Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **14** (2026) 1–10
DOI: 10.47443/cm.2026.059

Research Article

Crosswise Operators on Banach Spaces of Bounded Functions

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(Received: 13 July 2026. Received in revised form: 15 August 2026. Accepted: 24 August 2026. Published online: 28 August 2026.)

Abstract

In this paper, the notion of crosswise operator is introduced and studied. Through this class of bounded linear operators, certain aspects of the geometry of Banach spaces of bounded functions are studied and classified.

Keywords: Banach space; bounded linear operator; bounded function; supporting vector; supporting functional.

2020 Mathematics Subject Classification: 46B20.

1. Introduction

The quest to understand norm-attaining operators and their supporting vectors has been a foundational pillar of functional analysis since its beginnings [3]. This prolific area of research has yielded cornerstone results such as James' theorem [18,19], the Bishop–Phelps theorem [4,5], the Bishop–Phelps–Bollobás theorem [7], and many others [1,6,9,20].

A parallel and deeply related problem, central to both pure and applied mathematics, is that of minimizing the norm of a vector over a specific subset of a normed space, which is in fact intimately related to the norm-attaining problem if the minimization is over convex sets and understood as the distance from the set to the zero vector. This minimum-norm problem is a fundamental task in optimization theory with wide-ranging applications in fields like physics, statistics, and engineering.

These optimization problems can be approached either geometrically or analytically. Among the most remarkable applications of these minimization problems, there are the optimal-design problems for transcranial magnetic stimulation (TMS) coils and magnetic resonance imaging (MRI) coils [17,22,25,32], as well as the improvement of classical statistical tools such as principal components and others [16].

The geometric study of minimum-norm problems probably started with the metric notion of proximinality [28,29]. This notion, in the context of real normed spaces, led to remarkable results such as the famous James' characterization of reflexivity in terms of norm-attaining functionals [18]. A few years later, James [19] found the existence of a non-complete normed space in which every functional is norm-attaining (bear in mind that the fact that a functional is norm-attaining is equivalent to that of the corresponding unit hyperplane to have a minimum-norm element). Later on, Blatter [6] proved that every normed space in which each closed convex subset has a minimum-norm element must be complete. This result was later improved in [2, Theorem 2.1]. Other interesting results related to proximinality and minimum-norm elements can be found in [8,13,24,26].

2. Preliminaries

All vector spaces considered throughout this paper will be over the reals and nonzero by default. However, most of our results can be easily adapted to the complex case. All linear operators considered will also be assumed nonzero and bounded by default. All subspaces considered will be subobjects of the corresponding category, in the sense that, for instance, subspaces of normed spaces will be assumed to be linear subspaces, whereas subspaces of a Banach space will be assumed to be closed linear subspaces (like in [10, Lemma 2] where the involved linear operator is supposed to have closed range). Basic notions of operator theory and geometry of Banach spaces will be used as tools throughout the paper. The reader is referred to [14,21] for a wider perspective on those two branches of functional analysis.

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If X is a normed space, then for a given $x \in X$ and $\delta > 0$, $\mathsf{B}_X(x, \delta)$, $\mathsf{U}_X(x, \delta)$, and $\mathsf{S}_X(x, \delta)$, stand for the (closed) ball of center x and radius δ, the open ball of center x and radius δ, and the sphere of center x and radius δ, respectively. When $x = 0$ and $\delta = 1$, the closed unit ball, the open unit ball, and the unit sphere are denoted as usual, B_X, U_X, and S_X, respectively.

If X, Y are normed spaces, a linear operator $T : X \to Y$ is said to be bounded provided that it has the finite operator norm

$$\|T\| := \sup\{\|T(x)\| : x \in \mathsf{B}_X\} < \infty. \tag{1}$$

Notice that a linear operator is bounded if and only if it is continuous. The set of supporting vectors of T is defined as

$$\operatorname{suppv}(T) := \{x \in X : \|T(x)\| = \|T\|\|x\|\}. \tag{2}$$

Observe that zero is always a supporting vector of no interest. There is a noticeable subset of supporting vectors of a bounded linear functional. Indeed, the dual space of X consists of all bounded linear functionals and is usually denoted by X^*. If $x^* \in X^*$, then

$$\operatorname{suppv}_1(x^*) := \{x \in X : x^*(x) = \|x^*\|\|x\|\} \tag{3}$$

is a noticeable subset of $\operatorname{suppv}(x^*)$.

Given a subset A of a normed space X, we will be considering the set $\arg\min(A) := \{x \in A : \|x\| \leq \|a\| \ \forall\, a \in A\}$. Note that $\arg\min(A)$ may be empty. An interesting property satisfied by $\arg\min$ is the fact that $\arg\min(A) \cap B \subseteq \arg\min(A \cap B)$, and if $\arg\min(A) \cap B \neq \varnothing$, then $\arg\min(A) \cap B = \arg\min(A \cap B)$. This applies to the case where X, Y are normed spaces, $T, S : X \to Y$ are bounded linear operators, $y_0 \in Y$, and $a \geq 0$ to conclude that

$$\arg\min\left(T^{-1}(y_0)\right) \cap \{x \in X : \|S(x)\| \leq a\} \subseteq \arg\min\left(T^{-1}(y_0) \cap \{x \in X : \|S(x)\| \leq a\}\right),$$

and if $\arg\min\left(T^{-1}(y_0)\right) \cap \{x \in X : \|S(x)\| \leq a\} \neq \varnothing$, then

$$\arg\min\left(T^{-1}(y_0)\right) \cap \{x \in X : \|S(x)\| \leq a\} = \arg\min\left(T^{-1}(y_0) \cap \{x \in X : \|S(x)\| \leq a\}\right).$$

