Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **13** (2026) 92–101
DOI: 10.47443/cm.2026.014

Research Article

Exponentially (s, P)-Convex Functions and Related Integral Inequalities

Nemanja Vučićević*

Institute of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Kragujevac, Serbia

(Received: 2 February 2026. Received in revised form: 27 May 2026. Accepted: 14 June 2026. Published online: 14 July 2026.)

Abstract

Convexity theory, as one of the central topics in mathematical analysis and optimization, has become a prominent research direction. Various types of convexity play an important role not only in analysis but also in many other areas, where they provide effective tools for deriving inequalities and establishing useful estimates. In this paper, a new generalization of exponential convexity is introduced. Several basic algebraic properties of the proposed class of functions are established. The corresponding Hermite–Hadamard type inequalities, together with their refined versions, are also derived. Moreover, the applicability of the obtained Hermite–Hadamard inequalities is demonstrated by establishing a number of inequalities between classical special means, thereby highlighting the potential of the introduced notion of convexity. Finally, several possible directions for further research in this area are discussed.

Keywords: exponentially (s, P)-convex function; Hermite–Hadamard inequalities; convexity.

2020 Mathematics Subject Classification: 26A51, 26D10, 26D15.

1. Introduction

The study of convexity of functions is one of the fundamental topics in modern mathematics, both from a theoretical perspective and in terms of practical applications. Its role is particularly prominent in the development and investigation of inequalities, as well as in the formalization of optimization problems.

In recent years, new notions of convexity have been intensively developed, motivated by the need to extend classical concepts and adapt them to various classes of functions and real-world problems. In this direction, several generalizations, such as m-convexity, s-convexity, and (s, m)-convexity (see [1,5,7,15,19,23]), have been introduced and have served as a starting point for further "exponential" modifications of these notions. A central emphasis in such studies is generally placed on Hermite–Hadamard type integral inequalities, since they clearly reflect the impact of new definitions and enable meaningful comparisons with well-known results.

In parallel, fractional calculus has become an important research area in which these new convexity classes have proved to be highly effective. The interplay between these topics allows one to formulate and prove new integral bounds and inequalities, thereby enriching the theory while also providing tools applicable in many disciplines, including physics, biomedicine, and signal processing. Moreover, the literature also explores connections between such inequalities and various means, including arithmetic, geometric, and other means, as well as special classes of functions and polynomials that provide a convenient framework for testing new convexity properties; several interesting applications can be found in [2,3,11,12,17,21].

The main motivation for this paper arises precisely from these contemporary trends. Instead of following the classical exponential approach (see [4,8,9,16,18,20,22,24]), this study focuses on exponential (s, P)-convexity (see Definition 3.1), which naturally combines parametric flexibility through s with a P-type structure (see also [6,10,13,14,25]). This framework makes it possible to place familiar ideas from convexity theory in a broader setting and to derive new and sharper forms of Hermite–Hadamard type inequalities, together with further consequences.

2. Preliminaries

In this section, the definitions of convexity and several of its extensions are presented. Two known results are also recalled.

Definition 2.1 (Convex function). *Let $I \subseteq \mathbb{R}$ be an interval. A function $F : I \to \mathbb{R}$ is called* convex *if for all $x, y \in I$ and for every $t \in [0, 1]$, it holds that $F(tx + (1-t)y) \leq tF(x) + (1-t)F(y)$.*

*E-mail address: nemanja.vucicevic@pmf.kg.ac.rs

Definition 2.2 (see [1]). *Let $s, m \in (0,1]$. A function $F : [0,\infty) \to \mathbb{R}$ is called* (s,m)-convex *if for all $x, y \in [0,\infty)$, $s, m \in (0,1]$ and for every $t \in [0,1]$, it holds that*

$$F(tx + m(1-t)y) \leq t^s F(x) + m(1-t)^s F(y).$$

In Definition 2.2, by setting $s = 1$, we obtain the class of m-convex functions. Properties of the functions defined in this way, as well as their applications, can be found in [15, 19, 23].

The following three definitions formalize exponential convexity, m-exponential convexity, and exponential m-convexity. The interested reader is referred to the cited papers for properties and examples.

Definition 2.3 (see [9]). *Let $I \subseteq \mathbb{R}$. A function $F : I \to \mathbb{R}$ is called* exponentially convex *if for all $x, y \in I$ and $t \in [0,1]$, it holds that*

$$F(tx + (1-t)y) \leq (e^t - 1)F(x) + \left(e^{1-t} - 1\right) F(y).$$

Definition 2.4 (see [4]). *A nonnegative function $F : I \to \mathbb{R}$ is called* (s,m)-exponential type convex *for some fixed $s, m \in (0,1]$, if*

$$F(tx + m(1-t)y) \leq (e^{st} - 1)F(x) + m\left(e^{(1-t)s} - 1\right) F(y).$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 2.5 (see [19]). *Let $m \in [0,1]$, and let $I \subseteq \mathbb{R}$ be an m-convex set. Then, a real-valued function $F : I \to \mathbb{R}$ is said to be* exponentially m-convex *if*

$$e^{F((1-t)x+mty)} \leq (1-t)e^{F(x)} + mte^{F(y)},$$

for all $x, y \in I$ and $t \in [0,1]$.

In what follows, we introduce the notion of an (s,P)-convex function; for $s = 1$, it reduces to the well-known P-convexity [6].

Definition 2.6 (see [14]). *Let $s \in (0,1]$. A function $F : I \subseteq \mathbb{R} \to \mathbb{R}$ is called an* (s,P)-function *if*

$$F(tx + (1-t)y) \leq \left(t^s + (1-t)^s\right)\left[F(x) + F(y)\right], \tag{1}$$

for all $x, y \in I$ and $t \in [0,1]$.

