}(\ln f)(b_1)\right).$$

Therefore,

$$I_1 = \frac{[f(z)]^{\frac{(z-b_1)^\alpha}{b_2-b_1}}[f(b_1)]^{\frac{(1-\alpha)\Gamma(\alpha)}{b_2-b_1}}}{\left[{}^{AB}I^{\alpha,*}_{z^-} f(b_1)\right]^{\frac{B(\alpha)\Gamma(\alpha)}{b_2-b_1}}}.$$

Now, set

$$I_2 = \left[\exp\left(\int_0^1 t^\alpha \ln f^*\big(tz+(1-t)b_2\big)\,dt\right)\right]^{\frac{(b_2-z)^{\alpha+1}}{b_2-b_1}}.$$

Since

$$\frac{d}{dt}\ln f\big(tz+(1-t)b_2\big) = -(b_2-z)\ln f^*\big(tz+(1-t)b_2\big),$$

integration by parts yields

$$\int_0^1 t^\alpha \ln f^*\big(tz+(1-t)b_2\big)\,dt = -\frac{\ln f(z)}{b_2-z} + \frac{\alpha}{b_2-z}\int_0^1 t^{\alpha-1}\ln f\big(tz+(1-t)b_2\big)\,dt.$$

Using the change of variable

$$x = tz+(1-t)b_2, \qquad dt = -\frac{dx}{b_2-z},$$

we obtain

$$\begin{aligned}\int_0^1 t^{\alpha-1}\ln f\big(tz+(1-t)b_2\big)\,dt &= \frac{1}{(b_2-z)^\alpha}\int_z^{b_2}(b_2-x)^{\alpha-1}\ln f(x)\,dx\\ &= \frac{\Gamma(\alpha)}{(b_2-z)^\alpha} I^{\alpha}_{z^+}(\ln f)(b_2).\end{aligned}$$

Therefore,

$$\ln I_2 = -\frac{(b_2-z)^\alpha}{b_2-b_1}\ln f(z) + \frac{\Gamma(\alpha+1)}{b_2-b_1} I^{\alpha}_{z^+}(\ln f)(b_2).$$

Using

$${}^{AB}I^{\alpha}_{z^+} g(b_2) = \frac{1-\alpha}{B(\alpha)} g(b_2) + \frac{\alpha}{B(\alpha)} I^{\alpha}_{z^+} g(b_2),$$

we get

$$\alpha I^{\alpha}_{z^+} g(b_2) = B(\alpha)\,{}^{AB}I^{\alpha}_{z^+} g(b_2) - (1-\alpha)g(b_2).$$

Thus,

$$\ln I_2 = -\frac{(b_2-z)^\alpha}{b_2-b_1}\ln f(z) - \frac{(1-\alpha)\Gamma(\alpha)}{b_2-b_1}\ln f(b_2) + \frac{B(\alpha)\Gamma(\alpha)}{b_2-b_1}\,{}^{AB}I^{\alpha}_{z^+}(\ln f)(b_2).$$

Hence,

$$I_2 = \frac{\left[{}^{AB}I^{\alpha,*}_{z^+} f(b_2)\right]^{\frac{B(\alpha)\Gamma(\alpha)}{b_2-b_1}}}{[f(z)]^{\frac{(b_2-z)^\alpha}{b_2-b_1}}[f(b_2)]^{\frac{(1-\alpha)\Gamma(\alpha)}{b_2-b_1}}}.$$

Finally, dividing I_1 by I_2, we obtain

$$\frac{I_1}{I_2} = \frac{[f(z)]^{\frac{(z-b_1)^\alpha+(b_2-z)^\alpha}{b_2-b_1}}[f(b_1)f(b_2)]^{\frac{(1-\alpha)\Gamma(\alpha)}{b_2-b_1}}}{\left[{}^{AB}I^{\alpha,*}_{z^-} f(b_1)\,{}^{AB}I^{\alpha,*}_{z^+} f(b_2)\right]^{\frac{B(\alpha)\Gamma(\alpha)}{b_2-b_1}}},$$

which proves the desired identity. □

Remark 3.1. *The role of Lemma 3.1 is to express the underlying Atangana–Baleanu Ostrowski structure in terms of multiplicative fractional integrals and the multiplicative derivative. This logarithmic correspondence will be used throughout the subsequent estimates.*