If, in addition, X, Y are Banach spaces, $\ker(S) = \{0\}$, and $S(X)$ is closed in Y, then the existence of an element $x_0 \in T^{-1}(y_0) \cap \{x \in X : \|S(x)\| \leq a\}$ forces that

$$\frac{\|y_0\|}{\|T\|} \leq \left\|S^{-1}\right\| a. \tag{4}$$

Indeed, on the one hand, $\|y_0\| = \|T(x_0)\| \leq \|T\|\|x_0\|$. On the other hand, in accordance with the open mapping theorem, $S : X \to S(X)$ is an isomorphism, so there exists its inverse that can be denoted by $S^{-1} : S(X) \to X$. Then

$$\|x_0\| = \left\|S^{-1}(S(x_0))\right\| \leq \left\|S^{-1}\right\| \|S(x_0)\| \leq \left\|S^{-1}\right\| a.$$

As a consequence,

$$\frac{\|y_0\|}{\|T\|} \leq \|x_0\| \leq \left\|S^{-1}\right\| a.$$

It is worth mentioning that, in [31], $\arg\min(A)$ was estimated, where $A := \{x \in X : \|R_k(x)\| \geq a_k,\ k = 1, \ldots, l\}$ and, for $k = 1, \ldots, l$, $a_k \geq 0$ and $R_k : X \to Y$ are bounded linear operators between real normed spaces X, Y. Notice that the previous set A is not, in general, convex.

3. Results

Our first result is a technical lemma upon which we will rely later on. Observe that a left change of variable simply consists in applying a left inverse for a bounded linear operator (if it admits one). Notice that the existence of a left inverse in infinite dimensions does not imply that the bounded linear operator is an isomorphism.

Lemma 3.1. *Let X, Y, Z be normed spaces. Let $J : X \to Z$ and $T : X \to Y$ be bounded linear operators. Let $y_0 \in Y$. If the bounded linear operator $R : Z \to X$ is a left-inverse of J, then $J\left(T^{-1}(y_0)\right) = (T \circ R)^{-1}(y_0) \cap J(X)$. In particular,*

$$\arg\min\left(J\left(T^{-1}(y_0)\right)\right) = \arg\min\left((T \circ R)^{-1}(y_0) \cap J(X)\right) \supseteq \arg\min\left((T \circ R)^{-1}(y_0)\right) \cap J(X)$$

and if $\arg\min\left((T \circ R)^{-1}(y_0)\right) \cap J(X) \neq \varnothing$*, then*

$$\arg\min\left(J\left(T^{-1}(y_0)\right)\right) = \arg\min\left((T \circ R)^{-1}(y_0) \cap J(X)\right) = \arg\min\left((T \circ R)^{-1}(y_0)\right) \cap J(X).$$

Proof. Fix an arbitrary $x_0 \in X$ such that $J(x_0) \in T^{-1}(y_0)$. Then $(T \circ R)(J(x_0)) = T(R(J(x_0))) = T(x_0) = y_0$, meaning that $J(x_0) \in (T \circ R)^{-1}(y_0) \cap J(X)$. Conversely, take any $z \in (T \circ R)^{-1}(y_0) \cap J(X)$. Then $z = J(x)$ for some $x \in X$. All that remains to show is that $x \in T^{-1}(y_0)$. Indeed, $R(z) \in T^{-1}(y_0)$, and hence $y_0 = T(R(z)) = T(R(J(x))) = T(x)$. □

In view of Lemma 3.1 and under its settings, the main focus can then be directed toward estimating $\arg\min\left(T^{-1}(y_0)\right)$.

Proposition 3.1. *Let X, Y be normed spaces. Let $T : X \to Y$ be a bounded linear operator. Let $y_0 \in Y$. Then the following statements hold.*

1. *If $T^{-1}(y_0) \neq \varnothing$, then $\frac{\|y_0\|}{\|T\|} \leq \inf\left\{\|x\| : x \in T^{-1}(y_0)\right\}$.*
2. $\operatorname{suppv}(T) \cap T^{-1}(y_0) \subseteq \arg\min\left(T^{-1}(y_0)\right)$.
3. *If $\operatorname{suppv}(T) \cap T^{-1}(y_0) \neq \varnothing$, then $\frac{\|y_0\|}{\|T\|} = \min\left\{\|x\| : x \in T^{-1}(y_0)\right\}$.*
4. *If $\frac{\|y_0\|}{\|T\|} = \min\left\{\|x\| : x \in T^{-1}(y_0)\right\}$, then $\operatorname{suppv}(T) \cap T^{-1}(y_0) = \arg\min\left(T^{-1}(y_0)\right) \neq \varnothing$.*

Proof. The proof is itemized in accordance with the statement.

1. If $x \in T^{-1}(y_0)$, then
$$\|y_0\| = \|T(x)\| \leq \|T\|\|x\|,$$
meaning that
$$\frac{\|y_0\|}{\|T\|} \leq \|x\|, \tag{5}$$
which implies the desired result.

2. Fix arbitrary elements $x_0 \in \operatorname{suppv}(T) \cap T^{-1}(y_0)$ and $x \in T^{-1}(y_0)$. We have that
$$\|y_0\| = \|T(x_0)\| = \|T\|\|x_0\|,$$
which implies that
$$\frac{\|y_0\|}{\|T\|} = \|x_0\|. \tag{6}$$
By gluing together (5) and (6), we obtain that
$$\|x_0\| = \frac{\|y_0\|}{\|T\|} \leq \|x\|. \tag{7}$$
As a consequence, $x_0 \in \arg\min\left(T^{-1}(y_0)\right)$.

3. By assumption, there exists $x_0 \in \operatorname{suppv}(T) \cap T^{-1}(y_0)$. By (6), we conclude that
$$\frac{\|y_0\|}{\|T\|} = \|x_0\| = \min\left\{\|x\| : x \in T^{-1}(y_0)\right\}.$$

4. By assumption, we can take $x_0 \in T^{-1}(y_0)$ such that
$$\frac{\|y_0\|}{\|T\|} = \|x_0\|.$$
Then
$$\|y_0\| = \|T(x_0)\| \leq \|T\|\|x_0\| = \|y_0\|,$$
which implies that $x_0 \in \operatorname{suppv}(T)$. Finally, if $x \in \arg\min\left(T^{-1}(y_0)\right)$, then
$$\|x\| = \|x_0\| = \frac{\|y_0\|}{\|T\|}$$
and the same argument applies to show that $x \in \operatorname{suppv}(T)$.

