Contributions to Mathematics
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Contrib. Math. **13** (2026) 75–83
DOI: 10.47443/cm.2026.027

Research Article

Self-Identifying Codes in Direct Products of Complete Graphs With Paths and Cycles*

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(Received: 26 March 2026. Received in revised form: 8 May 2026. Accepted: 25 May 2026. Published online: 16 June 2026.)

Abstract

For a vertex u in a graph G, let $N[u]$ denote its closed neighborhood (i.e., the set consisting of u and its neighbors). Let K_m, P_n, and C_n denote the complete graph of order m, the path of order n, and the cycle of order n, respectively. A subset S of the vertex set of G is said to be a *self-identifying code* if for every vertex v, $N[v] \cap S \neq \emptyset$ and $\bigcap_{c \in N[v] \cap S} N[c] = \{v\}$. In this paper, we study self-identifying codes in the direct products $K_m \times P_n$ and $K_m \times C_n$. We obtain bounds on the minimum size of such codes that are linear in n, with coefficients depending on m, and show that these bounds are asymptotically tight.

Keywords: self-identifying code; direct product; complete graph; path; cycle.

2020 Mathematics Subject Classification: 05C69, 05C76, 68R99.

1. Introduction

Locating codes were introduced by Slater in 1988 [16] in the context of fault detection in nuclear power plants. In 1998, Karpovsky, Chakrabarty, and Levitin [13] introduced the concept of identifying codes, which has since been extensively studied in various graph families, including Cartesian products [2], rook's graphs (i.e., $K_m \square K_n$) [3], vertex-transitive graphs [4], and binary Hamming spaces [7]. Subsequent developments include t-robust 1-identifying codes [5] and complexity analyses [8]. Jean and Lobstein [9] maintain a comprehensive bibliography of over 500 articles on detection systems, including identifying codes, locating-dominating codes, and their fault-tolerant variants.

In [11], the concept of *self-identifying codes* was introduced (previously called $(1, \leq 1)^+$-identifying codes in earlier work [6]). Recent studies focus on self-identifying codes in specific graph families, including square grids [6], circulants [12, 17], and cubic graphs [14]. Very recently, Jean and Seo [10] proved that determining the minimum size of a self-identifying code is NP-complete in general graphs, and studied the code in cubic graphs and infinite grids. Notably, Shinde and Waphare [15] determined the minimum size of an identifying code for $K_m \times P_n$.

In this paper, we give sufficient conditions for a subset $S \subseteq V(K_m \times G)$ to be a self-identifying code (which yield an upper bound on the minimum size of such a code) and necessary conditions (which yield a lower bound), thereby deriving bounds for the minimum size of a self-identifying code in $K_m \times G$ with $G \in \{P_n, C_n\}$. Crucially, the density of the smallest self-identifying code in $K_m \times P_n$ asymptotically approaches that of the identifying code established by Shinde and Waphare [15].

Theorem 1.1. *For $m \geq 3$, the asymptotic density of the smallest self-identifying code in $K_m \times P_n$ (as $n \to \infty$, $n \geq 7$) and in $K_m \times C_n$ (as $n \to \infty$, $n \geq 3$) is $1/3$.*

The paper is organized as follows. Section 2 introduces the necessary notation and preliminaries. In Section 3, we present the bounds for $K_m \times P_n$, and in Section 4 we treat the case $K_m \times C_n$. The proofs of the upper bounds rely on explicit constructions (the sufficient conditions), while the lower bounds follow from combinatorial constraints (the necessary conditions). The density result is discussed in the final part of each section.

2. Terminology and Notation

A simple undirected graph G is defined as an ordered pair $(V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ the edge set. Given graphs G and H, their *direct product* $G \times H$ (see Figure 2.1) has vertex set $V(G) \times V(H)$ and edge set

$$E(G \times H) = \{(g_1, h_1)(g_2, h_2) \mid g_1 g_2 \in E(G) \text{ and } h_1 h_2 \in E(H)\}.$$

*This paper is a slightly revised version of the preprint *arXiv*:2512.22033v1 [math.CO].
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For clarity, we adopt the following notation (see Figure 2.1):

- $V(K_m) = \{v_0, v_1, \ldots, v_{m-1}\}$, $V(P_n) = \{0, 1, \ldots, n-1\}$, and $V(C_n) = \{0, 1, \ldots, n-1\}$ with edges between consecutive indices modulo n.
- The i-th *row* R_i in $K_m \times P_n$ (or $K_m \times C_n$) is $R_i = \{(v_i, j) \mid j \in V(P_n)\}$ (or $j \in V(C_n)$).
- The j-th *column* C_j in $K_m \times P_n$ (or $K_m \times C_n$) is $C_j = \{(v_i, j) \mid v_i \in V(K_m)\}$.

