

Research Article

**Asymptotic Expansions for a Class of High-Order Nonlinear Difference Equations With General Decay Rate**Qin Diao<sup>1,2</sup>, Yong-Guo Shi<sup>2,\*</sup>, Na Li<sup>1,2</sup>, Kelin Li<sup>1</sup><sup>1</sup>College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, Sichuan, P. R. China<sup>2</sup>College of Mathematics and Big Data, Neijiang Normal University, Neijiang, Sichuan, P. R. China

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© 2026 the authors. This is an open-access article under the CC BY (International 4.0) license ([www.creativecommons.org/licenses/by/4.0/](http://www.creativecommons.org/licenses/by/4.0/)).**Abstract**

This paper investigates the asymptotic behavior of solutions to the higher-order nonlinear difference equation

$$x_{n+1} = x_n f(x_{n-k}), \quad n > k, \quad k \in \mathbb{N},$$

where the continuous function  $f$ , defined on the open interval  $(a, \infty)$  with  $a \geq 0$ , admits an asymptotic expansion of the form

$$f(x) = 1 + L_1 x^{-\alpha} + L_2 x^{-2\alpha} + L_3 x^{-3\alpha} + o(x^{-3\alpha}),$$

where  $\alpha > 0$ ,  $L_1 > 0$ , and  $L_2, L_3 \in \mathbb{R}$ . The paper extends previous work beyond the case  $\alpha = 1$ . The main contribution is the derivation of a detailed asymptotic expansion for the solution sequence  $x_n$  up to several terms, including logarithmic corrections, revealing intricate and previously unknown dependencies on the decay rate  $\alpha$ , the delay parameter  $k$ , and the expansion coefficients  $L_i$  of  $f$ . The analysis is based on an iterative refinement approach and employs tools such as the Stolz–Cesàro theorem and Taylor expansions.

**Keywords:** delay logistic model; difference equation; asymptotic expansion.**2020 Mathematics Subject Classification:** 39A12, 11B37, 41A60.**1. Introduction**

The asymptotic analysis of nonlinear difference equations plays a fundamental role in understanding the long-term behavior of discrete dynamical systems arising in population biology, economics, probability theory, and other applied sciences. In particular, first-order recurrences of the form  $x_{n+1} = x_n f(x_n)$  have been extensively studied. For instance, when

$$f(x) = 1 - ax^k + o(x^k),$$

where  $a > 0$  and  $k > 0$ , it is known that  $x_n \sim (akn)^{-\frac{1}{k}}$  as  $n \rightarrow \infty$  [7]. For second-order equations, Wong and Li [9, 10] derived detailed asymptotic expansions, while Cao and Li [3] later developed a general theory of uniform asymptotic expansions near turning points. A comprehensive overview of asymptotic methods for linear difference equations can be found in [8]. For higher-order nonlinear recurrences, Stević [5] studied equations of the type

$$f(x, y) = px + (1 - p)y - \sum_{s=m}^{\infty} \mathcal{K}_s(x, y),$$

where  $0 < p < 1$ ,  $m > 1$ , and established that

$$x_n \sim \left( \frac{2-p}{(m-1) \sum_{i=0}^m a_{i,m}} \right)^{1/(m-1)} \cdot \frac{1}{m^{-1} \sqrt[n]{n}}.$$

In a different direction, Berg and Stević [1, 2] analyzed

$$y_n = \frac{y_{n-k}}{1 + y_{n-1} \cdots y_{n-k+1}}, \quad n \in \mathbb{N}, \quad k \in \{2, 3, \dots\},$$

and obtained the expansion

$$y_n = \left( \frac{k}{(k-1)n} \right)^{1/(k-1)} \left( 1 + \frac{a \ln n}{n} + \frac{b \ln^2 n}{n^2} + o\left( \frac{\ln^2 n}{n^2} \right) \right).$$

\*Corresponding author ([scumat@163.com](mailto:scumat@163.com)).

In this paper, we focus on the asymptotic behavior of the following class of higher-order nonlinear difference equations:

$$x_{n+1} = x_n f(x_{n-k}), \quad n > k, \quad \text{for some } k \in \mathbb{N}, \tag{1}$$

where the function  $f$  admits an asymptotic expansion of the form  $f(x) = 1 + L_1 x^{-\alpha} + L_2 x^{-2\alpha} + L_3 x^{-3\alpha} + o(x^{-3\alpha})$ ,  $x \rightarrow \infty$ ,  $\alpha > 0$ ,  $L_1 > 0$ ,  $L_2, L_3 \in \mathbb{R}$ . While the case  $\alpha = 1$  is well understood [4], many applied models, such as the generalized delay logistic model discussed in Section 4, naturally lead to equations with a general decay rate  $\alpha \neq 1$ . However, the asymptotic analysis for general  $\alpha > 0$  presents significant new challenges, since the exponents in the expansion of  $f(x)$  no longer align naturally with the harmonic series, and logarithmic correction terms arise in a more complex manner. The aim of this paper is to bridge this gap. Our work extends these results to arbitrary  $\alpha > 0$ , employing a more general and systematic approach that allows for broader applications. We establish detailed asymptotic expansions for  $x_n$  up to several terms, including logarithmic corrections, and illustrate the results with applications to generalized delay logistic models. The proofs rely on careful iterative refinements and the use of auxiliary sequences, combined with asymptotic tools such as the Stolz–Cesàro theorem and Taylor expansions.