□

As an immediate corollary of Proposition 3.1, the following result is obtained.

Corollary 3.1. *Let X be a normed space. Let $x^* \in X^* \setminus \{0\}$. Let $t_0 \in \mathbb{R} \setminus \{0\}$. Then*

$$\arg\min\left((x^*)^{-1}(t_0)\right) = \left\{\frac{t_0}{\|x^*\|\|x\|}x : x \in \operatorname{suppv}_1(x^*) \setminus \{0\}\right\}. \tag{8}$$

From now on, we will focus on Banach spaces of bounded functions endowed with the sup norm [15].

Remark 3.1. *Let $X := \mathcal{B}(\Lambda, \mathbb{R})$ be a Banach space of real-valued bounded functions on a nonempty set Λ endowed with the sup norm $\|\cdot\|_\infty$. The canonical elements of X are given by the characteristic functions*

$$\begin{array}{rccl} e_\lambda : & \Lambda & \to & \mathbb{R} \\ & \gamma & \mapsto & e_\lambda(\gamma) := \left\{\begin{array}{ll} 1 & \gamma = \lambda \\ 0 & \gamma \neq \lambda \end{array}\right. \end{array} \tag{9}$$

for every $\lambda \in \Lambda$. The coordinate functionals are given by

$$\begin{array}{rccl} e^*_\lambda : & X & \to & \mathbb{R} \\ & x & \mapsto & e^*_\lambda(x) := x(\lambda) \end{array} \tag{10}$$

for every $\lambda \in \Lambda$. For simplicity, we will denote $\mathcal{B}(\Lambda)$ instead of $\mathcal{B}(\Lambda, \mathbb{R})$.

The following notion, which is one of the crosswise operators, is novel and central to this work.

Definition 3.1 (Crosswise operator)**.** *Let Λ be a nonempty set and let $X := \mathcal{B}(\Lambda)$. Let Y be a normed space. A bounded linear operator $T : X \to Y$ is said to be crosswise provided that $\ker(T) \nsubseteq \ker(e^*_\lambda)$ for every $\lambda \in \Lambda$.*

The next theorem provides a necessary and sufficient condition to estimate $\arg\min\left(T^{-1}(y_0)\right)$ when $X := \mathcal{B}(\Lambda)$ and T is not crosswise. For the crosswise case, the next theorem provides a necessary condition to find $\arg\min\left(T^{-1}(y_0)\right)$.

Theorem 3.1. *Let Λ be a nonempty set and let $X := \mathcal{B}(\Lambda)$. Let Y be a normed space. Let $T : X \to Y$ be a bounded linear operator. Let $y_0 \in Y$. Then:*

1. *Suppose that $\ker(T) \subseteq \ker\left(e^*_{\lambda_0}\right)$ for some $\lambda_0 \in \Lambda$. Then*

$$\left\{x \in T^{-1}(y_0) : \|x\|_\infty = |x(\lambda_0)|\right\} \subseteq \arg\min\left(T^{-1}(y_0)\right). \tag{11}$$

If, in addition, $\left\{x \in T^{-1}(y_0) : \|x\|_\infty = |x(\lambda_0)|\right\} \neq \varnothing$, then

$$\arg\min\left(T^{-1}(y_0)\right) = \left\{x \in T^{-1}(y_0) : \|x\|_\infty = |x(\lambda_0)|\right\}. \tag{12}$$

2. *Suppose that T is crosswise. Then*

$$\arg\min\left(T^{-1}(y_0)\right) \subseteq \left\{x \in T^{-1}(y_0) : \forall\, \lambda \in \Lambda \text{ if } \|x\|_\infty = |x(\lambda)|\,, \text{ then } \|x - x(\lambda)e_\lambda\|_\infty = \|x\|_\infty\right\}. \tag{13}$$

Proof. The proof is itemized in accordance with the statement of the theorem.

1. Fix an arbitrary $x_0 \in T^{-1}(y_0)$ in such a way that $\|x_0\|_\infty = |x_0(\lambda_0)|$. We take any $x \in T^{-1}(y_0)$. We observe that $x_0 - x \in \ker(T) \subseteq \ker\left(e^*_{\lambda_0}\right)$. Therefore, $x_0(\lambda_0) - x(\lambda_0) = e^*_{\lambda_0}(x_0 - x) = 0$, meaning that

$$\|x_0\|_\infty = |x_0(\lambda_0)| = |x(\lambda_0)| \leq \|x\|_\infty.$$

As a consequence, $x_0 \in \arg\min\left(T^{-1}(y_0)\right)$. Next, suppose that there exists $x_0 \in T^{-1}(y_0)$ such that $\|x_0\|_\infty = |x_0(\lambda_0)|$. We will prove (12). Indeed, fix an arbitrary $x \in \arg\min\left(T^{-1}(y_0)\right)$. Observe that $x_0, x \in T^{-1}(y_0)$, meaning that $x_0 - x \in \ker(T) \subseteq \ker\left(e^*_{\lambda_0}\right)$. Therefore, $x_0(\lambda_0) - x(\lambda_0) = e^*_{\lambda_0}(x_0 - x) = 0$. Since both x_0 and x lie in $\arg\min\left(T^{-1}(y_0)\right)$, we conclude that

$$|x(\lambda_0)| \leq \|x\|_\infty = \|x_0\|_\infty = |x_0(\lambda_0)| = |x(\lambda_0)|.$$