The *open neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent with v, and the *closed neighborhood* is defined as $N[v] = \{v\} \cup N(v)$. A nonempty subset $D \subseteq V(G)$ is called a *dominating set* if $N[v] \cap D \neq \emptyset$ for every vertex $v \in V(G)$. A nonempty subset $S \subseteq V(G)$ is called a *separating set* if $N[u] \cap S \neq N[v] \cap S$ for all distinct vertices $u, v \in V(G)$. A nonempty subset $C \subseteq V(G)$ is called an *identifying code* if it is both a dominating set and a separating set.

Definition 2.1. *A nonempty subset $S \subseteq V(G)$ is a* self-identifying code *if for each $v \in V(G)$, $N[v] \cap S \neq \emptyset$, and*

$$\bigcap_{c \in N[v] \cap S} N[c] = \{v\}. \tag{1}$$

The following equivalent definition is given in [11].

Definition 2.2. *A nonempty subset $S \subseteq V(G)$ is a self-identifying code if for all distinct $u, v \in V(G)$,*

$$(N[u] \cap S) \setminus (N[v] \cap S) \neq \emptyset. \tag{2}$$

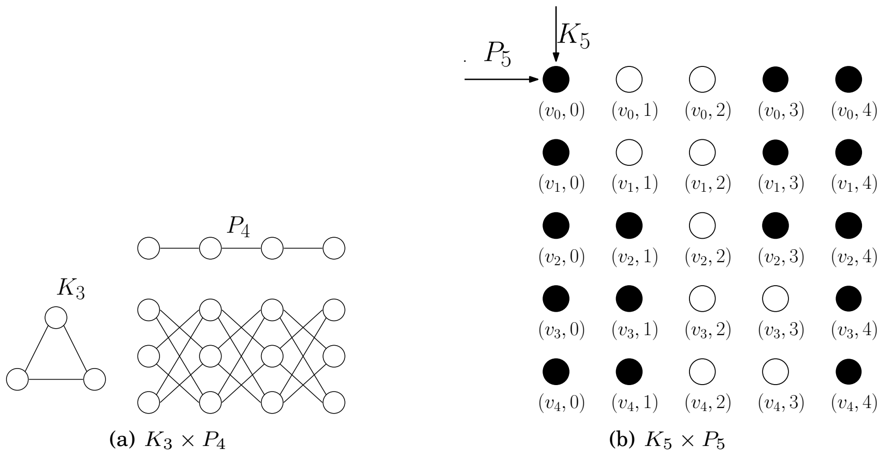


Figure 2.1: The direct product $K_3 \times P_4$, and a self-identifying code of $K_5 \times P_5$ (black vertices).

In the figures, vertices in S are represented by black circles. For clarity, edges are omitted in subsequent figures. Terminology not defined here follows Bondy and Murty [1] and West [18]. Let $[a, b] = \{a, a+1, \ldots, b\}$.

The minimum sizes of identifying and self-identifying codes in G are denoted by $\gamma^{\text{ID}}(G)$ and $\gamma^{\text{SID}}(G)$, respectively. Here, we have $\gamma^{\text{SID}}(G) \geq \gamma^{\text{ID}}(G)$. If G has a self-identifying code, then G is called a *self-identifiable graph*. The following characterization of self-identifiable graphs is given in [11].

Lemma 2.1 (see [11]). *A graph G is self-identifiable if and only if for all distinct $u, v \in V(G)$,*

$$N[u] \setminus N[v] \neq \emptyset. \tag{3}$$

Note that complete graphs K_m (for $m \geq 2$) are not self-identifiable, since all vertices have identical closed neighborhoods.

Lemma 2.2. *If a connected graph G admits a self-identifying code S, then $|N(v) \cap S| \geq 2$ for every $v \in V(G)$.*

Proof. Suppose $|N(u) \cap S| \leq 1$ for some $u \in V(G)$. If $|N(u) \cap S| = 1$, let $N(u) \cap S = \{w\}$. Then $\{u, w\} \subseteq \bigcap_{c \in N[u] \cap S} N[c]$, contradicting Definition 2.1. If $|N(u) \cap S| = 0$, then $u \in S$ implies

$$\bigcap_{c \in N[u] \cap S} N[c] = N[u] \neq \{u\},$$

as G has no isolated vertices. □

Lemma 2.2 implies that graphs with leaves (e.g., P_n, $K_1 \times P_n$, $K_2 \times P_n$) are not self-identifiable. Thus, we focus on $K_m \times P_n$ and $K_m \times C_n$ for $m, n \geq 3$.

Lemma 2.3. *For $m, n \geq 3$, both $K_m \times P_n$ and $K_m \times C_n$ are self-identifiable graphs.*

Proof. By Lemma 2.1, it suffices to show $N[u] \setminus N[w] \neq \emptyset$ for any distinct $u, w \in V(K_m \times P_n)$. Let $u = (v_i, j)$ and $w = (v_{i'}, j')$. If $j = j'$, then u and w are in the same column and they differ only in the first coordinate, so they are not adjacent; choosing a neighbor of u in an adjacent column yields a vertex in $N[u] \setminus N[w]$. If $j \neq j'$, one can similarly find a suitable neighbor. The proof for $K_m \times C_n$ is analogous, taking indices modulo n. □

Our main results provide bounds on γ^{SID} for the graphs given in Lemma 2.3.

