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Research Article

On the Direct Limit of Quotient Hypergroups

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Abstract

This paper investigates direct limits and quotients in the category of canonical hypergroups. Consider a direct system consisting of canonical hypergroups G_i and homomorphisms φ_{ij} between them. Also assume a compatible family of subhypergroups H_i, one for each index i. Let $\widehat{G}$ denote the direct limit of the system G_i, and let $\widehat{H}$ denote the direct limit of the system H_i. Our main result is a natural isomorphism between the direct limit of the quotient hypergroups G_i/H_i^* and the quotient $\widehat{G}/\widehat{H}$. A key step in the proof is showing that equivalence in the direct limit comes from equivalence at some finite stage of the system. This extends classical direct limit theorems from groups and modules to hypergroups, offering new tools for hypergroup structure theory.

Keywords: direct system; direct limit; compatible; quotient direct limit.

2020 Mathematics Subject Classification: 20N20.

1. Introduction

Numerous mathematical theories, including probability theory, fuzzy set theory [13], rough set theory [9, 10], vague set theory [2], and interval mathematics [3], have been established to model uncertainties encountered in various fields. Among these, the theory of soft sets, introduced by Molodtsov [7], has gained recognition as a powerful mathematical tool for addressing uncertainties. Soft set theory offers a novel approach to handling uncertainty, effectively circumventing many of the challenges associated with conventional theoretical frameworks. It is important to highlight that algebraic hyperstructures can be considered a natural generalization of classical algebraic structures.

In classical algebraic structures, the composition of two elements results in a single element. In contrast, in algebraic hyperstructures, the composition of two elements produces a set. The theory of hyperstructures was first introduced by F. Marty in 1934. At the 8th Congress of Scandinavian Mathematicians, Marty formally defined the concept of a hypergroup and subsequently demonstrated its applications in various fields, including group theory, algebraic functions, and rational fractions [5].

The concept of direct limits, also referred to as inductive limits, holds a fundamental position in various branches of mathematics, including algebra, category theory, and topology. Direct limits offer a systematic framework for constructing a unified object from a collection of mathematical structures interconnected through a directed system. This construction is particularly valuable when addressing infinite processes or unifying structures that are defined in a localized manner [4,6,11]. The following natural question here arises: how do these classical constructions, particularly those involving quotient structures, generalize to the hypergroup setting?

In this paper, we address this question by systematically investigating direct systems of canonical hypergroups and their associated quotient constructions. Specifically, we consider a direct system (G_i, φ_{ij}) of canonical hypergroups and a compatible family $\{H_i\}_{i\in I}$ of subhypergroups. Our central aim is to establish the conditions under which the fundamental isomorphism

$$\lim_{i\in I} G_i/H_i^* \cong \widehat{G}/\widehat{H},$$

holds. This result represents a natural generalization of a cornerstone principle in algebra, asserting that the operation of taking direct limits commutes with the formation of quotient structures.

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2. Preliminaries

We begin by reviewing some fundamental concepts and classical results related to hypergroups.

A hypergroupoid $(H, \circ)$ is defined as a non-empty set H equipped with a hyperoperation $\circ$, which is a mapping:

$$\circ : H \times H \longrightarrow \mathcal{P}^*(H),$$

where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of H. For $x \in H$ and non-empty subsets $A, B \subseteq H$, the following properties hold:

$$A \circ B = \bigcup_{a \in A,\ b \in B} a \circ b, \quad A \circ x = A \circ \{x\}, \quad x \circ B = \{x\} \circ B.$$

We now call $(H, \circ)$ a hypergroup if for all $x, y, z \in H$, we have

$$x \circ (y \circ z) = (x \circ y) \circ z,$$

and $x \circ H = H \circ x = H$ (see [12]).

Definition 2.1. *We say that a hypergroup $(H, \circ)$ is a canonical hypergroup if*

(1) it is commutative;

(2) it has a scalar identity, which means that

$$\exists\, e \in H : \forall\, x \in H, \quad x \circ e = e \circ x = x;$$

(3) every element has a unique inverse, which means that for all $x \in H$, there exists a unique $x^{-1} \in H$ such that

$$e \in x \circ x^{-1} \cap x^{-1} \circ x;$$

(4) it is reversible, which means that if $x \in y \circ z$, then there exists the inverse y^{-1} of y and z^{-1} of z such that

$$z \in y^{-1} \circ x, \quad y \in x \circ z^{-1}.$$

Definition 2.2. *Let $(H, \circ)$ be a canonical hypergroup and let $N \subseteq H$. Then, N is called a subcanonical hypergroup if $(N, \circ)$ is a canonical hypergroup.*

Example 2.1. *Let $\mathbb{R}$ denote the set of real numbers. Then, $\mathbb{R}$ forms a semihypergroup with respect to the following hyperoperation:*

$$x \circ y = \{z \in \mathbb{R} : n \leq z < n + 1\},$$

where $x, y \in \mathbb{R}$ and $n = \max\{[x], [y]\}$.