The remainder of the paper is organized as follows. Section 2 presents preliminary results and technical lemmas. Section 3 contains the main theorems on asymptotic expansions. Section 4 provides examples and applications. Concluding remarks are given in Section 5.

## 2. Preliminaries

In this section, we establish two general facts about asymptotic expansions and prove Lemma 2.1, which gives the first-order behavior of  $x_n$  and the leading logarithmic correction term.

For notational convenience, we denote the  $\alpha$ -th power of  $f(x)$ , i.e.,  $(f(x))^\alpha$ , by  $f^\alpha(x)$ .

**Fact 2.1.** *Let  $g : (0, b) \rightarrow (0, 1)$  with  $b > 0$ , and suppose that as  $x \rightarrow 0^+$ ,  $g(x) = 1 - l_1 x^\alpha + l_2 x^{2\alpha} + l_3 x^{3\alpha} + o(x^{3\alpha})$ , where  $\alpha > 0$ ,  $l_1 > 0$ , and  $l_2, l_3 \in \mathbb{R}$ . Then*

$$\frac{1}{g(x)} = 1 + l_1 x^\alpha + (l_1^2 - l_2) x^{2\alpha} + (l_1^3 - 2l_1 l_2 - l_3) x^{3\alpha} + o(x^{3\alpha}).$$

In particular, by substituting  $x$  with  $1/x$ , we obtain as  $x \rightarrow \infty$ ,

$$\left(g\left(\frac{1}{x}\right)\right)^{-1} = 1 + \frac{l_1}{x^\alpha} + \frac{l_1^2 - l_2}{x^{2\alpha}} + \frac{l_1^3 - 2l_1 l_2 - l_3}{x^{3\alpha}} + o\left(\frac{1}{x^{3\alpha}}\right).$$

**Fact 2.2.** *Let  $\alpha > 0$  and  $a, b, c, d \in \mathbb{R}$ . Then*

$$\begin{aligned} & \left(1 + \frac{a \ln n}{n} + \frac{b}{n} + \frac{c \ln n}{n^2} + \frac{d}{n^2} + o\left(\frac{1}{n^2}\right)\right)^{-\frac{1}{\alpha}} \\ &= 1 - \frac{a}{\alpha} \cdot \frac{\ln n}{n} - \frac{b}{\alpha n} + \frac{(1 + \alpha)a^2}{2\alpha^2} \cdot \frac{\ln^2 n}{n^2} + \frac{(1 + \alpha)ab - \alpha c}{\alpha^2} \cdot \frac{\ln n}{n^2} + \frac{(1 + \alpha)b^2 - 2\alpha d}{2\alpha^2 n^2} + o\left(\frac{1}{n^2}\right). \end{aligned}$$

The following lemma provides the first two terms in the asymptotic expansion of the solution to Equation (1).

**Lemma 2.1.** *Let the sequence  $\{x_n\}_{n \geq k+1}$  be defined by Equation (1). Then the following assertions hold:*

(i)  $\lim_{n \rightarrow \infty} x_n = \infty$  and  $\lim_{n \rightarrow \infty} \frac{x_n}{n^{1/\alpha}} = (\alpha L_1)^{1/\alpha}$ ;

(ii)  $x_n^\alpha = \alpha L_1 n + A \ln n + o(\ln n)$ , where

$$A = \frac{\alpha L_2 + \left(\binom{\alpha}{2} + \alpha^2 k\right) L_1^2}{\alpha L_1}.$$

**Proof.** (i) Note that  $x_{k+2} = x_{k+1} f(x_1)$ . Since  $x_1 > a$ ,  $x_k > 0$ , and  $f(x_1) > 1$ , it follows by induction that  $x_n > 0$  and  $f(x_{n-k}) > 1$  for all  $n \geq k + 1$ . Consequently,

$$x_{n+1} - x_n = x_n (f(x_{n-k}) - 1) > 0, \tag{2}$$

which implies that  $\{x_n\}$  is strictly increasing.

Next, we prove that  $\{x_n\}$  is unbounded. Suppose it is bounded above. By the monotone convergence theorem,  $x_n \rightarrow l$  with  $l > a \geq 0$ . Taking limits in (2) yields  $l(f(l) - 1) = 0$ , contradicting  $l > 0$  and  $f(l) > 1$ . Thus,  $\{x_n\}$  is unbounded and diverges to infinity.