2. Suppose on the contrary that there exists $x_0 \in T^{-1}(y_0)$ for which there exists $\lambda_0 \in \Lambda$ with $\|x_0\|_\infty = |x_0(\lambda_0)|$ but $\|x_0 - x_0(\lambda_0)e_{\lambda_0}\|_\infty < \|x_0\|_\infty$. The idea is to find a perturbation of x_0 with less ∞-norm. Observe that $x_0 \neq 0$. Since T is crosswise, there can be found $z \in \ker(T) \setminus \{0\}$ satisfying that

$$\|z\|_\infty < \min\{\|x_0\|_\infty, \|x_0\|_\infty - \|x_0 - x_0(\lambda_0)e_{\lambda_0}\|_\infty\}$$

and $z(\lambda_0) \neq 0$ and has the same sign as $x_0(\lambda_0)$. Notice that $T(x_0 - z) = T(x_0) - T(z) = y_0$. Therefore, it only remains to show that $\|x_0 - z\|_\infty < \|x_0\|_\infty$. Indeed, if $x_0(\lambda_0) > 0$, then $0 < z(\lambda_0) < x_0(\lambda_0)$, hence $0 < x_0(\lambda_0) - z(\lambda_0) < x_0(\lambda_0) = \|x_0\|_\infty$. If $x_0(\lambda_0) < 0$, then $x_0(\lambda_0) < z(\lambda_0) < 0$, hence $0 < z(\lambda_0) - x_0(\lambda_0) < -x_0(\lambda_0) = \|x_0\|_\infty$. In any case, we have $|x_0(\lambda_0) - z(\lambda_0)| < \|x_0\|_\infty$. On the other hand, if $\lambda \in \Lambda \setminus \{\lambda_0\}$, then

$$|x_0(\lambda) - z(\lambda)| \leq |x_0(\lambda)| + |z(\lambda)| \leq \|x_0 - x_0(\lambda_0)e_{\lambda_0}\|_\infty + \|z\|_\infty < \|x_0\|_\infty.$$

As a consequence, $\|x_0 - z\|_\infty < \|x_0\|_\infty$, meaning that $x_0 \notin \arg\min\left(T^{-1}(y_0)\right)$.

□

The following example shows that, in the case of crosswise operators, the elements of $\arg\min\left(T^{-1}(y_0)\right)$ do not necessarily attain their ∞-norm at every component.

Example 3.1. *Consider the bounded linear operator $T : \ell_\infty^3 \to \ell_\infty^2$ induced by the matrix*

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

This operator is crosswise because its kernel is $\{(-\alpha, \alpha, -\alpha) : \alpha \in \mathbb{R}\}$. Take $y_0 := (1, 2)$. Notice that

$$T^{-1}(y_0) = \{(1-\alpha, \alpha, 2-\alpha) : \alpha \in \mathbb{R}\}.$$

Let us find $\min_{\alpha\in\mathbb{R}} \|(1-\alpha, \alpha, 2-\alpha)\|_\infty$. If $\alpha \leq 0$ or $\alpha \geq 2$, then $\|(1-\alpha, \alpha, 2-\alpha)\|_\infty \geq 2$. If $0 < \alpha < 1$ or $1 < \alpha < 2$, then $\|(1-\alpha, \alpha, 2-\alpha)\|_\infty > 1$. Finally, if $\alpha = 1$, then $\|(1-\alpha, \alpha, 2-\alpha)\|_\infty = \|(0, 1, 1)\|_\infty = 1$. As a consequence,

$$\arg\min\left(T^{-1}(y_0)\right) = \{(0, 1, 1)\}$$

and $(0, 1, 1)$ does not attain its ∞-norm at all components.

Theorem 3.2 is a generalization of Theorem 3.1, but it only occurs in very exceptional cases. The following notation will be employed in the next theorem: if Λ is a nonempty set, $x_0 \in \mathcal{B}(\Lambda)$, and $\varnothing \neq \Gamma \subseteq \Lambda$, then the restriction of x_0 to Γ is denoted by

$$r_{x_0,\Gamma}(\lambda) := \begin{cases} x_0(\lambda) & \lambda \in \Gamma, \\ 0 & \lambda \in \Lambda \setminus \Gamma. \end{cases}$$

Theorem 3.2. *Let Λ be a nonempty set and let $X := \mathcal{B}(\Lambda)$. Let Y be a normed space. Let $T : X \to Y$ be a crosswise bounded linear operator. Let $y_0 \in Y$. Let $x_0 \in \arg\min\left(T^{-1}(y_0)\right)$ and $\varnothing \neq \Gamma \subseteq \Lambda$ such that $|x_0(\gamma)| = \|x_0\|_\infty$ for all $\gamma \in \Gamma$. If there exists $z \in \ker(T)$ so that $\inf_{\gamma\in\Gamma} |z(\gamma)| > 0$ and $\operatorname{sgn}(z(\gamma)) = \operatorname{sgn}(x_0(\gamma))$ for all $\gamma \in \Gamma$, then $\|x_0 - r_{x_0,\Gamma}\|_\infty = \|x_0\|_\infty$.*

Proof. Suppose on the contrary that

$$\|x_0 - r_{x_0,\Gamma}\|_\infty < \|x_0\|_\infty.$$

Since $T(x_0 - z) = T(x_0) - T(z) = y_0$, a contradiction will be reached by proving that $\|x_0 - z\|_\infty < \|x_0\|_\infty$. Indeed, let us observe first that

$$\|x_0 - z\|_\infty = \max\{\sup\{|x_0(\gamma) - z(\gamma)| : \gamma \in \Gamma\}, \sup\{|x_0(\lambda) - z(\lambda)| : \lambda \in \Lambda \setminus \Gamma\}\}.$$

Therefore, it will be shown that

$$\sup\{|x_0(\gamma) - z(\gamma)| : \gamma \in \Gamma\} < \|x_0\|_\infty \tag{14}$$

and

$$\sup\{|x_0(\lambda) - z(\lambda)| : \lambda \in \Lambda \setminus \Gamma\} < \|x_0\|_\infty. \tag{15}$$