The following definition serves as the starting point for the new class of convex functions introduced in this paper. By appropriately generalizing this definition, we obtain a new type of convexity.

Definition 2.7 (see [13]). *A non-negative function $F : I \subseteq \mathbb{R} \to \mathbb{R}$ is called an* exponential type P-function *if for every $m, n \in I$ and $t \in [0,1]$,*

$$F\big(tm + (1-t)n\big) \leq \left(e^t + e^{1-t} - 2\right)\big(F(m) + F(n)\big). \tag{2}$$

The properties established in the following two lemmas are used in the proofs of some key results of this paper.

Lemma 2.1 (see [22]). *Let $0 < m < n$ and $0 < k \leq 1$. Let $F : \left[m, \frac{n}{k}\right] \to \mathbb{R}$ be differentiable on $\left(m, \frac{n}{k}\right)$ such that $F' \in L_1\left[m, \frac{n}{k}\right]$. Then,*

$$\frac{F(m) + F\left(\frac{n}{k}\right)}{2} - \frac{k}{n - km}\int_m^{\frac{n}{k}} F(v)\,dv = \frac{n - km}{2k}\int_0^1 (1-2t)F'\left(tm + (1-t)\frac{n}{k}\right)dt.$$

Lemma 2.2 (see [22]). *Let $0 < m < n$ and $0 < k \leq 1$. Let $F : [km, n] \to \mathbb{R}$ be differentiable on (km, n) such that $F' \in L_1[km, n]$. Then,*

$$\frac{1}{n - km}\int_{km}^n F(v)\,dv - F\left(\frac{km + n}{2}\right) = (n - km)\left[-\int_0^1 tF'(tn + k(1-t)m)\,dt + \int_{1/2}^1 F'(tn + k(1-t)m)\,dt\right].$$

3. Main Results

In the following definition, we introduce the notion of exponential (s,P)-convexity, which combines P-convexity and the parameter s within the framework of exponential-type convexity.

Definition 3.1 (Exponentially (s,P)-convex function). *Let $I \subseteq \mathbb{R}$ be an interval and let $s \in (0,1]$. A function $F : I \to [0,\infty)$ is said to be* exponentially (s,P)-convex *if, for all $m, n \in I$ and $t \in [0,1]$, the following inequality holds:*

$$F\left(tm + (1-t)n\right) \leq \left(e^{st} + e^{s(1-t)} - 2\right)\left(F(m) + F(n)\right). \tag{3}$$

Remark 3.1. *The set of all exponentially (s,P)-convex functions are denoted by $ECF_{(s,P)}$.*

Remark 3.2. *For $s=1$, Definition 3.1 reduces to the class of exponential type P-functions given in (2).*

Example 3.1. *Let $F:\mathbb{R}\to[0,\infty)$ be a function defined by $F(m)=m^2+1$. Since the map $u\mapsto u^2$ is convex, we have $(tm+(1-t)n)^2\le tm^2+(1-t)n^2\le m^2+n^2$, and consequently,*

$$F(tm+(1-t)n)=(tm+(1-t)n)^2+1\le m^2+n^2+1\le m^2+n^2+2=F(m)+F(n).$$

Put $\varphi_s(t)=e^{st}+e^{s(1-t)}-2$. The minimum of $\varphi_s(t)$ on $[0,1]$ is attained at $t=\frac{1}{2}$, hence

$$\varphi_s(t)\ge\varphi_s\left(\frac{1}{2}\right)=2e^{s/2}-2.$$

If $s\ge 2\ln\frac{3}{2}$, then $2e^{s/2}-2\ge 1$, and thus $\varphi_s(t)\ge 1$ for every $t\in[0,1]$. Therefore,

$$F(tm+(1-t)n)\le F(m)+F(n)\le\varphi_s(t)\big(F(m)+F(n)\big),$$

which implies that F is exponentially (s,P)-convex on $\mathbb{R}$ for every $s\in\left[2\ln\frac{3}{2},\,1\right]$.

In the following results, some basic algebraic properties of the functions belonging to the class $ECF_{(s,P)}$ are established.

Proposition 3.1. *Let F be exponentially (s,P)-convex on I. Let $k\ne 0$ and n be real numbers such that $J=\{x\mid kx+n\in I\}$ is an interval. Then, the function $G:J\to[0,\infty)$ defined by $G(x)=F(kx+n)$ is exponentially (s,P)-convex on J.*

Proof. For $x,y\in J$ and $t\in[0,1]$, we have

$$\begin{aligned}G\big(tx+(1-t)y\big)&=F\big(k(tx+(1-t)y)+n\big)=F\big(t(kx+n)+(1-t)(ky+n)\big)\\&\le\left(e^{ts}+e^{s(1-t)}-2\right)\big(F(kx+n)+F(ky+n)\big)=\left(e^{ts}+e^{s(1-t)}-2\right)\big(G(x)+G(y)\big),\end{aligned}$$

as desired. □

Theorem 3.1. *Let $F,G:I\to[0,\infty)$ be exponentially (s,P)-convex functions. Then, the following statements hold.*

(i) The function $F+G$ is exponentially (s,P)-convex.

(ii) If $\alpha\ge 0$, then αF is exponentially (s,P)-convex.

Proof. Let $m,n\in I$ and $t\in[0,1]$.