Theorem 2.1. *For $m \geq 3$ and $n \geq 7$,*

$$\left\lceil \frac{n+1}{3} \right\rceil (m+2) - 2 \leq \gamma^{\mathrm{SID}}(K_m \times P_n) \leq \left\lceil \frac{n+1}{3} \right\rceil (m+3) + m.$$

Theorem 2.2. *For $m, n \geq 3$,*

$$\left\lfloor \frac{n}{3} \right\rfloor (m+2) \leq \gamma^{\mathrm{SID}}(K_m \times C_n) \leq \left\lceil \frac{n}{3} \right\rceil (m+3) + 3.$$

In the following two sections, we provide the proofs of these results.

3. Bounds on Self-Identifying Codes of $K_m \times P_n$

We first present several useful lemmas, corollaries, and preliminary results in Subsection 3.1, which establish the lower bound. Subsequently, in Subsection 3.2, we construct a self-identifying code for $K_m \times P_n$ with $m, n \geq 3$ to derive the upper bound.

For $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$, we define

- $R_i = \{(v_i, j) : j \in V(P_n)\}$ as the i-th row,
- $C_j = \{(v_i, j) : v_i \in V(K_m)\}$ as the j-th column.

3.1. Necessary Conditions for a Self-Identifying Code in $K_m \times P_n$

This subsection establishes properties of self-identifying codes in $K_m \times P_n$ using row and column structures. Let S denote a self-identifying code in $K_m \times P_n$ with $m, n \geq 3$.

Lemma 3.1. *The boundary columns satisfy $C_0 \subseteq S$ and $C_{n-1} \subseteq S$.*

Proof. Assume that there exists $i \in [0, m-1]$ such that $(v_i, 0) \in C_0 \setminus S$. If there exists $i' \neq i$ with $(v_{i'}, 1) \notin S$, then

$$\{(v_i, 1), (v_{i'}, 1)\} \subseteq \bigcap_{c \in N[(v_{i'}, 1)] \cap S} N[c],$$

contradicting Definition 2.1. Hence $C_1 \setminus \{(v_i, 1)\} \subseteq S$. Then $N[(v_i, 0)] \cap S \subseteq N[(v_i, 2)] \cap S$, contradicting Definition 2.2. Therefore $C_0 \subseteq S$. The same argument gives $C_{n-1} \subseteq S$. □

By Lemma 3.1 and Lemma 2.2, we have the following corollary.

Corollary 3.1. *The near-boundary columns satisfy $|C_1 \cap S| \geq 3$ and $|C_{n-2} \cap S| \geq 3$.*

Lemma 3.1 ensures boundary constraints, while Corollary 3.1 gives columns near boundaries. For internal columns, we establish the following result.

Lemma 3.2. *Let $n \geq 5$, $i \in [0, m-1]$, $j \in [2, n-3]$, and $(v_i, j) \notin S$. Then,*

(1) *$S \cap (C_{j-1} \cup C_{j+1}) \cap R_{i'} \neq \emptyset$ for all $i' \neq i$, i.e., $|N[(v_i, j)] \cap S| \geq m-1$,*

(2) *$S \cap C_{j-1} \neq \emptyset$ and $S \cap C_{j+1} \neq \emptyset$.*

Proof. Since $(v_i, j) \notin S$, we have $N[(v_i, j)] \subseteq C_{j-1} \cup C_{j+1}$. If there exists $i' \neq i$ such that $\{(v_{i'}, j-1), (v_{i'}, j+1)\} \cap S = \emptyset$, then

$$\{(v_i, j), (v_{i'}, j)\} \subseteq \bigcap_{c \in N[(v_i, j)] \cap S} N[c],$$

contradicting Definition 2.1. Hence for every $i' \neq i$, we have $S \cap (C_{j-1} \cup C_{j+1}) \cap R_{i'} \neq \emptyset$, which gives $|N[(v_i, j)] \cap S| \geq m-1$. If $C_{j-1} \cap S = \emptyset$, then $N[(v_i, j)] \cap S \subseteq N[(v_i, j+2)] \cap S$, contradicting Definition 2.2. Similarly, $S \cap C_{j+1} \neq \emptyset$. □

By Lemma 3.2 and Lemma 2.2, we have the following corollary.