Example 2.2. *Let $(\mathbb{Z}, +)$ be the ring of integers. Then, $(\mathbb{Z}, \oplus)$ is a hypergroup by the following hyperoperation:*

$$n \oplus n = \{t \in \mathbb{Z} : t \leq n\}, \quad n \oplus m = \max\{n, m\},$$

where $n, m \in \mathbb{Z}$ and $n \neq m$.

Definition 2.3. *A map $\varphi : G_1 \longrightarrow G_2$ is called homomorphism, where G_1 and G_2 are hypergroups, when for every $x, y \in G_1$,*

$$\varphi(x \circ y) = \varphi(x) \circ \varphi(y).$$

A homomorphism φ is called an isomorphism when φ is onto and one-to-one. If two hypergroups G_1 and G_2 are isomorphic, we denote this by $G_1 \cong G_2$.

In the context of hypergroups, regular relations and strongly regular relations are important concepts in algebraic hyperstructure theory. Below are their definitions.

Definition 2.4 (see [1]). *Let A and B be non-empty subsets and ρ be an equivalence relation on hypergroup $(H, \circ)$. Then,*

$$A \overline{\rho} B \Longrightarrow \ \forall a \in A\ \exists\, b \in B :\ a \rho b, \ \text{ and } \ \forall\, b \in B\ \exists\, a \in A :\ a \rho b.$$

Definition 2.5 (see [1])**.** *Let A and B be non-empty subsets and ρ be an equivalence relation on hypergroup $(H, \circ)$. Then,*

$$A\overline{\rho}B \Longrightarrow \forall a \in A, b \in B,\ a\rho b.$$

Definition 2.6 (see [1])**.** *Let ρ be an equivalence relation on a semihypergroup $(H, \circ)$.*

(1) The relation ρ is called a regular relation on a left (respectively, on a right) if, for every $x \in H$, the relation $a\,\rho\,b$ implies

$$(x \circ a)\,\overline{\rho}\,(x \circ b) \qquad (\textit{respectively, } (a \circ x)\,\overline{\rho}\,(b \circ x))\,.$$

(2) The relation ρ is called a strongly regular relation on a left (respectively, on a right) if, for every $x \in H$, the relation $a\,\rho\,b$ implies

$$(x \circ a)\,\overline{\overline{\rho}}\,(x \circ b) \qquad \left(\textit{respectively, } (a \circ x)\,\overline{\overline{\rho}}\,(b \circ x)\right).$$

Theorem 2.1 (see [1])**.** *Let $(H, \circ)$ be a semihypergroup and ρ be an equivalence relation on H. Then, the following statements hold.*

(1) If ρ is a regular relation on H, then $H/\rho = \{\rho(x) : x \in H\}$ is a semihypergroup with respect to the following hyperoperation:

$$\rho(x) \odot \rho(y) = \{\rho(z) : z \in x \circ y\}.$$

(2) If the hyperoperation $\odot$ is well-defined on H/ρ, then ρ is regular.

Corollary 2.1 (see [1])**.** *Let $(H, \circ)$ be a hypergroup and ρ an equivalence relation on H. Then, ρ is regular if and only if the quotient H/ρ forms a hypergroup under the induced operation.*

Definition 2.7. *Let $(H, +)$ be a canonical hypergroup and N be a subcanonical hypergroup of $(H, +)$. Then, we define the following relation:*

$$x \sim y \Longleftrightarrow \exists\, h \in N :\ x \in y + h.$$

Proposition 2.1. *Let $(H, +)$ be a canonical hypergroup and N be a subcanonical hypergroup of $(H, +)$. Then, the relation $\sim$ is an equivalence relation.*

Proof. The proof is straightforward. □

Proposition 2.2. *Let $(H, +)$ be a canonical hypergroup and N be a subcanonical hypergroup of $(H, +)$. Then, the relation $\sim$ is a regular relation on H.*

Proof. Suppose $x \sim y$. For any $z \in H$, we need to show that for every $u \in x + z$ there exists $v \in y + z$ with $u \sim v$, and conversely. This follows from the compatibility of the hyperoperation with the relation, which can be verified using the properties of canonical hypergroups. □

The subcanonical hypergroup N induces an equivalence relation, denoted by N^*. For an element $x \in H$, its equivalence class under this relation is denoted by $N^*(x)$.