(i) Since F and G are exponentially (s,P)-convex, we have

$$F(tm+(1-t)n)\le\left(e^{st}+e^{s(1-t)}-2\right)(F(m)+F(n)),$$

and

$$G(tm+(1-t)n)\le\left(e^{st}+e^{s(1-t)}-2\right)(G(m)+G(n)).$$

Adding these inequalities gives

$$(F+G)(tm+(1-t)n)\le\left(e^{st}+e^{s(1-t)}-2\right)((F+G)(m)+(F+G)(n)).$$

(ii) If $\alpha\ge 0$, then

$$\begin{aligned}(\alpha F)(tm+(1-t)n)&=\alpha F(tm+(1-t)n)\\&\le\alpha\left(e^{st}+e^{s(1-t)}-2\right)(F(m)+F(n))\\&=\left(e^{st}+e^{s(1-t)}-2\right)((\alpha F)(m)+(\alpha F)(n)).\end{aligned}$$

Hence, αF is exponentially (s,P)-convex. □

Theorem 3.2. *Let $I,J\subseteq\mathbb{R}$ be intervals. Let $G:I\to J$ be a convex function and let $F:J\to[0,\infty)$ be a nondecreasing exponentially (s,P)-convex function. Then, $F\circ G:I\to[0,\infty)$ is exponentially (s,P)-convex.*

Proof. Let $m, n \in I$ and $t \in [0,1]$. Since G is convex and F is nondecreasing, we have $G(tm+(1-t)n) \le tG(m)+(1-t)G(n)$, and therefore

$$F(G(tm+(1-t)n)) \le F(tG(m)+(1-t)G(n)).$$

Since F is exponentially (s,P)-convex on J, it follows that

$$F(tG(m)+(1-t)G(n)) \le \left(e^{st}+e^{s(1-t)}-2\right)\left(F(G(m))+F(G(n))\right).$$

Hence,

$$(F\circ G)(tm+(1-t)n) \le \left(e^{st}+e^{s(1-t)}-2\right)\left((F\circ G)(m)+(F\circ G)(n)\right),$$

which proves the claim. □

Theorem 3.3. *Let $\{F_i\}_{i\in A}$ be a family of exponentially (s,P)-convex functions on an interval I, and define*

$$F(m)=\sup_{i\in A} F_i(m).$$

Let $C=\{m\in I:\ F(m)<\infty\}$. If C is an interval, then F is exponentially (s,P)-convex on C.

Proof. Let $m,n\in C$ and $t\in[0,1]$. Since each F_i is exponentially (s,P)-convex, we have

$$F_i(tm+(1-t)n) \le \left(e^{st}+e^{s(1-t)}-2\right)\left(F_i(m)+F_i(n)\right).$$

Taking the supremum over $i\in A$, we obtain

$$\begin{aligned}
F(tm+(1-t)n) &= \sup_{i\in A} F_i(tm+(1-t)n)\\
&\le \left(e^{st}+e^{s(1-t)}-2\right)\sup_{i\in A}\left(F_i(m)+F_i(n)\right)\\
&\le \left(e^{st}+e^{s(1-t)}-2\right)\left(\sup_{i\in A}F_i(m)+\sup_{i\in A}F_i(n)\right)\\
&= \left(e^{st}+e^{s(1-t)}-2\right)\left(F(m)+F(n)\right).
\end{aligned}$$

Thus, F is exponentially (s,P)-convex on C. □

Theorem 3.4. *Let $m<n$. Let $F:[m,n]\to[0,\infty)$ be exponentially (s,P)-convex. If F is symmetric with respect to $\frac{m+n}{2}$ (i.e., $F(v)=F(m+n-v)$ for every $v\in[m,n]$), then the following inequality holds:*

$$\frac{1}{4(e-1)}\,F\left(\frac{m+n}{2}\right) \le 2(e-1)\big(F(m)+F(n)\big).$$

Proof. Let $u\in[m,n]$ be an arbitrary point. Since $e^{st}+e^{(1-t)s}-2\le 2e-2=2(e-1)$, it follows that

$$F(u)=F\left(\frac{u-m}{n-m}\,n+\frac{n-u}{n-m}\,m\right) \le \left(e^{s\left(\frac{u-m}{n-m}\right)}+e^{s\left(\frac{n-u}{n-m}\right)}-2\right)\big(F(m)+F(n)\big) \le 2(e-1)\big(F(m)+F(n)\big).$$

Furthermore,

$$F\left(\frac{m+n}{2}\right)=F\left(\frac12 u+\frac12(m+n-u)\right) \le 2(e-1)\big(F(u)+F(m+n-u)\big)=2(e-1)\cdot 2F(u)=4(e-1)F(u).$$

Combining the previous two inequalities, we obtain

$$F\left(\frac{m+n}{2}\right) \le 4(e-1)\cdot 2(e-1)\big(F(m)+F(n)\big),$$

that is,

$$\frac{1}{4(e-1)}\,F\left(\frac{m+n}{2}\right) \le 2(e-1)\big(F(m)+F(n)\big),$$

as required. □

In the sequel, we investigate the newly defined type of convexity through extensions of the Hermite–Hadamard inequality, assuming that $F\in L_1([m,n])$ is integrable on $[m,n]$.