Theorem 3.1. *Let $f : [b_1, b_2] \to (0, \infty)$ be a function such that $\ln f \in AC[b_1, b_2]$. Assume that f is multiplicatively differentiable on (b_1, b_2) and that f^* admits a positive extension to $[b_1, b_2]$ satisfying $f^* \in L^*[b_1, b_2]$. Suppose that $|\ln f^*|$ is convex on $[b_1, b_2]$. Then, for $z \in (b_1, b_2)$ and $0 < \alpha \leq 1$, it holds that*

$$|\ln \Omega^*_{AB}(f; z)| \leq \frac{(z-b_1)^{\alpha+1}}{b_2-b_1}\left[\frac{|\ln f^*(z)|}{\alpha+2} + \frac{|\ln f^*(b_1)|}{(\alpha+1)(\alpha+2)}\right] + \frac{(b_2-z)^{\alpha+1}}{b_2-b_1}\left[\frac{|\ln f^*(z)|}{\alpha+2} + \frac{|\ln f^*(b_2)|}{(\alpha+1)(\alpha+2)}\right],$$

where

$$\Omega^*_{AB}(f; z) = \frac{[f(z)]^{\frac{(z-b_1)^\alpha + (b_2-z)^\alpha}{b_2-b_1}} [f(b_1)f(b_2)]^{\frac{(1-\alpha)\Gamma(\alpha)}{b_2-b_1}}}{\left[{}^{AB}I^{\alpha,*}_{z^-} f(b_1)\, {}^{AB}I^{\alpha,*}_{z^+} f(b_2)\right]^{\frac{B(\alpha)\Gamma(\alpha)}{b_2-b_1}}}.$$

Proof. By Lemma 3.1, taking logarithms gives

$$\ln \Omega^*_{AB}(f; z) = \frac{(z-b_1)^{\alpha+1}}{b_2-b_1}\int_0^1 t^\alpha \ln f^*\big(tz + (1-t)b_1\big)\, dt - \frac{(b_2-z)^{\alpha+1}}{b_2-b_1}\int_0^1 t^\alpha \ln f^*\big(tz + (1-t)b_2\big)\, dt.$$

Hence, by the triangle inequality,

$$|\ln \Omega^*_{AB}(f; z)| \leq \frac{(z-b_1)^{\alpha+1}}{b_2-b_1}\int_0^1 t^\alpha \left|\ln f^*\big(tz + (1-t)b_1\big)\right| dt + \frac{(b_2-z)^{\alpha+1}}{b_2-b_1}\int_0^1 t^\alpha \left|\ln f^*\big(tz + (1-t)b_2\big)\right| dt.$$

Since $|\ln f^*|$ is convex on $[b_1, b_2]$, we have

$$\left|\ln f^*\big(tz + (1-t)b_1\big)\right| \leq t|\ln f^*(z)| + (1-t)|\ln f^*(b_1)|$$

and

$$\left|\ln f^*\big(tz + (1-t)b_2\big)\right| \leq t|\ln f^*(z)| + (1-t)|\ln f^*(b_2)|.$$

Therefore,

$$\int_0^1 t^\alpha \left|\ln f^*\big(tz + (1-t)b_1\big)\right| dt \leq |\ln f^*(z)|\int_0^1 t^{\alpha+1}\, dt + |\ln f^*(b_1)|\int_0^1 t^\alpha(1-t)\, dt,$$

and similarly,

$$\int_0^1 t^\alpha \left|\ln f^*\big(tz + (1-t)b_2\big)\right| dt \leq |\ln f^*(z)|\int_0^1 t^{\alpha+1}\, dt + |\ln f^*(b_2)|\int_0^1 t^\alpha(1-t)\, dt.$$