Corollary 3.2. *Let $n \geq 5$ and $j \in [2, n-3]$.*

(1) *If $j \in [3, n-4]$ and $C_j \cap S = \emptyset$, then $(C_{j-1} \cup C_{j+1}) \subseteq S$ for $n \geq 7$.*

(2) *If $|C_j \cap S| = 1$, then*

- $|(C_1 \cup C_3) \cap S| \geq m$ *for* $j = 2$;
- $|(C_{n-4} \cup C_{n-2}) \cap S| \geq m$ *for* $j = n-3$;
- $|(C_{j-1} \cup C_{j+1}) \cap S| \geq m+1$ *for* $j \in [3, n-4]$.

(3) *If $2 \leq |C_j \cap S| \leq m-2$, then $|(C_{j-1} \cup C_{j+1}) \cap S| \geq m$ and $(C_{j-1} \cup C_{j+1}) \cap S \cap R_i \neq \emptyset$ for all i.*

(4) *If $|C_j \cap S| = m-1$, then $|(C_{j-1} \cup C_{j+1}) \cap S| \geq m-1$.*

(5) *If $|C_j \cap S| = m$, then $|(C_{j-1} \cup C_{j+1}) \cap S| \geq 3$.*

By Corollary 3.2, there exists a special lower bound for three consecutive columns.

Corollary 3.3. *Let $n \geq 5$, $i \in [0, m-1]$, and $j \in [2, n-3]$.*

(1) *For $n = 5$, $|(C_1 \cup C_2 \cup C_3) \cap S| \geq m$.*

(2) *For $n \geq 6$, $j \in [2, n-3]$, $|(C_{j-1} \cup C_j \cup C_{j+1}) \cap S| \geq m+1$.*

(3) *For $n \geq 7$ and $j \in [3, n-4]$, $|(C_{j-1} \cup C_j \cup C_{j+1}) \cap S| \geq m+2$.*

Proof of Theorem 2.1 (Lower Bound). By Lemma 3.1, Corollary 3.1, and Corollary 3.3, we have

$$|(C_0 \cup C_1) \cap S| \geq m+3,$$

$$|(C_{n-2} \cup C_{n-1}) \cap S| \geq m+3,$$

$$|(C_{j-1} \cup C_j \cup C_{j+1}) \cap S| \geq m+2.$$

Therefore,

$$\gamma^{\mathrm{SID}}(K_m \times P_n) \geq \left\lceil \frac{n+1}{3} \right\rceil (m+2) - 2.$$

□

Having established a lower bound via necessary conditions, we now turn to an upper bound by constructing an explicit self-identifying code.

3.2. Construction of a Self-Identifying Code in $K_m \times P_n$

In this subsection, we construct a self-identifying code for $K_m \times P_n$, which yields an upper bound on $\gamma^{\mathrm{SID}}(K_m \times P_n)$ for $m \geq 3$ and $n \geq 7$. We begin with a sufficient condition for self-identifying codes.

Observation 3.1. *Let $S \subseteq V(K_m \times P_n)$. If the following conditions hold for all $i \in [0, m-1]$ and $j \in [0, n-1]$, then S is a self-identifying code of $K_m \times P_n$:*

- $|C_1 \cap S| \geq 3$ *and* $|C_{n-2} \cap S| \geq 3$;
- *For $j \in [1, n-2]$ and for each vertex $(v_i, j) \in C_j$:*
  - *if $(v_i, j) \in S$, then there exist two distinct indices $i_1, i_2 \in [0, m-1] \setminus \{i\}$ such that $(C_{j-1} \cup C_{j+1}) \cap S \cap R_{i'} \neq \emptyset$ for $i' \in \{i_1, i_2\}$;*
  - *if $(v_i, j) \notin S$, then there exist two distinct indices $i_1, i_2 \in [0, m-1] \setminus \{i\}$ such that $(C_{j-1} \cup C_{j+1}) \cap S \cap R_{i'} \neq \emptyset$, and additionally $C_{j-1} \cap S \neq \emptyset$, $C_{j+1} \cap S \neq \emptyset$;*

By Observation 3.1, we construct an upper bound of $\gamma^{\mathrm{SID}}(K_m \times P_n)$ in Theorem 2.1.

Proof of Theorem 2.1 (Upper Bound). For non-negative integers m, n, t, k, and $k = \lfloor \frac{n}{3} \rfloor$. We define

$$\mathcal{C}_t = \begin{cases} C_{3t-1} \cup C_{3t} \cup C_{3t+1} & \text{for } 1 \le t \le k-2, \\ C_{n-4} \cup C_{n-3} & \text{for } t = k-1 \text{ and } n = 3k, \\ C_{n-5} \cup C_{n-4} \cup C_{n-3} & \text{for } t = k-1 \text{ and } n = 3k+1, \\ C_{n-6} \cup C_{n-5} \cup C_{n-4} \cup C_{n-3} & \text{for } t = k-1 \text{ and } n = 3k+2. \end{cases}$$