Theorem 3.5. *Let $F : [m, n] \to [0, \infty)$ be an exponentially (s, P)-convex function. If $m < n$ and $F \in L[m, n]$, then the following holds:*

$$\frac{1}{4\left(\sqrt{e^s}-1\right)} F\left(\frac{m+n}{2}\right) \le \frac{1}{n-m}\int_m^n F(v)\,dv \le \frac{2}{s}\big(F(m)+F(n)\big)\left(e^s-1-s\right).$$

Proof. For $t \in [0, 1]$, we have

$$\frac{m+n}{2} = \frac{1}{2}\big(tm+(1-t)n\big) + \frac{1}{2}\big(tn+(1-t)m\big).$$

Hence,

$$\begin{aligned} F\left(\frac{m+n}{2}\right) &= F\left(\frac{1}{2}\big(tm+(1-t)n\big) + \frac{1}{2}\big(tn+(1-t)m\big)\right) \\ &\le \left(e^{\frac{s}{2}}+e^{\frac{s}{2}}-2\right)\Big(F\big(tm+(1-t)n\big)+F\big(tn+(1-t)m\big)\Big) \\ &= 2\left(\sqrt{e^s}-1\right)\Big(F\big(tm+(1-t)n\big)+F\big(tn+(1-t)m\big)\Big). \end{aligned}$$

Integrating over $t \in [0, 1]$, we obtain

$$F\left(\frac{m+n}{2}\right) \le 4\left(\sqrt{e^s}-1\right)\int_0^1 F\big(tm+(1-t)n\big)\,dt = \frac{4\left(\sqrt{e^s}-1\right)}{n-m}\int_m^n F(v)\,dv,$$

which yields

$$\frac{1}{4\left(\sqrt{e^s}-1\right)} F\left(\frac{m+n}{2}\right) \le \frac{1}{n-m}\int_m^n F(v)\,dv.$$

Now,

$$\frac{1}{n-m}\int_m^n F(v)\,dv = \int_0^1 F\big(tm+(1-t)n\big)\,dt \le \big(F(m)+F(n)\big)\int_0^1\left(e^{st}+e^{s(1-t)}-2\right)dt.$$

Since

$$\int_0^1 e^{st}\,dt = \frac{e^s-1}{s} \quad \text{and} \quad \int_0^1 e^{s(1-t)}\,dt = \frac{e^s-1}{s},$$

we obtain

$$\int_0^1\left(e^{st}+e^{s(1-t)}-2\right)dt = \frac{2(e^s-1)}{s}-2 = \frac{2}{s}\left(e^s-1-s\right),$$

and therefore,

$$\frac{1}{n-m}\int_m^n F(v)\,dv \le \frac{2}{s}\big(F(m)+F(n)\big)\left(e^s-1-s\right),$$

as required. □

In the following theorems, in addition to establishing improved and generalized versions of the Hermite–Hadamard inequality, we also present, for practical purposes, the corresponding corollaries obtained directly by setting the constant $k = 1$. In this way, the basic forms of these inequalities for exponentially (s, P)-convex functions are obtained. Although sharper estimates are established in this paper, the basic versions are also included, since they are often more convenient for various applications.

Theorem 3.6. *Let $0 < m < n$ and $0 < k \le 1$. Let F be differentiable on an open interval containing $\left[m, \frac{n}{k}\right]$, and assume that $F' \in L_1\left[m, \frac{n}{k}\right]$. If $|F'|^q$ is exponentially (s, P)-convex on $\left[m, \frac{n}{k}\right]$ for some $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\left|\frac{F(m)+F\left(\frac{n}{k}\right)}{2} - \frac{k}{n-km}\int_m^{\frac{n}{k}} F(v)\,dv\right| \le \frac{n-km}{2k}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s}\left(e^s-1-s\right)\left(|F'(m)|^q+\left|F'\left(\tfrac{n}{k}\right)\right|^q\right)\right)^{\frac{1}{q}}.$$

Proof. By Lemma 2.1 and the fact that $|F'|^q$ is exponentially (s, P)-convex, we obtain

$$\begin{aligned} \left|\frac{F(m)+F\left(\frac{n}{k}\right)}{2} - \frac{k}{n-km}\int_m^{\frac{n}{k}} F(v)\,dv\right| &= \frac{n-km}{2k}\left|\int_0^1 (1-2t)\,F'\left(tm+(1-t)\frac{n}{k}\right)\,dt\right| \\ &\le \frac{n-km}{2k}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|F'\left(tm+(1-t)\frac{n}{k}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\le \frac{n-km}{2k}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left(e^{ts}+e^{s(1-t)}-2\right)dt\right)^{\frac{1}{q}}\left(|F'(m)|^q+\left|F'\left(\tfrac{n}{k}\right)\right|^q\right)^{\frac{1}{q}} \\ &= \frac{n-km}{2k}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s}\left(e^s-1-s\right)\right)^{\frac{1}{q}}\left(|F'(m)|^q+\left|F'\left(\tfrac{n}{k}\right)\right|^q\right)^{\frac{1}{q}}, \end{aligned}$$

which completes the proof. □

Corollary 3.1. *Let $0 < m < n$. Let F be differentiable on an open interval containing $[m, n]$, and assume that $F' \in L_1[m, n]$. If $|F'|^q$ is exponentially (s, P)-convex on $[m, n]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\left|\frac{F(m)+F(n)}{2} - \frac{1}{n-m}\int_m^n F(x)\,dx\right| \leq \frac{n-m}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s}\left(e^s-1-s\right)\left(|F'(m)|^q+|F'(n)\,|^q\right)\right)^{\frac{1}{q}}.$$

Theorem 3.7. *Let $0 < m < n$ and $0 < k \leq 1$. Let F be differentiable on an open interval containing $\left[m, \frac{n}{k}\right]$, and assume that $F' \in L_1\left[m, \frac{n}{k}\right]$. Suppose that $|F'|^q$ is exponentially (s, P)-convex on $\left[m, \frac{n}{k}\right]$ for a fixed $q \geq 1$. Then, the following holds:*

$$\left|\frac{F(m)+F\left(\frac{n}{k}\right)}{2} - \frac{k}{n-km}\int\limits_m^{\frac{n}{k}} F(v)dv\right| \leq \frac{n-km}{2k}\cdot\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\cdot\left(|F'(m)|^q+\left|F'\left(\frac{n}{k}\right)\right|^q\right)^{\frac{1}{q}}\left(\frac{2e^s(s-2)+8\sqrt{e^s}-s^2-2s-4}{s^2}\right)^{\frac{1}{q}}.$$