Since

$$\lim_{x \rightarrow \infty} x^\alpha (f(x) - 1) = L_1 > 0,$$

it follows that  $\lim_{x \rightarrow \infty} f(x) = 1$ . From the recurrence relation  $x_{n+1}/x_n = f(x_{n-k})$ , we obtain  $\lim_{n \rightarrow \infty} x_{n+1}/x_n = 1$ . This implies that for any  $i, j \in \mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} \frac{x_{n-i}}{x_{n-j}} = 1. \tag{3}$$

We have

$$\lim_{x \rightarrow \infty} x^\alpha (f^\alpha(x) - 1) = \alpha L_1.$$

Applying the Stolz–Cesàro theorem, we compute

$$\lim_{n \rightarrow \infty} \frac{x_n^\alpha}{n} = \lim_{n \rightarrow \infty} (x_{n+1}^\alpha - x_n^\alpha) = \lim_{n \rightarrow \infty} \frac{x_n^\alpha}{x_{n-k}^\alpha} \cdot x_{n-k}^\alpha (f^\alpha(x_{n-k}) - 1) = \alpha L_1.$$

Since  $x_n \rightarrow \infty$ , one expects  $x_n^\alpha \sim \alpha L_1 n$ , which implies  $x_n \sim (\alpha L_1 n)^{1/\alpha}$ . Hence

$$\lim_{n \rightarrow \infty} \frac{x_n}{n^{1/\alpha}} = (\alpha L_1)^{1/\alpha}.$$

(ii) We obtain the refined limit

$$\lim_{x \rightarrow \infty} x^{2\alpha} \left( f^\alpha(x) - 1 - \frac{\alpha L_1}{x^\alpha} \right) = \binom{\alpha}{2} L_1^2 + \alpha L_2. \tag{4}$$

We proceed by defining an auxiliary sequence  $y_n = x_n^\alpha - \alpha L_1 n$  and examining its difference using a Taylor expansion of the recurrence relation. Then

$$\begin{aligned} y_{n+1} - y_n &= x_{n+1}^\alpha - x_n^\alpha - \alpha L_1 \\ &= x_n^\alpha (f^\alpha(x_{n-k}) - 1) - \alpha L_1 \\ &= x_n^\alpha \left( f^\alpha(x_{n-k}) - 1 - \frac{\alpha L_1}{x_{n-k}^\alpha} \right) + \frac{\alpha L_1 (x_n^\alpha - x_{n-k}^\alpha)}{x_{n-k}^\alpha}. \end{aligned}$$

Observing that

$$x_n^\alpha - x_{n-k}^\alpha = \sum_{i=1}^k (x_{n-i+1}^\alpha - x_{n-i}^\alpha) = \sum_{i=1}^k x_{n-i}^\alpha (f^\alpha(x_{n-i-k}) - 1),$$

we rewrite the difference as

$$y_{n+1} - y_n = x_n^\alpha \left( f^\alpha(x_{n-k}) - 1 - \frac{\alpha L_1}{x_{n-k}^\alpha} \right) + \alpha L_1 \sum_{i=1}^k \frac{x_{n-i}^\alpha}{x_{n-k}^\alpha} (f^\alpha(x_{n-i-k}) - 1).$$

Multiplying both sides by  $x_n^\alpha$  yields

$$x_n^\alpha (y_{n+1} - y_n) = \frac{x_n^{2\alpha}}{x_{n-k}^{2\alpha}} \cdot x_{n-k}^{2\alpha} \left( f^\alpha(x_{n-k}) - 1 - \frac{\alpha L_1}{x_{n-k}^\alpha} \right) + \alpha L_1 \sum_{i=1}^k \frac{x_{n-i}^\alpha \cdot x_n^\alpha}{x_{n-k}^\alpha \cdot x_{n-i-k}^\alpha} \cdot x_{n-i-k}^\alpha (f^\alpha(x_{n-i-k}) - 1).$$

Taking the limit as  $n \rightarrow \infty$  and using (3) and (4), we find

$$\lim_{n \rightarrow \infty} x_n^\alpha (y_{n+1} - y_n) = \binom{\alpha}{2} L_1^2 + \alpha L_2 + \alpha^2 k L_1^2.$$

Since  $\lim_{n \rightarrow \infty} x_n^\alpha/n = \alpha L_1$ , it follows that

$$\lim_{n \rightarrow \infty} \frac{y_{n+1} - y_n}{1/n} = \lim_{n \rightarrow \infty} \frac{x_n^\alpha (y_{n+1} - y_n)}{x_n^\alpha/n} = \frac{\binom{\alpha}{2} L_1^2 + \alpha L_2 + \alpha^2 k L_1^2}{\alpha L_1} = A.$$

Applying the Stolz–Cesàro theorem again, we obtain

$$\lim_{n \rightarrow \infty} \frac{y_n}{\sum_{j=1}^{n-1} \frac{1}{j}} = \lim_{n \rightarrow \infty} \frac{y_{n+1} - y_n}{1/n} = A.$$

Since  $\sum_{j=1}^{n-1} \frac{1}{j} \sim \ln n$ , we conclude that  $y_n = A \ln n + o(\ln n)$ , which completes the proof. □

To derive the third term in the asymptotic expansion of  $x_n^\alpha$ , we require the following estimate.