For the partition $V(K_m \times P_n) = C_0 \cup C_1 \cup \left(\bigcup_{t=1}^{k-2} \mathcal{C}_t\right) \cup \mathcal{C}_{k-1} \cup C_{n-2} \cup C_{n-1}$, we construct a subset S of $V(K_m \times P_n)$ with $S = S_1 \cup S_2 \cup S_3 \cup S_4$, where

$$S_1 = (C_0 \cup C_{n-1}) \cap S = C_0 \cup C_{n-1},$$

$$S_2 = (C_1 \cup C_{n-2}) \cap S = \{(v_0, 1), (v_1, 1), (v_2, 1), (v_0, n-2), (v_1, n-2), (v_2, n-2)\},$$

$$S_3 = \left(\bigcup_{t=1}^{k-2} \mathcal{C}_t\right) \cap S = \begin{cases} \bigcup_{t=1,3,5,\ldots,k-3} (A_t \cup C_{3t} \cup C_{3t+3}) & \text{if } k \text{ even}, \\ \left(\bigcup_{t=1,3,5,\ldots,k-4} A'_t \cup C_{3t} \cup C_{3t+3}\right) \cup A'' \cup C_{3k-6} & \text{if } k \text{ odd}, \end{cases}$$

$$S_4 = \mathcal{C}_{k-1} \cap S = \begin{cases} \{(v_0, n-3), (v_1, n-3), (v_2, n-3)\} \cup C_{n-4} & \text{if } n = 3k, \\ \{(v_2, n-5), (v_0, n-3), (v_1, n-3)\} \cup C_{n-4} & \text{if } n = 3k+1, \\ \{(v_2, n-6), (v_0, n-4), (v_1, n-4)\} \cup C_{n-5} \cup C_{n-3} & \text{if } n = 3k+2, \end{cases}$$

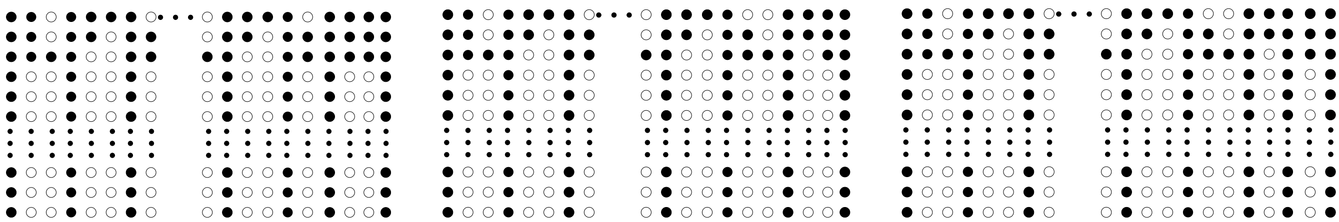
$$A_t = \{(v_2, 3t-1), (v_0, 3t+1), (v_1, 3t+1), (v_0, 3t+2), (v_1, 3t+4), (v_2, 3t+4)\},$$

$$A'_t = \{(v_0, 3t-1), (v_1, 3t+1), (v_2, 3t+1), (v_2, 3t+2), (v_0, 3t+4), (v_1, 3t+4)\},$$

$$A'' = \{(v_0, 3k-7), (v_1, 3k-5), (v_2, 3k-5)\}.$$

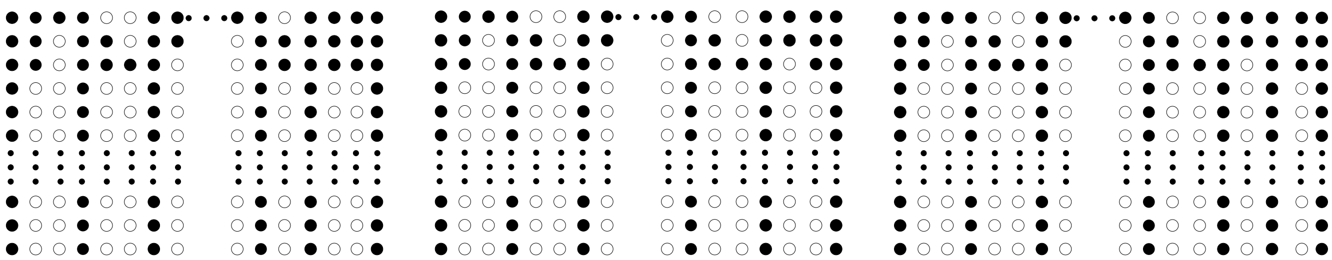
By Observation 3.1, S is a self-identifying code of $K_m \times P_n$ (see Figures 3.1 and 3.2). Thus

$$\gamma^{\mathrm{SID}}(K_m \times P_n) \le \begin{cases} (k+1)(m+3) & \text{if } n = 3k, \\ (k+1)(m+3) & \text{if } n = 3k+1, \\ (k+1)(m+3) + m & \text{if } n = 3k+2. \end{cases}$$



(a) $n = 3k$, k is even (b) $n = 3k+1$, k is even (c) $n = 3k+2$, k is even

Figure 3.1: A self-identifying code of $K_m \times P_n$, where n is even.