Proof. We start from the left-hand side and rewrite it using Lemma 2.1. Next, we apply the power-mean inequality to estimate the resulting integral, and finally we invoke the exponential (s, P)-convexity of $|F'|^q$ to complete the proof.

$$\begin{aligned}
\left|\frac{F(m)+F\left(\frac{n}{k}\right)}{2} - \frac{k}{n-km}\int\limits_m^{\frac{n}{k}} F(v)dv\right| &\leq \frac{n-km}{2k}\int\limits_0^1 |1-2t|\left|F'\left(tm+(1-t)\frac{n}{k}\right)\right|dt \\
&\leq \frac{n-km}{2k}\left(\int\limits_0^1|1-2t|dt\right)^{1-\frac{1}{q}}\cdot\left(\int\limits_0^1|1-2t|\left|F'\left(tm+(1-t)\frac{n}{k}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{n-km}{2k}\left(\int\limits_0^1|1-2t|dt\right)^{1-\frac{1}{q}} \\
&\quad\cdot\left(\int\limits_0^1|1-2t|\left(e^{s\cdot t}+e^{s\cdot(1-t)}-2\right)\left(|F'(m)|^q+\left|F'\left(\frac{n}{k}\right)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\leq \frac{n-km}{2k}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(|F'(m)|^q+\left|F'\left(\frac{n}{k}\right)\right|^q\right)^{\frac{1}{q}}\left(\int\limits_0^1|1-2t|\left(e^{s\cdot t}+e^{s\cdot(1-t)}-2\right)dt\right)^{\frac{1}{q}} \\
&= \frac{n-km}{2k}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{2e^s(s-2)+8\sqrt{e^s}-s^2-2s-4}{s^2}\right)^{\frac{1}{q}}\left(|F'(m)|^q+\left|F'\left(\frac{n}{k}\right)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Corollary 3.2. *Let $0 < m < n$. Let F be differentiable on an open interval containing $[m, n]$, and assume that $F' \in L_1[m, n]$. Suppose that $|F'|^q$ is exponentially (s, P)-convex on $[m, n]$ for a fixed $q \geq 1$. Then, the following inequality holds:*

$$\left|\frac{F(m)+F(n)}{2} - \frac{1}{n-m}\int_m^n F(v)\,dv\right| \leq \frac{n-m}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(|F'(m)|^q+|F'(n)|^q\right)^{\frac{1}{q}}\left(\frac{2e^s(s-2)+8\sqrt{e^s}-s^2-2s-4}{s^2}\right)^{\frac{1}{q}}.$$

Theorem 3.8. *Let $0 < m < n$ and $0 < k \leq 1$. Let F be differentiable on an open interval containing $[km, n]$, and assume that $F' \in L_1[km, n]$. Suppose that $|F'|^q$ is exponentially (s, P)-convex on $[km, n]$ for some $q > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\left|\frac{1}{n-km}\int_{km}^n F(u)\,du - F\left(\frac{km+n}{2}\right)\right| \leq (n-km)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s}(e^s-1-s)A_q\right)^{\frac{1}{q}} + \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{s}(e^s-1-s)A_q\right)^{\frac{1}{q}}\right],$$

where $A_q = |F'(n)|^q + |F'(km)|^q$.

Proof. By Lemma 2.2, we have

$$\frac{1}{n-km}\int_{km}^n F(u)\,du - F\left(\frac{km+n}{2}\right) = (n-km)\left[-\int_0^1 t\,F'\big(tn+k(1-t)m\big)\,dt + \int_{1/2}^1 F'\big(tn+k(1-t)m\big)\,dt\right].$$

Taking absolute values, we obtain

$$\left|\frac{1}{n-km}\int_{km}^n F(u)\,du - F\left(\frac{km+n}{2}\right)\right| \leq (n-km)\left[\int_0^1 t\left|F'\big(tn+k(1-t)m\big)\right|dt + \int_{1/2}^1\left|F'\big(tn+k(1-t)m\big)\right|dt\right].$$

Applying Hölder's inequality to the first integral gives

$$\begin{aligned}\int_0^1 t\left|F'\big(tn+k(1-t)m\big)\right|dt &\le \left(\int_0^1 t^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|F'\big(tn+k(1-t)m\big)\right|^q dt\right)^{\frac{1}{q}}\\ &= \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left|F'\big(tn+k(1-t)m\big)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Similarly, for the second integral, we get

$$\begin{aligned}\int_{1/2}^1 \left|F'\big(tn+k(1-t)m\big)\right|dt &\le \left(\int_{1/2}^1 1^p\,dt\right)^{\frac{1}{p}}\left(\int_{1/2}^1 \left|F'\big(tn+k(1-t)m\big)\right|^q dt\right)^{\frac{1}{q}}\\ &= \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\int_{1/2}^1 \left|F'\big(tn+k(1-t)m\big)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|F'|^q$ is exponentially (s,P)-convex and $tn+k(1-t)m = tn+(1-t)km$, we have

$$\left|F'\big(tn+k(1-t)m\big)\right|^q \le \left(e^{st}+e^{s(1-t)}-2\right)\left(|F'(n)|^q+|F'(km)|^q\right).$$