**Proposition 2.1.** *Under the hypotheses of Lemma 2.1, we have*

$$\frac{\binom{\alpha}{2}L_1^2 + \alpha L_2}{x_{n-k}^{2\alpha}} x_n^\alpha + \alpha^2 L_1^2 \sum_{i=1}^k \frac{x_{n-i}^\alpha}{x_{n-k}^\alpha x_{n-i-k}^\alpha} = \frac{A}{n} - \frac{A^2}{\alpha L_1} \cdot \frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right),$$

where

$$A = \frac{\alpha L_2 + \left(\binom{\alpha}{2} + \alpha^2 k\right) L_1^2}{\alpha L_1}.$$

**Proof.** Fix  $i \in \mathbb{N}$ . By Lemma 2.1 and Equation (3), we have

$$\lim_{n \rightarrow \infty} \frac{x_{n-i}^\alpha - \alpha L_1 n}{\ln n} = A,$$

which implies

$$x_{n-i}^\alpha = \alpha L_1 n \left( 1 + \frac{A}{\alpha L_1} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right) \right).$$

It follows that

$$\frac{x_{n-i}^\alpha}{x_{n-k}^\alpha x_{n-i-k}^\alpha} = \frac{1}{\alpha L_1 n} \left( 1 - \frac{A}{\alpha L_1} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right) \right).$$

Consequently,

$$\begin{aligned} \alpha^2 L_1^2 \sum_{i=1}^k \frac{x_{n-i}^\alpha}{x_{n-k}^\alpha x_{n-i-k}^\alpha} &= \frac{\alpha L_1 k}{n} - kA \cdot \frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right), \\ \frac{\binom{\alpha}{2}L_1^2 + \alpha L_2}{x_{n-k}^{2\alpha}} x_n^\alpha &= \frac{\binom{\alpha}{2}L_1^2 + \alpha L_2}{\alpha L_1} \cdot \frac{1}{n} - \frac{A \left(\binom{\alpha}{2}L_1^2 + \alpha L_2\right)}{\alpha^2 L_1^2} \cdot \frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right). \end{aligned}$$

Summing the two expressions and simplifying using the definition of  $A$  yields the desired result. □

### 3. Main Results

In this section, we present the first four terms in the asymptotic expansion of the solution to Equation (1).

**Lemma 3.1.** *Let the sequence  $\{x_n\}_{n \geq k+1}$  be defined by Equation (1). Then there exists a constant  $C \in \mathbb{R}$  such that*

$$C = \lim_{n \rightarrow \infty} (x_n^\alpha - \alpha L_1 n - A \ln n),$$

where

$$A = \frac{\alpha L_2 + \left(\binom{\alpha}{2} + \alpha^2 k\right) L_1^2}{\alpha L_1}.$$

Moreover, the following asymptotic expansion holds

$$x_n^\alpha = \alpha L_1 n + A \ln n + C + B \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right), \tag{5}$$

where  $B = \frac{A^2}{\alpha L_1}$ .

**Proof.** Define  $z_n = y_n - A \ln n$ , where  $y_n = x_n^\alpha - \alpha L_1 n$ . For all  $n \geq 2k + 1$ , we have

$$\begin{aligned} z_{n+1} - z_n &= y_{n+1} - y_n - A \ln \left( 1 + \frac{1}{n} \right) \\ &= x_n^\alpha \left( f^\alpha(x_{n-k}) - 1 - \frac{\alpha L_1}{x_{n-k}^\alpha} \right) + \alpha L_1 \sum_{i=1}^k \frac{x_{n-i}^\alpha}{x_{n-k}^\alpha} (f^\alpha(x_{n-i-k}) - 1) - A \ln \left( 1 + \frac{1}{n} \right). \end{aligned}$$

After adding and subtracting the second-order term, we obtain

$$z_{n+1} - z_n = x_n^\alpha \left( f^\alpha(x_{n-k}) - 1 - \frac{\alpha L_1}{x_{n-k}^\alpha} - \frac{\binom{\alpha}{2} L_1^2 + \alpha L_2}{x_{n-k}^{2\alpha}} \right) + \alpha L_1 \sum_{i=1}^k \frac{x_{n-i}^\alpha}{x_{n-k}^\alpha} \left( f^\alpha(x_{n-i-k}) - 1 - \frac{\alpha L_1}{x_{n-i-k}^\alpha} \right) + \left[ \frac{\left(\binom{\alpha}{2} L_1^2 + \alpha L_2\right) x_n^\alpha}{x_{n-k}^{2\alpha}} + \alpha^2 L_1^2 \sum_{i=1}^k \frac{x_{n-i}^\alpha}{x_{n-k}^\alpha x_{n-i-k}^\alpha} - A \ln \left( 1 + \frac{1}{n} \right) \right].$$