Observe that the given kernel vector z may be replaced by a sufficiently small positive scalar multiple, which preserves the sign and positive-infimum hypotheses, in order to assure that

$$\|z\|_\infty < \|x_0\|_\infty - \|x_0 - r_{x_0,\Gamma}\|_\infty.$$

If $\lambda \in \Lambda \setminus \Gamma$, then

$$|x_0(\lambda) - z(\lambda)| \leq |x_0(\lambda)| + |z(\lambda)| \leq \|x_0 - r_{x_0,\Gamma}\|_\infty + \|z\|_\infty < \|x_0\|_\infty,$$

meaning that

$$\sup\{|x_0(\lambda) - z(\lambda)| : \lambda \in \Lambda \setminus \Gamma\} \leq \|x_0 - r_{x_0,\Gamma}\|_\infty + \|z\|_\infty < \|x_0\|_\infty,$$

achieving (15). Finally, if $\gamma \in \Gamma$, then the fact that $|x_0(\gamma)| = \|x_0\|_\infty$ together with

$$0 < l \leq |z(\gamma)| \leq \|z\|_\infty < \|x_0\|_\infty - \|x_0 - r_{x_0,\Gamma}\|_\infty \leq \|x_0\|_\infty$$

and $\operatorname{sgn}(z(\gamma)) = \operatorname{sgn}(x_0(\gamma))$ forces that

$$|x_0(\gamma) - z(\gamma)| = \|x_0\|_\infty - |z(\gamma)| \leq \|x_0\|_\infty - l < \|x_0\|_\infty,$$

meaning that

$$\sup\{|x_0(\gamma) - z(\gamma)| : \gamma \in \Gamma\} \leq \|x_0\|_\infty - l < \|x_0\|_\infty$$

achieving (14), where $l := \inf |z(\Gamma)|$. As a consequence, the following inclusion has been proved:

$$\begin{aligned}\arg\min\left(T^{-1}(y_0)\right) \quad\subseteq\quad & \Big\{x \in T^{-1}(y_0) : \text{ If } \exists\, \Gamma \in \mathcal{P}(\Lambda) \setminus \{\varnothing\} \text{ with } \|x\|_\infty = |x(\gamma)| \ \forall\, \gamma \in \Gamma \text{ and if } \exists\, z \in \ker(T) \text{ with} \\ & \inf|z(\Gamma)| > 0 \text{ and } \operatorname{sgn}(z(\gamma)) = \operatorname{sgn}(x(\gamma))\ \forall\, \gamma \in \Gamma, \text{ then } \|x - r_{x,\Gamma}\|_\infty = \|x\|_\infty\Big\}.\end{aligned}$$

□

The upcoming lemma is related to Theorem 3.1(2). However, first a little bit of notation is necessary.

Remark 3.2. *Let Λ be a set with more than two points. Fix an arbitrary $\lambda_0 \in \Lambda$. There is a natural linear projection*

$$\begin{array}{rrcl} Q_{\lambda_0}: & \mathcal{B}(\Lambda) & \to & \mathcal{B}(\Lambda \setminus \{\lambda_0\}) \\ & x & \mapsto & Q_{\lambda_0}(x) := x|_{\Lambda\setminus\{\lambda_0\}} \end{array} \tag{16}$$

satisfying that $\|x\|_\infty = \max\{\|Q_{\lambda_0}(x)\|_\infty, |x(\lambda_0)|\}$ for all $x \in \mathcal{B}(\Lambda)$. There is also a natural linear embedding

$$\begin{array}{rrcl} J_{\lambda_0}: & \mathcal{B}(\Lambda \setminus \{\lambda_0\}) & \to & \mathcal{B}(\Lambda) \\ & x & \mapsto & J_{\lambda_0}(x) \end{array}$$

where, for each $x \in \mathcal{B}(\Lambda \setminus \{\lambda_0\})$,

$$\begin{array}{rrcl} J_{\lambda_0}(x): & \Lambda & \to & \mathbb{R} \\ & \lambda & \mapsto & J_{\lambda_0}(x)(\lambda) := \begin{cases} x(\lambda) & \lambda \neq \lambda_0 \\ 0 & \lambda = \lambda_0. \end{cases} \end{array}$$

Note that $\|J_{\lambda_0}(x)\|_\infty = \|x\|_\infty$ for each $x \in \mathcal{B}(\Lambda \setminus \{\lambda_0\})$.

Lemma 3.2. *Let Λ be a set with more than two points and let $X := \mathcal{B}(\Lambda)$. Let Y be a normed space. Let $T : X \to Y$ be a crosswise bounded linear operator. Let $y_0 \in Y$. Let $x_0 \in \arg\min\left(T^{-1}(y_0)\right)$ with $|x_0(\lambda_0)| = \|x_0\|_\infty$ for some $\lambda_0 \in \Lambda$. Consider $X_{\lambda_0} := \mathcal{B}(\Lambda \setminus \{\lambda_0\})$,*

$$\begin{array}{rrcl} T_{\lambda_0}: & X_{\lambda_0} & \to & Y \\ & x & \mapsto & T_{\lambda_0}(x) := T\left(J_{\lambda_0}(x)\right), \end{array} \tag{17}$$

and $y_{\lambda_0} := T_{\lambda_0}\left(Q_{\lambda_0}(x_0)\right)$. If $x \in \arg\min\left(T_{\lambda_0}^{-1}(y_{\lambda_0})\right)$, then $J_{\lambda_0}(x) + x_0(\lambda_0)e_{\lambda_0} \in \arg\min\left(T^{-1}(y_0)\right)$.