Using

$$\int_0^1 t^{\alpha+1}\, dt = \frac{1}{\alpha+2}$$

and

$$\int_0^1 t^\alpha(1-t)\, dt = \frac{1}{(\alpha+1)(\alpha+2)},$$

we obtain the desired inequality. □

Remark 3.2. *Theorem 3.1 should be interpreted in connection with the ordinary Atangana–Baleanu estimates obtained by Kalsoom and Khan [11]. Their corresponding result is formulated for the derivative of an ordinary function under a preinvexity assumption, whereas the present estimate is expressed in the multiplicative framework through the logarithmic derivative $|\ln f^*|$ and ordinary convexity. Thus, the result provides the corresponding multiplicative formulation under the hypotheses adopted in the present paper.*

Corollary 3.1. *Let $f : [b_1, b_2] \to (0, \infty)$ be a function such that $\ln f \in AC[b_1, b_2]$. Assume that f is multiplicatively differentiable on (b_1, b_2) and that f^* admits a positive extension to $[b_1, b_2]$ satisfying $f^* \in L^*[b_1, b_2]$. If f^* is multiplicatively convex on $[b_1, b_2]$ and $f^*(x) \geq 1$ for all $x \in [b_1, b_2]$, then, for $z \in (b_1, b_2)$ and $0 < \alpha \leq 1$, it holds that*

$$|\ln \Omega^*_{AB}(f; z)| \leq \frac{(z-b_1)^{\alpha+1}}{b_2-b_1}\left[\frac{\ln f^*(z)}{\alpha+2} + \frac{\ln f^*(b_1)}{(\alpha+1)(\alpha+2)}\right] + \frac{(b_2-z)^{\alpha+1}}{b_2-b_1}\left[\frac{\ln f^*(z)}{\alpha+2} + \frac{\ln f^*(b_2)}{(\alpha+1)(\alpha+2)}\right],$$

where

$$\Omega^*_{AB}(f; z) = \frac{[f(z)]^{\frac{(z-b_1)^\alpha + (b_2-z)^\alpha}{b_2-b_1}} [f(b_1)f(b_2)]^{\frac{(1-\alpha)\Gamma(\alpha)}{b_2-b_1}}}{\left[{}^{AB}I^{\alpha,*}_{z^-} f(b_1)\, {}^{AB}I^{\alpha,*}_{z^+} f(b_2)\right]^{\frac{B(\alpha)\Gamma(\alpha)}{b_2-b_1}}}.$$

Proof. Since f^* is multiplicatively convex, $\ln f^*$ is convex on $[b_1, b_2]$. Moreover, the assumption $f^*(x) \geq 1$ implies that $\ln f^*(x) \geq 0$ for every $x \in [b_1, b_2]$. Hence,

$$|\ln f^*(x)| = \ln f^*(x), \qquad x \in [b_1, b_2].$$

Therefore, the assertion follows directly from Theorem 3.1. □

Theorem 3.2. *Let $f : [b_1, b_2] \to (0, \infty)$ be a function such that $\ln f \in AC[b_1, b_2]$. Assume that f is multiplicatively differentiable on (b_1, b_2) and that f^* admits a positive extension to $[b_1, b_2]$ satisfying $f^* \in L^*[b_1, b_2]$. Suppose that $|\ln f^*|^q$ is convex on $[b_1, b_2]$ for some $q > 1$. Then, for $z \in (b_1, b_2)$ and $0 < \alpha \leq 1$, the following inequality holds:*

$$|\ln \Omega^*_{AB}(f; z)| \leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \frac{(z-b_1)^{\alpha+1}}{b_2-b_1} \left[\frac{|\ln f^*(z)|^q}{\alpha+2} + \frac{|\ln f^*(b_1)|^q}{(\alpha+1)(\alpha+2)}\right]^{\frac{1}{q}}$$
$$+ \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \frac{(b_2-z)^{\alpha+1}}{b_2-b_1} \left[\frac{|\ln f^*(z)|^q}{\alpha+2} + \frac{|\ln f^*(b_2)|^q}{(\alpha+1)(\alpha+2)}\right]^{\frac{1}{q}},$$

where $\Omega^*_{AB}(f; z)$ is defined as in Theorem 3.1.