By Proposition 2.1, since  $\frac{1}{n} - \ln \left( 1 + \frac{1}{n} \right) = O \left( \frac{1}{n^2} \right) = o \left( \frac{\ln n}{n^2} \right)$ , the bracketed term satisfies

$$\begin{aligned} \frac{\left(\binom{\alpha}{2} L_1^2 + \alpha L_2\right) x_n^\alpha}{x_{n-k}^{2\alpha}} + \alpha^2 L_1^2 \sum_{i=1}^k \frac{x_{n-i}^\alpha}{x_{n-k}^\alpha x_{n-i-k}^\alpha} - A \ln \left( 1 + \frac{1}{n} \right) &= A \left( \frac{1}{n} - \ln \left( 1 + \frac{1}{n} \right) \right) - \frac{A^2}{\alpha L_1} \cdot \frac{\ln n}{n^2} + o \left( \frac{\ln n}{n^2} \right) \\ &= -\frac{A^2}{\alpha L_1} \cdot \frac{\ln n}{n^2} + o \left( \frac{\ln n}{n^2} \right). \end{aligned}$$

Now, we compute the limit

$$\begin{aligned} \lim_{n \rightarrow \infty} n^2 x_n^\alpha \left( f^\alpha(x_{n-k}) - 1 - \frac{\alpha L_1}{x_{n-k}^\alpha} - \frac{\binom{\alpha}{2} L_1^2 + \alpha L_2}{x_{n-k}^{2\alpha}} \right) &= \lim_{n \rightarrow \infty} \frac{n^2}{x_{n-k}^{2\alpha}} \cdot \frac{x_n^{3\alpha}}{x_{n-k}^{3\alpha}} \cdot x_{n-k}^{3\alpha} \left( f^\alpha(x_{n-k}) - 1 - \frac{\alpha L_1}{x_{n-k}^\alpha} - \frac{\binom{\alpha}{2} L_1^2 + \alpha L_2}{x_{n-k}^{2\alpha}} \right) \\ &= \frac{1}{\alpha^2 L_1^2} \left( \binom{\alpha}{3} L_1^3 + \alpha(\alpha - 1) L_1 L_2 + \alpha L_3 \right). \end{aligned}$$

Similarly, for each fixed  $i$ ,

$$\lim_{n \rightarrow \infty} n^2 \left( f^\alpha(x_{n-i-k}) - 1 - \frac{\alpha L_1}{x_{n-i-k}^\alpha} \right) = \frac{\binom{\alpha}{2} L_1^2 + \alpha L_2}{\alpha^2 L_1^2}.$$

Combining these results, we obtain

$$\lim_{n \rightarrow \infty} \frac{z_{n+1} - z_n}{\ln n/n^2} = -B,$$

where

$$B = \frac{A^2}{\alpha L_1}.$$

Since the series  $\sum_{n=1}^\infty \frac{\ln n}{n^2}$  converges, the telescoping series  $z_1 + \sum_{n=1}^\infty (z_{n+1} - z_n)$  converges absolutely. Define

$$C = z_1 + \sum_{n=1}^\infty (z_{n+1} - z_n) = \lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} (x_n^\alpha - \alpha L_1 n - A \ln n).$$

By the Stolz–Cesàro theorem,

$$\lim_{n \rightarrow \infty} \frac{z_n - C}{\ln n/n} = \lim_{n \rightarrow \infty} \frac{z_{n+1} - z_n}{(\ln(n+1))/(n+1) - (\ln n)/n} = \lim_{n \rightarrow \infty} \frac{z_{n+1} - z_n}{-\ln n/n^2} = B.$$

Hence,

$$z_n = C + B \cdot \frac{\ln n}{n} + o \left( \frac{\ln n}{n} \right).$$

Substituting back into the definition of  $y_n$  and  $z_n$  yields Equation (5). □

**Proposition 3.1.** *Under the hypotheses of Lemma 3.1, it holds that*

$$\frac{\left(\binom{\alpha}{2} L_1^2 + \alpha L_2\right) x_n^\alpha}{x_{n-k}^{2\alpha}} + \alpha^2 L_1^2 \sum_{i=1}^k \frac{x_{n-i}^\alpha}{x_{n-k}^\alpha x_{n-i-k}^\alpha} = \frac{A}{n} - B \cdot \frac{\ln n}{n^2} + \frac{E}{n^2} + o \left( \frac{1}{n^2} \right),$$

where

$$A = \frac{\alpha L_2 + \left(\binom{\alpha}{2} + \alpha^2 k\right) L_1^2}{\alpha L_1}, \quad B = \frac{A^2}{\alpha L_1}, \quad \text{and} \quad E = \frac{A(\alpha k L_1 - C)}{\alpha L_1}.$$

The proof follows arguments similar to those used in Proposition 2.1 and is therefore omitted.

Our primary extension of Popa’s work [4] is summarized in the following theorem, which holds for any  $\alpha > 0$ .