Proof. First, note that $J_{\lambda_0}(x) + x_0(\lambda_0)e_{\lambda_0} \in X = \mathcal{B}(\Lambda)$ and

$$\begin{aligned} T\left(J_{\lambda_0}(x) + x_0(\lambda_0)e_{\lambda_0}\right) &= T\left(J_{\lambda_0}(x)\right) + T\left(x_0(\lambda_0)e_{\lambda_0}\right) \\ &= T_{\lambda_0}(x) + T\left(x_0(\lambda_0)e_{\lambda_0}\right) \\ &= y_{\lambda_0} + T\left(x_0(\lambda_0)e_{\lambda_0}\right) \\ &= T_{\lambda_0}\left(Q_{\lambda_0}(x_0)\right) + T\left(x_0(\lambda_0)e_{\lambda_0}\right) \\ &= T\left(J_{\lambda_0}\left(Q_{\lambda_0}(x_0)\right)\right) + T\left(x_0(\lambda_0)e_{\lambda_0}\right) \\ &= T\left(J_{\lambda_0}\left(Q_{\lambda_0}(x_0)\right) + x_0(\lambda_0)e_{\lambda_0}\right) \\ &= T(x_0) \\ &= y_0. \end{aligned}$$

Also, $\|J_{\lambda_0}(x)+x_0(\lambda_0)e_{\lambda_0}\|_\infty = \max\{\|x\|_\infty, \|x_0\|_\infty\}$. At this stage, since $Q_{\lambda_0}(x_0) \in T_{\lambda_0}^{-1}(y_{\lambda_0})$, we have that

$$\|x\|_\infty \leq \|Q_{\lambda_0}(x_0)\|_\infty = \|x_0\|_\infty$$

by Theorem 3.1(2). Therefore,

$$\|J_{\lambda_0}(x)+x_0(\lambda_0)e_{\lambda_0}\|_\infty = \max\{\|x\|_\infty, \|x_0\|_\infty\} = \|x_0\|_\infty,$$

meaning that $J_{\lambda_0}(x)+x_0(\lambda_0)e_{\lambda_0} \in \arg\min\left(T^{-1}(y_0)\right)$. □

The next example shows that the bounded linear operator (17) is not necessarily crosswise even though so is T. In other words, crosswiseness is not a hereditary property for bounded linear operators.

Example 3.2. *Consider $X := \ell_\infty^3$ where Λ can be expressed as the three coordinates $\{x, y, z\}$ of the XYZ-space. Consider the bounded linear operator*

$$\begin{array}{rccl} T: & \mathbb{R}^3 & \to & \mathbb{R}^2 \\ & (x,y,z) & \mapsto & T(x,y,z) := (x-y, y-x) \end{array} \tag{18}$$

Note that T is crosswise since $\ker(T) = \{(x,x,z) : x, z \in \mathbb{R}\}$. However, if $\lambda_0 := x$, then

$$\begin{array}{rccl} T_{\lambda_0}: & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & (y,z) & \mapsto & T_{\lambda_0}(y,z) := T\left(J_{\lambda_0}(y,z)\right) = T(0,y,z) = (-y,y) \end{array} \tag{19}$$

is not a crosswise operator because $\ker\left(T_{\lambda_0}\right) = \{(0,z) : z \in \mathbb{R}\}$.

Our next theorem, which will be applied later on, is a particular case of Corollary 3.1 for $\mathcal{B}(\Lambda)$-spaces. At this stage, it is worth noting that, in [27], the supporting vectors for certain operators on ℓ_∞ are characterized. The next theorem does not restrict the cardinality of Λ to countable cardinalities (which is the case of ℓ_∞). On the other hand, [11] provides general topological and geometrical properties of the set of supporting vectors, whereas the next theorem describes a necessary inclusion satisfied by supporting vectors for functionals on $\mathcal{B}(\Lambda)$-spaces.

Theorem 3.3. *Let Λ be a nonempty set and let $X := \mathcal{B}(\Lambda)$. Let Y be a normed space. Let $x^* \in X^* \setminus \{0\}$ be a bounded linear functional. Then*

$$\operatorname{suppv}_1(x^*) \setminus \{0\} \subseteq \{x \in \mathcal{B}(\Lambda) \setminus \{0\} : \forall\, \lambda \in \Lambda \text{ if } x^*(e_\lambda) \neq 0, \text{ then } |x(\lambda)| = \|x\|_\infty \text{ and } \operatorname{sgn}(x(\lambda)) = \operatorname{sgn}(x^*(e_\lambda))\}. \tag{20}$$

If Λ is finite, then equality holds in (20).

Proof. We fix an arbitrary $x \in \operatorname{suppv}_1(x^*) \setminus \{0\}$. We take any $\lambda \in \Lambda$ such that $x^*(e_\lambda) \neq 0$. If either $|x(\lambda)| < \|x\|_\infty$ or $\operatorname{sgn}(x(\lambda)) \neq \operatorname{sgn}(x^*(e_\lambda))$, then it suffices to consider

$$z := x - x(\lambda)e_\lambda + \operatorname{sgn}(x^*(e_\lambda))\|x\|_\infty e_\lambda$$

to reach the contradiction that $\|z\|_\infty = \|x\|_\infty$ but $x^*(z) > x^*(x)$. Indeed,

$$z(\gamma) := \begin{cases} x(\gamma) & \gamma \in \Lambda \setminus \{\lambda\} \\ \operatorname{sgn}(x^*(e_\lambda))\|x\|_\infty & \gamma = \lambda \end{cases}$$

and hence $\|z\|_\infty = \|x\|_\infty$. Next,

$$\begin{aligned} x^*(z) &= x^*(x) - x(\lambda)x^*(e_\lambda) + \operatorname{sgn}(x^*(e_\lambda))\|x\|_\infty x^*(e_\lambda) \\ &= \|x^*\|\|x\|_\infty + \left(\operatorname{sgn}(x^*(e_\lambda))\|x\|_\infty - x(\lambda)\right)x^*(e_\lambda). \end{aligned}$$