(a) $n = 3k$, k is odd (b) $n = 3k+1$, k is odd (c) $n = 3k+2$, k is odd

Figure 3.2: A self-identifying code of $K_m \times P_n$, where n is odd.

The upper bound of $\gamma^{\text{SID}}(K_m \times P_n)$ for $m \geq 3$ and $3 \leq n \leq 6$, due to its technical intricacy, is deferred to the following subsection.

3.3. Self-Identifying Codes in $K_m \times P_n$ for Small n

This subsection establishes exact values of $\gamma^{\text{SID}}(K_m \times P_n)$ for $m \geq 3$ and $3 \leq n \leq 6$, extending the bounds from Sections 3.1–3.2.

Theorem 3.1. *For $K_3 \times P_n$ and $3 \leq n \leq 6$,*

$$\gamma^{\text{SID}}(K_3 \times P_n) = \begin{cases} 9 & n = 3, \\ 12 & n = 4, 5, \\ 14 & n = 6. \end{cases}$$

Proof. Lower bounds follow from Lemma 3.1 and Corollary 3.1. Upper bounds are achieved by explicit constructions (see Figure 3.3),

- $S = V(K_3 \times P_3)$ for $n = 3$,
- $S = V(K_3 \times P_4)$ for $n = 4$,
- $S = C_0 \cup C_1 \cup C_3 \cup C_4$ for $n = 5$,
- $S = C_0 \cup C_1 \cup C_4 \cup C_5 \cup \{(v_2, 2), (v_2, 3)\}$ for $n = 6$.

Verification uses Observation 3.1. □

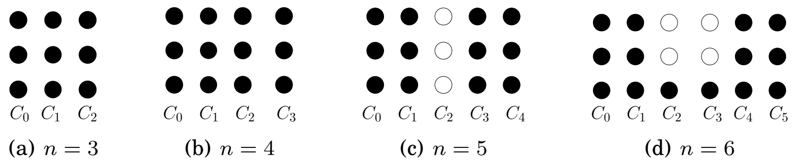


Figure 3.3: A self-identifying code of $K_3 \times P_n$ for $n = 3, 4, 5, 6$.

Theorem 3.2. *For $K_m \times P_n$ and $m \geq 4$,*

$$\gamma^{\text{SID}}(K_m \times P_n) = \begin{cases} 2m + 3 & n = 3, \\ 2m + 6 & n = 4. \end{cases}$$

Proof. Lower bounds follow from Lemma 3.1 and Corollary 3.1. Upper bounds are achieved by explicit constructions (see Figure 3.4),

- $S = C_0 \cup C_2 \cup \{(v_0, 1), (v_1, 1), (v_2, 1)\}$ for $n = 3$,
- $S = C_0 \cup C_3 \cup \{(v_0, 1), (v_1, 1), (v_2, 1), (v_0, 2), (v_1, 2), (v_2, 2)\}$ for $n = 4$.

Verification uses Observation 3.1. □

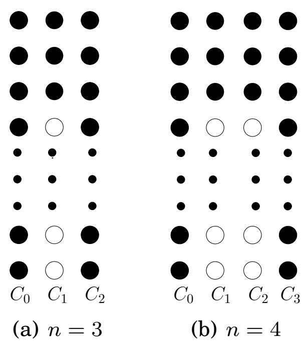


Figure 3.4: A self-identifying code of $K_m \times P_n$ for $n \in \{3, 4\}$ and $m \geq 4$.

Theorem 3.3. *For $K_m \times P_n$ and $m \in \{4,5\}$,*

$$\gamma^{\mathrm{SID}}(K_m \times P_n) = \begin{cases} 2m+6 & n=5, \\ 4m & n=6. \end{cases}$$

Proof. Lower bounds follow from Lemma 3.1 and Corollary 3.1. Upper bounds are achieved by explicit constructions (see Figure 3.5),

- $S = C_0 \cup C_4 \cup \{(v_0,1),(v_1,1),(v_2,1)\} \cup \{(v_{m-3},3),(v_{m-2},3),(v_{m-1},3)\}$ for $n=5$,
- $S = C_0 \cup C_5 \cup \{(v_0,1),(v_1,1),(v_2,1),(v_0,4),(v_1,4),(v_2,4)\} \cup \bigcup_{i=3}^{m-1}\{(v_i,2),(v_i,3)\}$ for $n=6$.

□

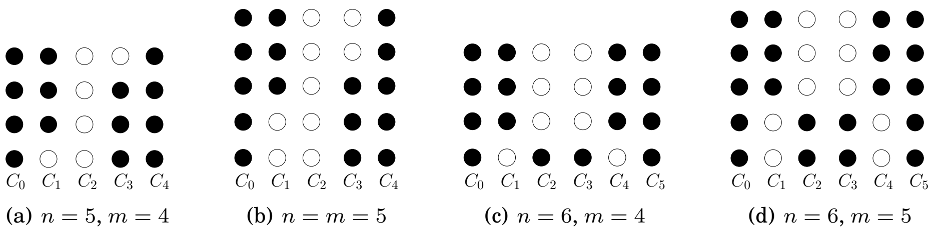


Figure 3.5: A self-identifying code of $K_m \times P_n$ for $n \in \{5,6\}$ and $m \in \{4,5\}$.