Proof. By Lemma 3.1, taking logarithms and using the triangle inequality, we obtain

$$|\ln \Omega^*_{AB}(f; z)| \leq \frac{(z-b_1)^{\alpha+1}}{b_2-b_1} \int_0^1 t^\alpha \left|\ln f^*\big(tz + (1-t)b_1\big)\right| dt + \frac{(b_2-z)^{\alpha+1}}{b_2-b_1} \int_0^1 t^\alpha \left|\ln f^*\big(tz + (1-t)b_2\big)\right| dt.$$

Applying the power-mean inequality to the first integral gives

$$\int_0^1 t^\alpha \left|\ln f^*\big(tz + (1-t)b_1\big)\right| dt \leq \left(\int_0^1 t^\alpha \, dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t^\alpha \left|\ln f^*\big(tz + (1-t)b_1\big)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|\ln f^*|^q$ is convex on $[b_1, b_2]$, we have

$$\left|\ln f^*\big(tz + (1-t)b_1\big)\right|^q \leq t|\ln f^*(z)|^q + (1-t)|\ln f^*(b_1)|^q.$$

Therefore,

$$\int_0^1 t^\alpha \left|\ln f^*\big(tz + (1-t)b_1\big)\right|^q dt \leq |\ln f^*(z)|^q \int_0^1 t^{\alpha+1} dt + |\ln f^*(b_1)|^q \int_0^1 t^\alpha(1-t)\, dt$$
$$= \frac{|\ln f^*(z)|^q}{\alpha+2} + \frac{|\ln f^*(b_1)|^q}{(\alpha+1)(\alpha+2)}.$$

Since

$$\int_0^1 t^\alpha \, dt = \frac{1}{\alpha+1},$$

it follows that

$$\int_0^1 t^\alpha \left|\ln f^*\big(tz + (1-t)b_1\big)\right| dt \leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left[\frac{|\ln f^*(z)|^q}{\alpha+2} + \frac{|\ln f^*(b_1)|^q}{(\alpha+1)(\alpha+2)}\right]^{\frac{1}{q}}.$$

Similarly,

$$\int_0^1 t^\alpha \left|\ln f^*\big(tz + (1-t)b_2\big)\right| dt \leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left[\frac{|\ln f^*(z)|^q}{\alpha+2} + \frac{|\ln f^*(b_2)|^q}{(\alpha+1)(\alpha+2)}\right]^{\frac{1}{q}}.$$

Substituting these estimates into the preceding bound yields the desired inequality. □

Remark 3.3. *A power-mean estimate based on the ordinary Atangana–Baleanu Ostrowski identity was also obtained by Kalsoom and Khan [11] under a preinvexity assumption on the q-th power of the derivative. Theorem 3.2 is formulated in the multiplicative setting and involves the logarithmic derivative $|\ln f^*|^q$. Moreover, the resulting bound is expressed in terms of the value at the intermediate point z together with the corresponding endpoint values. Hence, the present estimate is stated in a different multiplicative parametrization and should be viewed as the corresponding power-mean formulation within the multiplicative Atangana–Baleanu framework.*

Theorem 3.3. *Let $f : [b_1, b_2] \to (0, \infty)$ be a function such that $\ln f \in AC[b_1, b_2]$. Assume that f is multiplicatively differentiable on (b_1, b_2) and that f^* admits a positive extension to $[b_1, b_2]$ satisfying $f^* \in L^*[b_1, b_2]$. Suppose that $|\ln f^*|^q$ is convex on $[b_1, b_2]$ for some $q > 1$, and let $p > 1$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, for $z \in (b_1, b_2)$ and $0 < \alpha \leq 1$, the following inequality holds:

$$|\ln \Omega^*_{AB}(f; z)| \leq \frac{(z - b_1)^{\alpha+1}}{b_2 - b_1} \left[\frac{1}{p(\alpha p + 1)} + \frac{|\ln f^*(z)|^q + |\ln f^*(b_1)|^q}{2q} \right]$$
$$+ \frac{(b_2 - z)^{\alpha+1}}{b_2 - b_1} \left[\frac{1}{p(\alpha p + 1)} + \frac{|\ln f^*(z)|^q + |\ln f^*(b_2)|^q}{2q} \right],$$

where $\Omega^*_{AB}(f; z)$ is defined as in Theorem 3.1.