At this stage, observe that if either $|x(\lambda)| < \|x\|_\infty$ or $|x(\lambda)| = \|x\|_\infty$ but $\operatorname{sgn}(x(\lambda)) \neq \operatorname{sgn}(x^*(e_\lambda))$, then

$$\left(\operatorname{sgn}(x^*(e_\lambda))\|x\|_\infty - x(\lambda)\right)x^*(e_\lambda) > 0,$$

reaching the previously mentioned contradiction that

$$x^*(z) > \|x^*\|\|x\|_\infty = x^*(x).$$

Conversely, suppose that $\Lambda = \{\lambda_1, \dots, \lambda_k\}$ is finite and fix an arbitrary $x \in \mathcal{B}(\Lambda) \setminus \{0\}$ satisfying that

$$|x(\lambda)| = \|x\|_\infty \text{ and } \operatorname{sgn}(x(\lambda)) = \operatorname{sgn}\left(x^*(e_\lambda)\right)$$

for all $\lambda \in \Lambda$ such that $x^*(e_\lambda) \neq 0$. Take any $z \in \mathcal{B}(\Lambda)$ such that $\|z\|_\infty \leq \|x\|_\infty$. Observe that

$$\begin{aligned} |x^*(z)| &= \left| x^*\left(\sum_{i=1}^k z(\lambda_i) e_{\lambda_i}\right)\right| \leq \sum_{i=1}^k |z(\lambda_i)| \left|x^*\left(e_{\lambda_i}\right)\right| \leq \sum_{i=1}^k \|z\|_\infty \left|x^*\left(e_{\lambda_i}\right)\right| \\ &\leq \sum_{i=1}^k \|x\|_\infty \left|x^*\left(e_{\lambda_i}\right)\right| = \sum_{i\in I} \|x\|_\infty \left|x^*\left(e_{\lambda_i}\right)\right| = \sum_{i\in I} |x(\lambda_i)| \left|x^*\left(e_{\lambda_i}\right)\right| \\ &= \sum_{i\in I} x(\lambda_i) x^*\left(e_{\lambda_i}\right) = x^*\left(\sum_{i\in I} x(\lambda_i) e_{\lambda_i}\right) = x^*(x), \end{aligned}$$

where $I := \{i \in \{1, \dots, k\} : x^*(e_{\lambda_i}) \neq 0\}$. This shows that $x^*(x) = \|x^*\|\|x\|_\infty$. □

As a direct consequence of Corollary 3.1 together with Theorem 3.3, we obtain the following corollary.

Corollary 3.2. *Let Λ be a nonempty set and let $X := \mathcal{B}(\Lambda)$. Let Y be a normed space. Let $x^* \in X^* \setminus \{0\}$ be a bounded linear functional. Let $t_0 \in \mathbb{R} \setminus \{0\}$. Then*

$$\begin{aligned} \arg\min\left(\left(x^*\right)^{-1}(t_0)\right) \subseteq \Bigg\{ & \frac{t_0}{\|x^*\|\|x\|_\infty} x : x \in \mathcal{B}(\Lambda) \setminus \{0\} \text{ and } \forall\, \lambda \in \Lambda \text{ if } x^*(e_\lambda) \neq 0, \\ & \text{then } |x(\lambda)| = \|x\|_\infty \text{ and } \operatorname{sgn}(x(\lambda)) = \operatorname{sgn}\left(x^*(e_\lambda)\right) \Bigg\}. \end{aligned} \tag{21}$$

If Λ is finite, then equality holds in (21).

The final theorem of this paper provides an elegant estimate of $\arg\min\left(J\left(\mathsf{S}_X(0,\alpha)\right)\right)$, where X is a Hilbert space, $J : X \to \mathcal{B}(\Lambda)$ is a one-to-one bounded linear operator, $\alpha > 0$, and $\mathsf{S}_X(0,\alpha) := \{x \in X : \|x\|_2 = \alpha\}$ is the sphere of X of center 0 and radius α.

An example of such J is, for instance, the canonical inclusion of $X := \ell_2$ into $\mathcal{B}(\Lambda) := \ell_\infty$. In fact, $\ell_2 \subseteq \ell_\infty$ as sets and $\|x\|_\infty \leq \|x\|_2$ for all $x \in \ell_2$.

Theorem 3.4. *If $X := \ell_2^n$ and $J : \ell_2^n \to \ell_\infty^n$ is the canonical inclusion, then $\arg\min\left(J\left(\mathsf{S}_X(0,\alpha)\right)\right)$, $\alpha > 0$, consists of all the elements of the form*

$$\left(\pm\frac{\alpha}{\sqrt{n}}, \dots, \pm\frac{\alpha}{\sqrt{n}}\right). \tag{22}$$

If $X := \ell_2$ and $J : \ell_2 \to \ell_\infty$ is the canonical inclusion, then $\arg\min\left(J\left(\mathsf{S}_X(0,\alpha)\right)\right) = \varnothing$ and $\inf\{\|x\|_\infty : \|x\|_2 = \alpha\} = 0$.

Proof. First, assume that $X := \ell_2^n$. Notice that

$$\left\|\left(\pm\frac{\alpha}{\sqrt{n}}, \dots, \pm\frac{\alpha}{\sqrt{n}}\right)\right\|_2 = \alpha.$$

Next, if $y \in \ell_2^n$ with $\|y\|_2 = \alpha$ satisfies that $\|y\|_\infty < \|x\|_\infty$, where x is of the form given in (22), then

$$\alpha^2 = \|y\|_2^2 = \sum_{i=1}^n |y_i|^2 \leq \sum_{i=1}^n \|y\|_\infty^2 < \sum_{i=1}^n \|x\|_\infty^2 = \sum_{i=1}^n \left|\pm\frac{\alpha}{\sqrt{n}}\right|^2 = \alpha^2 \tag{23}$$

reaching a contradiction. As a consequence, $\arg\min\left(J\left(\mathsf{S}_X(0,\alpha)\right)\right)$ contains all the elements of the form (22). Conversely, if $y \in \ell_2^n$, $\|y\|_2 = \alpha$, and $\|y\|_\infty = \alpha/\sqrt{n}$, then

$$\alpha^2 = \sum_{i=1}^n |y_i|^2 \leq n\|y\|_\infty^2 = \alpha^2$$

forces $|y_i| = \alpha/\sqrt{n}$ for each $i = 1, \dots, n$. Therefore, $\arg\min\left(J\left(\mathsf{S}_X(0,\alpha)\right)\right)$ consists of all the elements of the form (22).