Theorem 3.4. *For $K_m \times P_n$ and $m \geq 6$,*

$$\gamma^{\mathrm{SID}}(K_m \times P_n) = \begin{cases} 3m & n=5, \\ 3m+6 & n=6. \end{cases}$$

Proof. Lower bounds follow from Lemma 3.1 and Corollary 3.1. Upper bounds are achieved by explicit constructions (see Figure 3.6),

- $S = C_0 \cup C_4 \cup \{(v_0,1),(v_1,1),(v_2,1)\} \cup \bigcup_{i=3}^{m-1}\{(v_i,3)\}$ for $n=5$,
- $S = C_0 \cup C_5 \cup C_2 \cup \{(v_0,1),(v_1,1),(v_2,1),(v_0,4),(v_1,4),(v_2,4)\}$ for $n=6$.

□

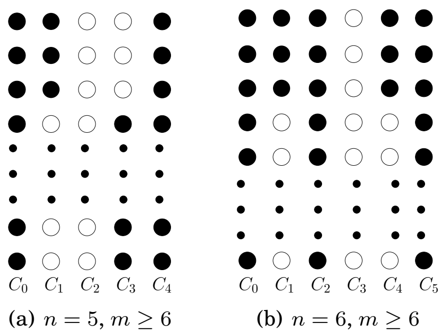


Figure 3.6: A self-identifying code of $K_m \times P_n$ for $n = 5, 6$ and $m \geq 6$.

We now have completed the proof of Theorem 2.1. □

4. Bounds on Self-Identifying Codes of $K_m \times C_n$

In this section, we consider self-identifying codes in $K_m \times C_n$ for $m, n \geq 3$. By applying a method similar to the one used in Section 3, we derive lower and upper bounds for $\gamma^{\mathrm{SID}}(K_m \times C_n)$, which together establish Theorem 2.2. We use the same notation for rows and columns as before: $R_i = \{(v_i, j) : j \in V(C_n)\}$ (the i-th row) and $C_j = \{(v_i, j) : v_i \in V(K_m)\}$ (the j-th column). In this section, all column indices are taken modulo n.

We assume that S is a self-identifying code of $K_m \times C_n$, where $m, n \geq 3$. The analysis is similar to the path case; in analogy with Corollary 3.3, we obtain the following lemma, which leads to the lower bound.

Lemma 4.1. *For any* $j \in [0, n-1]$, $|(C_{j-1} \cup C_j \cup C_{j+1}) \cap S| \geq m + 2$.

Corollary 4.1 (Lower Bound in Theorem 2.2)**.** *If* $m, n \geq 3$*, then*

$$\gamma^{\mathrm{SID}}(K_m \times C_n) \geq \left\lfloor \frac{n}{3} \right\rfloor (m+2).$$

Now, we construct a self-identifying code for $K_m \times C_n$, which yields an upper bound on $\gamma^{\mathrm{SID}}(K_m \times C_n)$ for $m, n \geq 3$. We begin with a sufficient condition for self-identifying codes.

Observation 4.1. *Let* $S \subseteq V(K_m \times C_n)$*. If the following conditions hold for all* $i \in [0, m-1]$ *and* $j \in [0, n-1]$*, then* S *is a self-identifying code of* $K_m \times C_n$:

- *For* $(v_i, j) \in S$*, there exist two distinct indices* $i_1, i_2 \in [0, m-1] \setminus \{i\}$ *such that* $(C_{j-1} \cup C_{j+1}) \cap S \cap R_{i'} \neq \emptyset$ *for* $i' \in \{i_1, i_2\}$.
- *For* $(v_i, j) \notin S$*, there exist two distinct indices* $i_1, i_2 \in [0, m-1] \setminus \{i\}$ *such that* $(C_{j-1} \cup C_{j+1}) \cap S \cap R_{i'} \neq \emptyset$*, and* $C_{j-1} \cap S \neq \emptyset$*,* $C_{j+1} \cap S \neq \emptyset$.

By Observation 4.1, we construct an upper bound of $\gamma^{\mathrm{SID}}(K_m \times C_n)$ in Theorem 2.2.