**Theorem 3.1.** *Let the sequence  $\{x_n\}_{n \geq k+1}$  be defined by Equation (1). Then there exists a constant  $C \in \mathbb{R}$  such that*

$$C = \lim_{n \rightarrow \infty} (x_n^\alpha - \alpha L_1 n - A \ln n),$$

and the following expansion holds

$$x_n^\alpha = \alpha L_1 n + A \ln n + C + B \cdot \frac{\ln n}{n} - \frac{D}{n} + o\left(\frac{1}{n}\right), \tag{6}$$

where

$$A = \frac{\alpha L_2 + \binom{\alpha}{2} + \alpha^2 k}{\alpha L_1} L_1^2, \quad B = \frac{A^2}{\alpha L_1}, \quad D = \frac{\binom{\alpha}{3} L_1^3 + \alpha(\alpha - 1)L_1 L_2 + \alpha L_3}{\alpha^2 L_1^2} + \frac{\binom{\alpha}{2} L_1^2 + \alpha L_2}{\alpha L_1} k + \frac{2E - 2B + A}{2},$$

and

$$E = \frac{A(\alpha k L_1 - C)}{\alpha L_1}.$$

**Proof.** We proceed by defining an auxiliary sequence  $w_n = z_n - C - B \cdot \frac{\ln n}{n}$ , for  $n \geq k + 1$ ,

$$\begin{aligned} w_{n+1} - w_n &= z_{n+1} - z_n - B \left( \frac{\ln(n+1)}{n+1} - \frac{\ln n}{n} \right) \\ &= x_n^\alpha \left( f^\alpha(x_{n-k}) - 1 - \frac{\alpha L_1}{x_{n-k}^\alpha} - \frac{\binom{\alpha}{2} L_1^2 + \alpha L_2}{x_{n-k}^{2\alpha}} \right) \\ &\quad + \alpha L_1 \sum_{i=1}^k \frac{x_{n-i}^\alpha}{x_{n-k}^\alpha} \left( f^\alpha(x_{n-i-k}) - 1 - \frac{\alpha L_1}{x_{n-i-k}^\alpha} \right) \\ &\quad + \left[ \frac{\binom{\alpha}{2} L_1^2 + \alpha L_2}{x_{n-k}^{2\alpha}} x_n^\alpha + \alpha^2 L_1^2 \sum_{i=1}^k \frac{x_{n-i}^\alpha}{x_{n-k}^\alpha x_{n-i-k}^\alpha} - r_n \right], \end{aligned}$$

where

$$r_n = A \ln \left( 1 + \frac{1}{n} \right) + B \left( \frac{\ln(n+1)}{n+1} - \frac{\ln n}{n} \right).$$

By Proposition 3.1 and the expansions

$$\begin{aligned} \ln \left( 1 + \frac{1}{n} \right) &= \frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right), \\ \frac{\ln(n+1)}{n+1} - \frac{\ln n}{n} &= -\frac{\ln n}{n^2} + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right), \end{aligned}$$

we find that the bracketed term satisfies

$$\frac{\binom{\alpha}{2} L_1^2 + \alpha L_2}{x_{n-k}^{2\alpha}} x_n^\alpha + \alpha^2 L_1^2 \sum_{i=1}^k \frac{x_{n-i}^\alpha}{x_{n-k}^\alpha x_{n-i-k}^\alpha} - r_n = \frac{2E - 2B + A}{2n^2} + o\left(\frac{1}{n^2}\right).$$

The remaining terms contribute a limit

$$\lim_{n \rightarrow \infty} n^2(w_{n+1} - w_n) = D = \frac{\binom{\alpha}{3} L_1^3 + \alpha(\alpha - 1)L_1 L_2 + \alpha L_3}{\alpha^2 L_1^2} + \frac{\binom{\alpha}{2} L_1^2 + \alpha L_2}{\alpha L_1} k + \frac{2E - 2B + A}{2}.$$

By the Stolz–Cesàro theorem,

$$\lim_{n \rightarrow \infty} \frac{-w_n}{1/n} = \lim_{n \rightarrow \infty} \frac{w_n - w_{n+1}}{1/(n+1) - 1/n} = D,$$

and hence,

$$w_n = -\frac{D}{n} + o\left(\frac{1}{n}\right).$$

Now, using the definitions of  $w_n$ ,  $z_n$ , and  $y_n$ , we obtain Equation (6). □

The following corollary is an application of Theorem 3.1 to a related class of difference equations.

**Corollary 3.1.** *Suppose  $x_{n+1} = x_n g(x_{n-k})$  for all  $n > k$ ,  $k \in \mathbb{N}$ , where  $g \in C(0, b)$ ,  $b > 0$ , and*

$$g(x) = 1 - l_1 x^\alpha + l_2 x^{2\alpha} + l_3 x^{3\alpha} + o(x^{3\alpha})$$

as  $x \rightarrow 0^+$ , with  $\alpha > 0$ ,  $l_1 > 0$ , and  $l_2, l_3 \in \mathbb{R}$ . Then there exists a constant  $C \in \mathbb{R}$  such that

$$C = \lim_{n \rightarrow +\infty} n^{(\alpha+1)/\alpha} \left( x_n - \frac{1}{(\alpha l_1)^{1/\alpha}} \cdot \frac{1}{n^{1/\alpha}} + \frac{A}{\alpha^2 l_1 (\alpha l_1)^{1/\alpha}} \cdot \frac{\ln n}{n^{(\alpha+1)/\alpha}} \right),$$