Therefore,

$$\int_0^1 \left|F'\big(tn+k(1-t)m\big)\right|^q dt \le A_q\int_0^1 \left(e^{st}+e^{s(1-t)}-2\right)dt = A_q\cdot\frac{2}{s}\left(e^s-1-s\right),$$

where $A_q = |F'(n)|^q+|F'(km)|^q$. Also,

$$\int_{1/2}^1 \left|F'\big(tn+k(1-t)m\big)\right|^q dt \le A_q\int_{1/2}^1 \left(e^{st}+e^{s(1-t)}-2\right)dt = A_q\cdot\frac{1}{s}\left(e^s-1-s\right).$$

Combining the above estimates, we obtain

$$\left|\frac{1}{n-km}\int_{km}^n F(u)\,du - F\left(\frac{km+n}{2}\right)\right| \le (n-km)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s}\left(e^s-1-s\right)A_q\right)^{\frac{1}{q}} + \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{s}\left(e^s-1-s\right)A_q\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Theorem 3.9. *Let $0<m<n$ and $0<k\le 1$. Let F be differentiable on an open interval containing $[km,n]$, and assume that $F'\in L_1[km,n]$. Suppose that $|F'|^q$ is exponentially (s,P)-convex on $[km,n]$ for some $q\ge 1$. Then, the following inequality holds:*

$$\left|\frac{1}{n-km}\int_{km}^n F(v)\,dv - F\left(\frac{km+n}{2}\right)\right| \le 2(n-km)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{e^s-s-1}{s}\left(|F'(km)|^q+|F'(n)|^q\right)\right]^{\frac{1}{q}}.$$

Proof. By Lemma 2.2, we have

$$\frac{1}{n-km}\int_{km}^n F(v)\,dv - F\left(\frac{km+n}{2}\right) = (n-km)\left[-\int_0^1 t\,F'\big(tn+k(1-t)m\big)\,dt + \int_{1/2}^1 F'\big(tn+k(1-t)m\big)\,dt\right].$$

Taking absolute values, we obtain

$$\left|\frac{1}{n-km}\int_{km}^n F(v)\,dv - F\left(\frac{km+n}{2}\right)\right| \le (n-km)\left[\int_0^1 t\left|F'\big(tn+k(1-t)m\big)\right|dt + \int_{1/2}^1 \left|F'\big(tn+k(1-t)m\big)\right|dt\right].$$

We estimate the first integral by the power-mean inequality:

$$\begin{aligned}\int_0^1 t\left|F'\big(tn+k(1-t)m\big)\right|dt &\le \left(\int_0^1 t\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|F'\big(tn+k(1-t)m\big)\right|^q dt\right)^{\frac{1}{q}}\\ &= \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|F'\big(tn+k(1-t)m\big)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|F'|^q$ is exponentially (s,P)-convex on $[km,n]$, we have

$$\left|F'\big(tn+k(1-t)m\big)\right|^q \le \left(e^{st}+e^{s(1-t)}-2\right)\left(|F'(n)|^q+|F'(km)|^q\right).$$

So, we obtain

$$\int_0^1 t\left|F'\big(tn+k(1-t)m\big)\right|^q dt \le \left(|F'(n)|^q+|F'(km)|^q\right)\int_0^1 t\left(e^{st}+e^{s(1-t)}-2\right) dt.$$

Therefore, by evaluating the corresponding integral, we have

$$\int_0^1 t\left|F'\big(tn+k(1-t)m\big)\right| dt \le \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{e^s-s-1}{s}\left(|F'(n)|^q+|F'(km)|^q\right)\right]^{\frac{1}{q}}.$$

Now, we estimate the second integral. Again, by the power-mean inequality, we have

$$\begin{aligned}\int_{1/2}^1 \left|F'\big(tn+k(1-t)m\big)\right| dt &\le \left(\int_{1/2}^1 1\,dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1 \left|F'\big(tn+k(1-t)m\big)\right|^q dt\right)^{\frac{1}{q}} \\ &= \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1 \left|F'\big(tn+k(1-t)m\big)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Using again the exponentially (s,P)-convexity of $|F'|^q$, we get

$$\int_{1/2}^1 \left|F'\big(tn+k(1-t)m\big)\right|^q dt \le \left(|F'(n)|^q+|F'(km)|^q\right)\int_{1/2}^1 \left(e^{st}+e^{s(1-t)}-2\right) dt.$$

Therefore, we obtain

$$\int_{1/2}^1 \left|F'\big(tn+k(1-t)m\big)\right| dt \le \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{e^s-s-1}{s}\left(|F'(n)|^q+|F'(km)|^q\right)\right]^{\frac{1}{q}}.$$

Combining the estimates for both integrals, we obtain

$$\left|\frac{1}{n-km}\int_{km}^n F(v)\,dv - F\left(\frac{km+n}{2}\right)\right| \le 2(n-km)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{e^s-s-1}{s}\left(|F'(km)|^q+|F'(n)|^q\right)\right]^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 3.3. *Let $0<m<n$. Let F be differentiable on an open interval containing $[m,n]$, and assume that $F'\in L_1[m,n]$. Suppose that $|F'|^q$ is exponentially (s,P)-convex on $[m,n]$ for some $q\ge 1$. Then, the following inequality holds:*

$$\left|\frac{1}{n-m}\int_m^n F(v)\,dv - F\left(\frac{m+n}{2}\right)\right| \le 2(n-m)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{e^s-s-1}{s}\left(|F'(m)|^q+|F'(n)|^q\right)\right]^{\frac{1}{q}}.$$

4. Applications to Special Means

Throughout this section, let $0<m<n$, and denote the classical means of two positive numbers m and n by

$$A(m,n)=\frac{m+n}{2},\qquad G(m,n)=\sqrt{mn},\qquad H(m,n)=\frac{2mn}{m+n},\qquad L(m,n)=\frac{n-m}{\ln n-\ln m}\quad (m\ne n).$$