Proof. By Lemma 3.1, taking logarithms and applying the triangle inequality, we obtain

$$|\ln \Omega^*_{AB}(f; z)| \leq \frac{(z - b_1)^{\alpha+1}}{b_2 - b_1} \int_0^1 t^\alpha \left| \ln f^*\big(tz + (1 - t)b_1\big) \right| \, dt$$
$$+ \frac{(b_2 - z)^{\alpha+1}}{b_2 - b_1} \int_0^1 t^\alpha \left| \ln f^*\big(tz + (1 - t)b_2\big) \right| \, dt.$$

By Young's inequality,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad a, b \geq 0,$$

with

$$a = t^\alpha, \qquad b = \left| \ln f^*\big(tz + (1 - t)b_1\big) \right|,$$

we obtain

$$\int_0^1 t^\alpha \left| \ln f^*\big(tz + (1 - t)b_1\big) \right| \, dt \leq \frac{1}{p} \int_0^1 t^{\alpha p} \, dt + \frac{1}{q} \int_0^1 \left| \ln f^*\big(tz + (1 - t)b_1\big) \right|^q \, dt.$$

Since $|\ln f^*|^q$ is convex on $[b_1, b_2]$, we have

$$\left| \ln f^*\big(tz + (1 - t)b_1\big) \right|^q \leq t |\ln f^*(z)|^q + (1 - t) |\ln f^*(b_1)|^q.$$

Therefore,

$$\int_0^1 \left| \ln f^*\big(tz + (1 - t)b_1\big) \right|^q \, dt \leq |\ln f^*(z)|^q \int_0^1 t \, dt + |\ln f^*(b_1)|^q \int_0^1 (1 - t) \, dt$$
$$= \frac{|\ln f^*(z)|^q + |\ln f^*(b_1)|^q}{2}.$$

Moreover,

$$\int_0^1 t^{\alpha p} \, dt = \frac{1}{\alpha p + 1}.$$

Hence,

$$\int_0^1 t^\alpha \left| \ln f^*\big(tz + (1 - t)b_1\big) \right| \, dt \leq \frac{1}{p(\alpha p + 1)} + \frac{|\ln f^*(z)|^q + |\ln f^*(b_1)|^q}{2q}.$$

Similarly,

$$\int_0^1 t^\alpha \left| \ln f^*\big(tz + (1 - t)b_2\big) \right| \, dt \leq \frac{1}{p(\alpha p + 1)} + \frac{|\ln f^*(z)|^q + |\ln f^*(b_2)|^q}{2q}.$$

Substituting the last two estimates into the preceding inequality completes the proof. □

Remark 3.4. *A related Young-type estimate was obtained by Kalsoom and Khan [11] in the ordinary Atangana–Baleanu setting under a preinvexity assumption on $|Z'|^q$. In contrast, Theorem 3.3 is expressed in the multiplicative framework through the logarithmic derivative $|\ln f^*|^q$ and ordinary convexity. The resulting estimate involves the value at the intermediate point z together with the corresponding endpoint values. Thus, the theorem should be interpreted as the corresponding Young-type formulation within the multiplicative Atangana–Baleanu framework.*

4. Special Cases and Comparison with Previous Results

The following consequences are obtained from Theorem 3.1 by particular choices of the point z and the fractional order α.

Corollary 4.1. *Let the assumptions of Theorem 3.1 hold. Then*

$$\left|\ln \Omega^*_{AB}\left(f;\frac{b_1+b_2}{2}\right)\right| \leq \frac{(b_2-b_1)^\alpha}{2^{\alpha+1}}\left[\frac{2}{\alpha+2}\left|\ln f^*\left(\frac{b_1+b_2}{2}\right)\right| + \frac{|\ln f^*(b_1)|+|\ln f^*(b_2)|}{(\alpha+1)(\alpha+2)}\right].$$

Proof. Taking

$$z = \frac{b_1+b_2}{2}$$

in Theorem 3.1, we have

$$z-b_1 = b_2-z = \frac{b_2-b_1}{2}.$$

Substituting these relations into the estimate of Theorem 3.1 and simplifying yields the stated inequality. □