and the following asymptotic expansion holds

$$\begin{aligned} x_n &= \frac{1}{(\alpha l_1)^{1/\alpha}} \cdot \frac{1}{n^{1/\alpha}} - \frac{A}{\alpha^2 l_1 (\alpha l_1)^{1/\alpha}} \cdot \frac{\ln n}{n^{(\alpha+1)/\alpha}} - \frac{C}{\alpha^2 l_1 (\alpha l_1)^{1/\alpha}} \cdot \frac{1}{n^{(\alpha+1)/\alpha}} \\ &+ \frac{(1+\alpha)A^2}{2\alpha^4 l_1^2 (\alpha l_1)^{1/\alpha}} \cdot \frac{\ln^2 n}{n^{(2\alpha+1)/\alpha}} + \frac{(1+\alpha)AC - \alpha^2 l_1 B}{\alpha^4 l_1^2 (\alpha l_1)^{1/\alpha}} \cdot \frac{\ln n}{n^{(2\alpha+1)/\alpha}} \\ &+ \frac{(1+\alpha)C^2 + 2\alpha^2 l_1 D}{2\alpha^4 l_1^2 (\alpha l_1)^{1/\alpha}} \cdot \frac{1}{n^{(2\alpha+1)/\alpha}} + o\left(\frac{\ln n}{n^{(2\alpha+1)/\alpha}}\right), \end{aligned}$$

where

$$\begin{aligned} A &= \frac{\left(\binom{\alpha}{2} + \alpha^2 k + \alpha\right) l_1^2 - \alpha l_2}{\alpha l_1^3}, & B &= \frac{\left(\left(\binom{\alpha}{2} + \alpha^2 k + \alpha\right) l_1^2 - \alpha l_2\right)^2}{\alpha^3 l_1^7}, \\ E &= \frac{\left(\left(\binom{\alpha}{2} + \alpha^2 k + \alpha\right) l_1^2 - \alpha l_2\right) (2\alpha k l_1 - C)}{\alpha^2 l_1^4}, \end{aligned}$$

and

$$D = \frac{\left(\binom{\alpha}{3} + \alpha^2\right) l_1^3 + (-\alpha^2 - \alpha) l_1 l_2 + \alpha l_3}{\alpha^2 l_1^2} + \frac{\left(\left(\binom{\alpha}{2} + \alpha\right) l_1^2 - \alpha l_2\right) k}{\alpha l_1} + \frac{2E - 2B + A}{2}.$$

**Proof.** Define  $f(x) = (g(1/x))^{-1}$ . Then  $f$  is continuous and  $f(x) > 1$  for all  $x > b^{-1} > 0$ . Let  $\nu_n = x_n^{-1}$  for  $n \geq 1$ . Then the recurrence becomes

$$\nu_{n+1} = \nu_n f(\nu_{n-k})$$

for  $n > k$ . By Fact 2.1,

$$f(x) = 1 + \frac{L_1}{x^\alpha} + \frac{L_2}{x^{2\alpha}} + \frac{L_3}{x^{3\alpha}} + o\left(\frac{1}{x^{3\alpha}}\right)$$

as  $x \rightarrow \infty$ , where  $L_1 = l_1$ ,  $L_2 = l_1^2 - l_2$ ,  $L_3 = l_1^3 - 2l_1 l_2 - l_3$ . Applying Theorem 3.1 to  $\{\nu_n\}$ , we obtain

$$\nu_n^\alpha = \alpha L_1 n + A \ln n + C + B \cdot \frac{\ln n}{n} - \frac{D}{n} + o\left(\frac{1}{n}\right).$$

The asymptotic expansion for  $x_n = 1/\nu_n$  is now obtained by inverting the corresponding asymptotic series. □

### 4. Examples

As applications of our results, we present asymptotic expansions for two generalized models.

**Example 4.1.** *Let  $\alpha > 0$  and  $k \in \mathbb{N}$ . Define the sequence by  $x_1, \dots, x_k > 0$  and  $x_{n+1} = x_n(1 - x_{n-k}^\alpha)$ , for all  $n > k$ . Then, there exists  $C \in \mathbb{R}$  such that*

$$C = \lim_{n \rightarrow +\infty} n^{(\alpha+1)/\alpha} \left( x_n - \frac{1}{\alpha^{1/\alpha}} \cdot \frac{1}{n^{1/\alpha}} + \frac{A}{\alpha^{(2\alpha+1)/\alpha}} \cdot \frac{\ln n}{n^{(\alpha+1)/\alpha}} \right).$$