Remark 4.1. *If $F: I\to[0,\infty)$ is P-convex on an interval I, that is,*

$$F(tm+(1-t)n)\le F(m)+F(n),$$

for all $m,n\in I$ and $t\in[0,1]$, and if

$$s\in\left[2\ln\frac{3}{2},\,1\right],$$

then F is exponentially (s,P)-convex on I. Indeed, for such s, we have

$$e^{st}+e^{s(1-t)}-2\ge 1,\qquad t\in[0,1].$$

Therefore,

$$F(tm+(1-t)n)\le F(m)+F(n)\le\left(e^{st}+e^{s(1-t)}-2\right)\left(F(m)+F(n)\right).$$

Corollary 4.1. *Let* $0 < m < n$ *and* $s \in \left[2\ln\frac{3}{2},\, 1\right]$. *Then,*

$$\frac{1}{4\left(\sqrt{e^s}-1\right)}\, G(m,n) \le L(m,n) \le \frac{4}{s}\left(e^s-1-s\right) A(m,n).$$

Proof. Take $F(v) = e^v$ on $[\ln m, \ln n]$. Since $F \ge 0$ and F is convex, it is P-convex; hence, by Remark 4.1, F is exponentially (s, P)-convex for $s \in \left[2\ln\frac{3}{2}, 1\right]$. Applying Theorem 3.5 to the interval $[\ln m, \ln n]$ yields

$$\frac{1}{4(\sqrt{e^s}-1)}\, e^{\frac{\ln m+\ln n}{2}} \le \frac{1}{\ln n - \ln m}\int_{\ln m}^{\ln n} e^v\, dv \le \frac{2}{s}\left(e^{\ln m} + e^{\ln n}\right)\left(e^s-1-s\right).$$

Noting that $e^{\frac{\ln m+\ln n}{2}} = \sqrt{mn} = G(m,n)$,

$$\frac{1}{\ln n - \ln m}\int_{\ln m}^{\ln n} e^v\, dv = \frac{n-m}{\ln n - \ln m} = L(m,n) \quad \text{and} \quad e^{\ln m} + e^{\ln n} = m+n = 2A(m,n),$$

we obtain the stated inequality. □

Corollary 4.2. *Let* $0 < m < n$ *and* $s \in \left[2\ln\frac{3}{2},\, 1\right]$. *Then,*

$$\frac{s}{4\left(e^s-1-s\right)}\, H(m,n) \le L(m,n) \le 4\left(\sqrt{e^s}-1\right) A(m,n).$$

Proof. Apply Theorem 3.5 to $F(v) = \frac{1}{v}$ on $[m, n]$. Since $F \ge 0$ and F is convex on $(0, \infty)$, it is P-convex, and Remark 4.1 guarantees that F is exponentially (s, P)-convex for $s \in \left[2\ln\frac{3}{2}, 1\right]$. Hence,

$$\frac{1}{4(\sqrt{e^s}-1)} \cdot \frac{1}{\frac{m+n}{2}} \le \frac{1}{n-m}\int_m^n \frac{1}{v}\, dv \le \frac{2}{s}\left(\frac{1}{m}+\frac{1}{n}\right)\left(e^s-1-s\right).$$

Using

$$\frac{1}{n-m}\int_m^n \frac{1}{v}\, dv = \frac{\ln n - \ln m}{n-m} = \frac{1}{L(m,n)}, \qquad \frac{1}{m}+\frac{1}{n} = \frac{m+n}{mn} = \frac{2}{H(m,n)}, \qquad \frac{1}{\frac{m+n}{2}} = \frac{1}{A(m,n)},$$

we get

$$\frac{1}{4(\sqrt{e^s}-1)} \cdot \frac{1}{A(m,n)} \le \frac{1}{L(m,n)} \le \frac{4\left(e^s-1-s\right)}{s\, H(m,n)}.$$

Taking reciprocals (all quantities are positive) gives the desired bounds. □

For $r > 0$, the *generalized logarithmic mean* $L_r(m,n)$ is defined by

$$L_r(m,n) = \left(\frac{n^{r+1} - m^{r+1}}{(r+1)(n-m)}\right)^{1/r} \qquad (m \ne n),$$

and note that $L_1(m,n) = A(m,n)$.

Corollary 4.3. *Let* $0 < m < n$*,* $r \ge 1$ *and* $s \in \left[2\ln\frac{3}{2},\, 1\right]$. *Then,*

$$\left(\frac{1}{4(\sqrt{e^s}-1)}\right)^{1/r} A(m,n) \le L_r(m,n) \le \left(\frac{4}{s}\left(e^s-1-s\right)\right)^{1/r} M_r(m,n),$$

where $M_r(m,n) = \left(\dfrac{m^r+n^r}{2}\right)^{1/r}$ *is the power mean of order* r.

Proof. Set $F(v) = v^r$ on $[m, n]$. For $r \ge 1$, F is convex and nonnegative on $(0, \infty)$. Hence, P-convex; by Remark 4.1, it is exponentially (s, P)-convex for $s \in \left[2\ln\frac{3}{2}, 1\right]$. Applying Theorem 3.5 gives

$$\frac{1}{4(\sqrt{e^s}-1)}\left(\frac{m+n}{2}\right)^r \le \frac{1}{n-m}\int_m^n v^r\, dv \le \frac{2}{s}\left(m^r+n^r\right)\left(e^s-1-s\right).$$

Since

$$\frac{1}{n-m}\int_m^n v^r\, dv = \frac{n^{r+1}-m^{r+1}}{(r+1)(n-m)} = L_r(m,n)^r,$$

taking the r-th root and using $m^r + n^r = 2M_r(m,n)^r$ yields the claim. □

5. Conclusion

In this paper, a new class of functions is introduced by generalizing exponentially convex functions with respect to the parameter s and combining them with P-convex functions. In this way, the existing class of exponentially P-convex functions is significantly extended. Several basic algebraic properties of this newly defined class are presented. In addition, a number of Hermite–Hadamard type inequalities, including both their basic and refined forms, are established. Applications to various classical means are also presented.