Finally, suppose that $X := \ell_2$. The vectors of the form

$$\left(\frac{\alpha}{\sqrt{n}}, \ldots, \frac{\alpha}{\sqrt{n}}, 0, 0, \ldots\right) \in \ell_2 \tag{24}$$

for $n \in \mathbb{N}$ generate a sequence of elements in the sphere $\mathsf{S}_{\ell_2}(0, \alpha)$ whose ∞-norms are decreasing to 0:

$$\left\|\left(\frac{\alpha}{\sqrt{n}}, \ldots, \frac{\alpha}{\sqrt{n}}, 0, 0, \ldots\right)\right\|_\infty = \frac{\alpha}{\sqrt{n}} \to 0 \text{ as } n \to \infty. \tag{25}$$

□

What follows next is a quick application of Corollary 3.2 to the practical problem of designing a near-field communication (NFC) antenna with improved manufacturability, reduced unwanted capacitive coupling, and cross-talk between PCB (printed circuit board) traces. To this end, we propose the minimization of the maximum current density in the antenna and thereby maximizing of the minimum spacing between PCB tracks. In addition, a maximum magnetic field strength at a given target location (representing the expected position of the receiving tag) is also considered in order to control the antenna's directionality. By using the magneto-quasistatic inverse boundary element formalism in [23, 30], this NFC antenna design problem can be written as

$$\left\{\begin{array}{l} \min \|J\psi\|_\infty \\ \max B_z\psi \end{array}\right. \tag{26}$$

where $\psi = (\psi_1, \psi_2, \ldots, \psi_N)^T$ is the vector containing the nodal values of the stream function, $J \in \mathbb{R}^{N\times N}$ is the absolute current density matrix, and $B_z \in \mathbb{R}^N$ is a vector of the z-component of the magnetic induction produced by each current element in the prescribed target location [12]. Introducing the following change of variables $x = J\psi$ and $a = B_z J^{-1}$ (absolute current density matrices in this setting are invertible), the optimization problem in (26) can be reformulated as

$$\left\{\begin{array}{l} \min \|(x_1, \ldots, x_N)\|_\infty \\ \max \sum_{j=1}^N a_j x_j \end{array}\right. \tag{27}$$

where $x = (x_1, \ldots, x_N)$ and $a = (a_1, \ldots, a_N)$ are both in $\mathbb{R}^N$. Note that, in this setting, the z-component of the magnetic induction B_z can be considered not zero, hence $a \neq 0$. According to our Corollary 3.2, let us consider $x \in \mathbb{R}^N$ defined by

$$x_j := \left\{\begin{array}{ll} \frac{|a_j|}{a_j} & a_j \neq 0 \\ 0 & a_j = 0 \end{array}\right. \quad j = 1, \ldots, N. \tag{28}$$

Note that $\|x\|_\infty = 1$ and $ax = \sum_{j=1}^N |a_j|$. Therefore, if $y \in \ell_\infty^N$ and $\|y\|_\infty < 1$, then

$$|ay| = \left|\sum_{j=1}^N a_j y_j\right| \leq \sum_{j=1}^N |a_j||y_j| < \sum_{j=1}^N |a_j| = ax.$$

This shows that x is a Pareto optimal solution of (27).

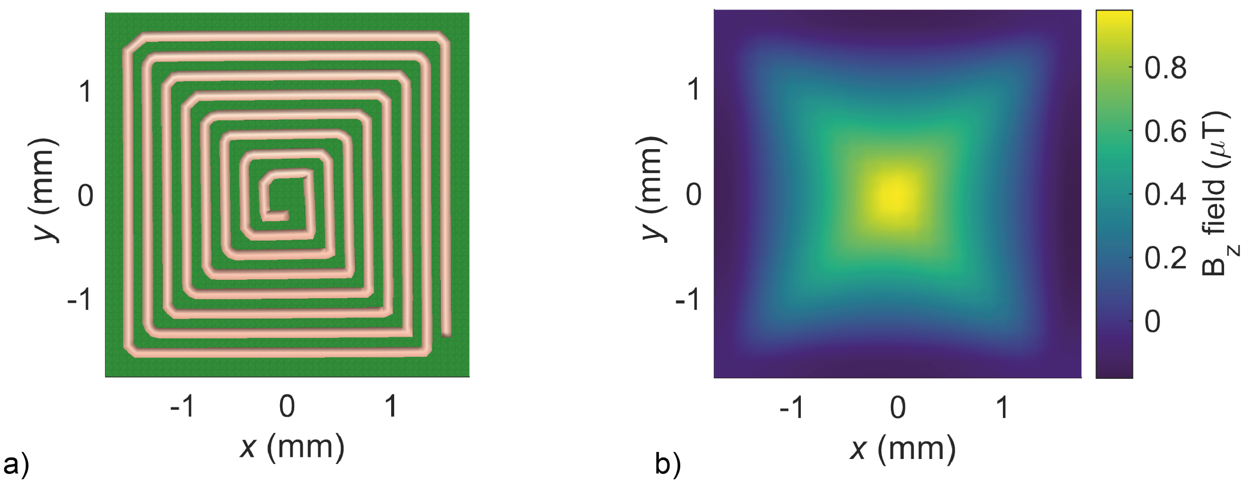

Figure 3.1: a) PCB track distribution solution of the NFC antenna. b) Color-coded map of the B_z-component of the magnetic field at $z = 0.25$ mm.

Let us illustrate this result by designing a square NFC antenna with side length 3.5 mm, which is produced to generate a maximum magnetic field at a distance of 0.25 mm above its center for 0.2 mA current peak. The PCB track pattern obtained from the solution of (26) is depicted in Figure 3.1(a), where a maximal spread of the PCB tracks can be seen. This antenna, when energized, produces the magnetic field displayed in Figure 3.1(b), where it is worth noting that the maximum value of B_z occurs at the target location.

Beyond NFC applications, the proposed methodology is directly applicable to the design of wireless power transfer coils, biomedical inductive links, or sensor excitation coils, where spatial selectivity is critical.

Acknowledgments

This paper is dedicated to our colleague Roberto Manito Fernández and his team, to whom we express our sincere gratitude for valuable conversations and discussions. The authors would also like to thank the reviewers, whose comments and suggestions helped improve this paper.

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