Moreover,

$$\begin{aligned} x_n &= \frac{1}{\alpha^{1/\alpha}} \cdot \frac{1}{n^{1/\alpha}} - \frac{A}{\alpha^{(2\alpha+1)/\alpha}} \cdot \frac{\ln n}{n^{(\alpha+1)/\alpha}} - \frac{C}{\alpha^{(2\alpha+1)/\alpha}} \cdot \frac{1}{n^{(\alpha+1)/\alpha}} \\ &+ \frac{(1+\alpha)A^2}{2\alpha^{(4\alpha+1)/\alpha}} \cdot \frac{\ln^2 n}{n^{(2\alpha+1)/\alpha}} + \frac{(1+\alpha)AC - \alpha^2 B}{\alpha^{(4\alpha+1)/\alpha}} \cdot \frac{\ln n}{n^{(2\alpha+1)/\alpha}} \\ &+ \frac{(1+\alpha)C^2 + 2\alpha^2 D}{2\alpha^{(4\alpha+1)/\alpha}} \cdot \frac{1}{n^{(2\alpha+1)/\alpha}} + o\left(\frac{\ln n}{n^{(2\alpha+1)/\alpha}}\right), \end{aligned}$$

where

$$A = \frac{2\alpha k + \alpha + 1}{2}, \quad B = \frac{(2\alpha k + \alpha + 1)^2}{4\alpha}, \quad \text{and} \quad D = \frac{4\binom{\alpha}{3} + 4\alpha^2 + 4k\alpha \left(\binom{\alpha}{2} + \alpha\right) + (2\alpha k + \alpha + 1)(2\alpha k - 2C - 1)\alpha}{4\alpha^2}.$$

The discrete delay logistic difference equation  $x_{n+1} = \frac{x_n}{1+\beta x_{n-k}}$ ,  $\beta > 0$ ,  $x_1, \dots, x_k > 0$ , was studied by Stević [6], who proved that

$$x_n = \frac{1}{\beta n} - \frac{k}{\beta} \cdot \frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right).$$

**Example 4.2.** Let  $\alpha > 0$ ,  $\beta > 0$ , and  $k \in \mathbb{N}$ . Define the sequence by  $x_1, \dots, x_k > 0$  and  $x_{n+1} = \frac{x_n}{1+\beta x_{n-k}^\alpha}$ , for all  $n > k$ . Then, there exists  $C \in \mathbb{R}$  such that

$$C = \lim_{n \rightarrow +\infty} n^{(\alpha+1)/\alpha} \left( x_n - \frac{1}{(\alpha\beta)^{1/\alpha}} \cdot \frac{1}{n^{1/\alpha}} + \frac{A}{\alpha^2\beta(\alpha\beta)^{1/\alpha}} \cdot \frac{\ln n}{n^{(\alpha+1)/\alpha}} \right).$$

Moreover,

$$\begin{aligned} x_n &= \frac{1}{(\alpha\beta)^{1/\alpha}} \cdot \frac{1}{n^{1/\alpha}} - \frac{A}{\alpha^2\beta(\alpha\beta)^{1/\alpha}} \cdot \frac{\ln n}{n^{(\alpha+1)/\alpha}} - \frac{C}{\alpha^2\beta(\alpha\beta)^{1/\alpha}} \cdot \frac{1}{n^{(\alpha+1)/\alpha}} \\ &+ \frac{(1+\alpha)A^2}{2\alpha^4\beta^2(\alpha\beta)^{1/\alpha}} \cdot \frac{\ln^2 n}{n^{(2\alpha+1)/\alpha}} + \frac{(1+\alpha)AC - \alpha^2\beta B}{\alpha^4\beta^2(\alpha\beta)^{1/\alpha}} \cdot \frac{\ln n}{n^{(2\alpha+1)/\alpha}} \\ &+ \frac{(1+\alpha)C^2 + 2\alpha^2\beta D}{2\alpha^4\beta^2(\alpha\beta)^{1/\alpha}} \cdot \frac{1}{n^{(2\alpha+1)/\alpha}} + o\left(\frac{\ln n}{n^{(2\alpha+1)/\alpha}}\right), \end{aligned}$$

where

$$A = \frac{\binom{\alpha}{2} + \alpha^2 k}{\alpha} \beta, \quad B = \frac{\left(\binom{\alpha}{2} + \alpha^2 k\right)^2 \beta}{\alpha^3}, \quad \text{and} \quad D = \frac{2\binom{\alpha}{3}\alpha\beta + 2\binom{\alpha}{2}\alpha^2\beta k + \left(\binom{\alpha}{2} + \alpha^2 k\right)(2\alpha^2\beta k - 2\alpha C - 2\binom{\alpha}{2}\beta + \alpha^2\beta)}{2\alpha^3}.$$

## 5. Concluding Remarks

We derived a detailed asymptotic expansion for solutions of the nonlinear difference equation  $x_{n+1} = x_n f(x_{n-k})$ , under a general asymptotic condition on  $f$ . Extending previously known results from the case  $\alpha = 1$  to arbitrary  $\alpha > 0$ , we established a unified framework that highlights the influence of the decay rate on the asymptotic behavior of solutions. The explicit coefficients in the expansion reveal a subtle interplay among the delay  $k$ , the parameter  $\alpha$ , and the expansion coefficients  $L_1$ ,  $L_2$ , and  $L_3$  of  $f$ .

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