Corollary 4.2. *Let the assumptions of Theorem 3.1 hold. If $\alpha = 1$, then, for every $z \in (b_1, b_2)$,*

$$|\ln \Omega^*_{AB}(f;z)| \leq \frac{(z-b_1)^2}{b_2-b_1}\left[\frac{|\ln f^*(z)|}{3} + \frac{|\ln f^*(b_1)|}{6}\right] + \frac{(b_2-z)^2}{b_2-b_1}\left[\frac{|\ln f^*(z)|}{3} + \frac{|\ln f^*(b_2)|}{6}\right].$$

Proof. Taking $\alpha = 1$ in Theorem 3.1 and using

$$\frac{1}{\alpha+2} = \frac{1}{3}, \qquad \frac{1}{(\alpha+1)(\alpha+2)} = \frac{1}{6},$$

we obtain the desired result. □

Corollary 4.3. *Let the assumptions of Theorem 3.1 hold. If*

$$z = \frac{b_1+b_2}{2} \qquad \textit{and} \qquad \alpha = 1,$$

then

$$\left|\ln \Omega^*_{AB}\left(f;\frac{b_1+b_2}{2}\right)\right| \leq \frac{b_2-b_1}{24}\left[4\left|\ln f^*\left(\frac{b_1+b_2}{2}\right)\right| + |\ln f^*(b_1)| + |\ln f^*(b_2)|\right].$$

Proof. Taking

$$z = \frac{b_1+b_2}{2}$$

in Corollary 4.2, we have

$$z-b_1 = b_2-z = \frac{b_2-b_1}{2}.$$

Hence,

$$\frac{(z-b_1)^2}{b_2-b_1} = \frac{(b_2-z)^2}{b_2-b_1} = \frac{b_2-b_1}{4}.$$

Substitution into Corollary 4.2 gives

$$\left|\ln \Omega^*_{AB}\left(f;\frac{b_1+b_2}{2}\right)\right| \leq \frac{b_2-b_1}{24}\left[4\left|\ln f^*\left(\frac{b_1+b_2}{2}\right)\right| + |\ln f^*(b_1)| + |\ln f^*(b_2)|\right],$$

which proves the assertion. □

Remark 4.1. *The limiting cases in Corollaries 4.2 and 4.3 are closely related to the fractional multiplicative Ostrowski estimates obtained by Meftah et al. [15]. In particular, after passing to logarithms, the case $\alpha = 1$ exhibits the same characteristic coefficient pattern $1/6, 1/3, 1/6$, while the midpoint specialization leads to the corresponding $1, 4, 1$ structure. Therefore, these corollaries are included to display the consistency of the present multiplicative Atangana–Baleanu formulation with previously known multiplicative Ostrowski-type estimates, rather than as independent novel inequalities.*

5. Conclusion

In this paper, Ostrowski-type inequalities have been investigated within the framework of multiplicative Atangana–Baleanu fractional calculus. By using the logarithmic correspondence between ordinary and multiplicative calculus, an Atangana–Baleanu Ostrowski identity has been expressed in multiplicative form. This formulation was then employed to derive estimates under convexity assumptions on $|\ln f^*|$ and $|\ln f^*|^q$ by means of direct convexity arguments, the power-mean inequality, and Young's inequality.

The relationship with previously established results has also been made explicit. In particular, the underlying identity is closely connected with the ordinary Atangana–Baleanu Ostrowski identity of Kalsoom and Khan [11], while the limiting case $\alpha = 1$ and the midpoint specialization exhibit coefficient structures consistent with the fractional multiplicative Ostrowski inequalities obtained by Meftah et al. [15]. Accordingly, the results presented here clarify how known additive Atangana–Baleanu estimates can be represented and analyzed within the multiplicative setting.

The present formulation may also be useful for investigating related inequalities under other generalized convexity assumptions and for multiplicative fractional integral operators associated with different kernels. Further studies may consider corresponding quadrature-type estimates, error bounds, and related problems in multiplicative fractional calculus.

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