For non-negative integers m, n, t, k, and $k = \lfloor \frac{n}{3} \rfloor$, we define a nonempty set $S \subseteq V(K_m \times C_n)$ as follows. Let

$$T = \{0, 2, 4, 6, \ldots, k-2\} \qquad \text{if } k \text{ is even},$$

$$T' = \{0, 2, 4, 6, \ldots, k-3\} \qquad \text{if } k \text{ is odd}.$$

If k is even, then let

$$S = \begin{cases} \bigcup_{t \in T} (B_t \cup C_{3t+1} \cup C_{3t+4}) & \text{if } n = 3k, \\ \bigcup_{t \in T} (B_t \cup C_{3t+1} \cup C_{3t+4}) \cup C_{3k} & \text{if } n = 3k+1, \\ \bigcup_{t \in T} (B_t \cup C_{3t+1} \cup C_{3t+4}) \cup B' \cup C_{3k+1} & \text{if } n = 3k+2, \end{cases}$$

If k is odd, then let

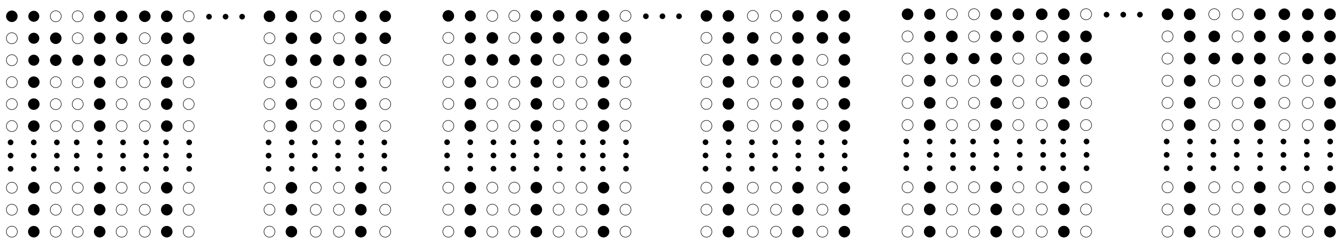
$$S = \begin{cases} \bigcup_{t \in T'} (B_t \cup C_{3t+1} \cup C_{3t+4}) \cup B'' \cup C_{3k-2} & \text{if } n = 3k, \\ \bigcup_{t \in T'} (B_t \cup C_{3t+1} \cup C_{3t+4}) \cup B'' \cup C_{3k-2} \cup C_{3k} & \text{if } n = 3k+1, \\ \bigcup_{t \in T'} (B_t \cup C_{3t+1} \cup C_{3t+4}) \cup B'' \cup C_{3k-2} \cup B' \cup C_{3k+1} & \text{if } n = 3k+2, \end{cases}$$

where

$$B_t = \{(v_0, 3t), (v_1, 3t+2), (v_2, 3t+2), (v_2, 3t+3), (v_0, 3t+5), (v_1, 3t+5)\},$$

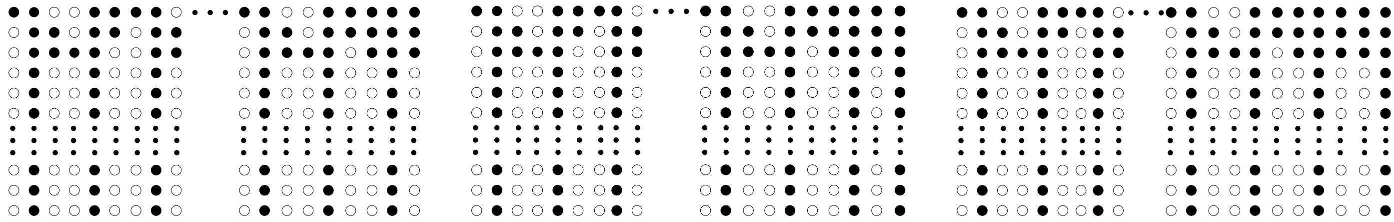
$$B' = \{(v_0, 3k), (v_1, 3k), (v_2, 3k)\},$$

$$B'' = \{(v_0, 3k-3), (v_1, 3k-3), (v_2, 3k-3), (v_0, 3k-1), (v_1, 3k-1), (v_2, 3k-1)\}.$$



(a) $n = 3k$, k is even (b) $n = 3k+1$, k is even (c) $n = 3k+2$, k is even

Figure 4.1: A self-identifying code of $K_m \times C_n$, where n is even.



(a) $n = 3k$, k is odd (b) $n = 3k + 1$, k is odd (c) $n = 3k + 2$, k is odd

Figure 4.2: A self-identifying code of $K_m \times C_n$, where n is odd.

By Observation 4.1, S is a self-identifying code of $K_m \times C_n$. Thus for $m, n \geq 3$,

$$\gamma^{\mathrm{SID}}(K_m \times C_n) \leq \left\lceil \frac{n}{3} \right\rceil (m+3) + 3.$$

This completes the proof of Theorem 2.2.

Acknowledgments

The authors thank the anonymous referees for their insightful suggestions, which improved the quality of this work. This research was supported by the Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ24A010014) and the Zhejiang Provincial College Students' Science and Technology Innovation Activity Plan (Xinmiao Talent Program) (Grant Nos. 2023R451014 and 2024R429A011).

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