The potential for further research in this direction is substantial from several viewpoints: investigating links with other related notions of convexity, pursuing additional generalizations, and employing this approach in the introduction of new convexity concepts. Moreover, as supported by a large body of recent literature, a particularly promising direction concerns applications in fractional calculus and the further use of this type of convexity within that framework.

Acknowledgment

This work was supported by the Serbian Ministry of Science, Technological Development and Innovation (Agreement No. 451-03-34/2026-03/ 200122).

References

[1] G. A. Anastassiou, Generalised fractional Hermite–Hadamard inequalities involving m-convexity and (s, m)-convexity, *Facta Univ. Ser. Math. Inform.* **28** (2013) 107–126.

[2] S. Aslan, A. O. Akdemir, Fractional inequalities of Milne type for twice differentiable m-convex and (α, m)-convex functions, *Electron. J. Math.* **10** (2025) 38–51.

[3] M. Avcı-Ardıç, A. O. Akdemir, E. Set, On new integral inequalities via geometric-arithmetic convex functions with applications, *Sahand Commun. Math. Anal.* **19** (2022) 1–14.

[4] S. I. Butt, A. Kashuri, J. Nasir, Hermite–Hadamard type inequalities via new exponential type convexity and their applications, *Filomat* **35** (2021) 1803–1822.

[5] S. S. Dragomir, On some new inequalities of Hermite–Hadamard type for m-convex functions, *Tamkang J. Math.* **33** (2002) 45–56.

[6] S. S. Dragomir, J. Pečarić, L. E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.* **21** (1995) 335–341.

[7] H. Hudzik, L. Maligranda, Some remarks on s-convex functions, *Aequationes Math.* **48** (1994) 100–111.

[8] M. Kadakal, Exponential convex functions with respect to s, *Sahand Commun. Math. Anal.* **21** (2024) 275–287.

[9] M. Kadakal, İ. İşcan, Exponential type convexity and some related inequalities, *J. Inequal. Appl.* **2020** (2020) #82.

[10] M. Kadakal, İ. İşcan, H. Kadakal, Some new integral inequalities for exponential type P-functions, *Ann. Univ. Craiova Math. Comput. Sci. Ser.* **51** (2024) 291–302.

[11] A. Kashuri, S. Iqbal, S. I. Butt, J. Nasir, K. S. Nisar, T. Abdeljawad, Trapezium-type inequalities for k-fractional integrals via new exponential-type convexity and their applications, *J. Math.* **2020** (2020) #8672710.

[12] A. Munir, H. Budak, A. Kashuri, I. Faiz, H. Kara, A. Q. Qayyum, Some new improvements of Hermite–Hadamard type inequalities using strongly (s, m)-convex functions with applications, *Sahand Commun. Math. Anal.* **22** (2025) 307–332.

[13] S. Numan, İ. İşcan, On exponential type P-functions, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **70** (2021) 497–509.

[14] S. Numan, İ. İşcan, On (s, P)-functions and related inequalities, *Sigma J. Eng. Nat. Sci.* **40** (2022) 585–592.

[15] S. Özcan, Hermite–Hadamard type inequalities for m-convex and (α, m)-convex functions, *J. Inequal. Appl.* **2020** (2020) #175.

[16] S. Özcan, Hermite–Hadamard type inequalities for exponential type multiplicatively convex functions, *Filomat* **37** (2023) 9777–9789.

[17] S. Panwar, R. M. Pandey, P. Rai, S. D. Purohit, D. L. Suthar, Hermite–Hadamard–Mercer's type inequalities for ABK-fractional integrals via strong convexity and its applications, *Discontin. Nonlinearity Complex.* **13** (2024) 663–674.

[18] X. Qiang, G. Farid, J. Pečarić, S. B. Akbar, Generalized fractional integral inequalities for exponentially (s, m)-convex functions, *J. Inequal. Appl.* **2020** (2020) #70.

[19] S. Rashid, F. Safdar, A. O. Akdemir, M. A. Noor, K. I. Noor, Some new fractional integral inequalities for exponentially m-convex functions via extended generalized Mittag-Leffler function, *J. Inequal. Appl.* **2019** (2019) #299.

[20] S. K. Sahoo, M. Tariq, H. Ahmad, J. Nasir, H. Aydi, A. Mukheimer, New Ostrowski-type fractional integral inequalities via generalized exponential-type convex functions and applications, *Symmetry* **13** (2021) #1429.

[21] M. Samraiz, A. Saleem, S. Naheed, Fractional integral inequalities for m-polynomial exponential type s-convex functions, *Filomat* **39** (2025) 1373–1388.

[22] M. Tariq, H. Ahmad, C. Cesarano, H. Abu-Zinadah, A. E. Abouelregal, S. Askar, Novel analysis of Hermite–Hadamard type integral inequalities via generalized exponential type m-convex functions, *Mathematics* **10** (2022) #31.

[23] S. Toader, The order of a star-convex function, *Bull. Appl. Comput. Math.* **85** (1998) 347–350.

[24] N. Vučićević, On m-exponential convexity with respect to s and its applications, *Sahand Commun. Math. Anal.* **23** (2026) 203–218.

[25] N. Vučićević, M. A. Noor, K. I. Noor, (s, m, P)-convexity: A new class of generalized convex functions, *Filomat*, In press.