Corollary 2.2. *Let $(H, +)$ be a canonical hypergroup and N be a subcanonical hypergroup of $(H, +)$. Then, the quotient*

$$H/N^* = \{N^*(x) : x \in H\},$$

forms a canonical hypergroup under the hyperoperation defined by

$$N^*(x) \oplus N^*(y) = \{N^*(z) : z \in x + y\}.$$

Proof. The result follows directly from the fact that N^* is a regular relation and from Theorem 2.1, which guarantees that the quotient of a canonical hypergroup by a regular relation is again a canonical hypergroup. □

3. Direct Limit of Hypergroups

The concepts of direct limits and direct systems are fundamental in category theory and algebra, providing a systematic framework for unifying a collection of objects and morphisms into a single, coherent structure. In this section, we investigate direct systems of hypergroups and canonical hypergroups.

A partially ordered set I is said to be a directed set if for each pair $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Definition 3.1 (see [8]). *Let $\{G_i\}_{i\in I}$ be a family of hypergroups indexed by a directed set I. For every pair $i, j \in I$ with $i \leq j$, there exists a homomorphism $\varphi_{ij} : G_i \longrightarrow G_j$ such that the following axioms are satisfied.*

(1) For every $i \in I$, the homomorphism φ_{ii} is the identity map on G_i.

(2) For every $i \leq j \leq k$, the homomorphisms satisfy the following equation:

$$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}.$$

Then, the pair (G_i, φ_{ij}), is called a direct system over directed set I.

Definition 3.2. *Let (G_i, φ_{ij}) be a direct system of hypergroups and $\{H_i\}_{i\in I}$ be a family where each H_i is a subhypergroup of G_i. Then, the family $\{H_i\}_{i\in I}$ is called compatible with the direct system, when for every $i \leq j$, we have $\varphi_{ij}(H_i) \subseteq H_j$.*

Proposition 3.1. *Let (G_i, φ_{ij}) be a direct system of hypergroups and $\{H_i\}_{i\in I}$ be a compatible family of subhypergroups. Then, (H_i, ψ_{ij}) is a direct system, where $\psi_{ij} = \varphi_{ij} |_{H_i}$ denotes the restriction of φ_{ij} to H_i.*

Proof. We verify the two defining properties of a direct system as follows.

1. For each $i \in I$, ψ_{ii} is the identity on H_i. This follows because φ_{ii} is the identity on G_i, so its restriction to H_i is the identity on H_i.
2. For $i \leq j \leq k$, we have $\psi_{jk} \circ \psi_{ij} = \psi_{ik}$. Indeed, for any $x \in H_i$, we have

$$(\psi_{jk} \circ \psi_{ij})(x) = \psi_{jk}(\varphi_{ij}(x)) = \varphi_{jk}(\varphi_{ij}(x)) = \varphi_{ik}(x) = \psi_{ik}(x),$$

where we used the composition property $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ from the original direct system.

Since both conditions are satisfied, (H_i, ψ_{ij}) forms a direct system. □

Definition 3.3 (see [8]). *Let (G_i, φ_{ij}) be a direct system over a directed set I. The direct limit of this system, denoted by $\lim_{i\in I} G_i$, is a hypergroup equipped with a family of homomorphisms $\alpha_i : G_i \longrightarrow \lim_{i\in I} G_i$ satisfying the compatibility condition $\alpha_j \circ \varphi_{ij} = \alpha_i$, where $i \leq j$. Moreover, for any subhypergroup G and homomorphism $f_i : G_i \longrightarrow G$ such that $f_i = f_j \circ \varphi_{ij}$, for all $i \leq j$, there is a unique homomorphism $\beta : \lim_{i\in I} G_i \longrightarrow G$ provided that $\beta \circ \alpha_i = f_i$.*

Let (G_i, φ_{ij}) be a direct system over a directed set I and let G_∞ be the disjoint union of G_i. We define a relation $\equiv$ on G_∞ as follows:

$$g_i \equiv g_j \iff \exists\, k \geq i, j : \quad \varphi_{ik}(g_i) = \varphi_{jk}(g_j),$$

where $g_i \in G_i$ and $g_j \in G_j$. It is clear that the relation $\equiv$ is equivalence. For $x \in G_\infty$, let $\widehat{x}$ denote its equivalence class. Suppose that $\widehat{G}$ is the set of all equivalence classes. For every $g_i \in G_i$ and $g_j \in G_j$, we define

$$\widehat{g}_i \odot \widehat{g}_j = \{\widehat{x} : x \in g_k \circ g_k^{'} : g_k = \varphi_{ik}(g_i), \;\; g_k^{'} = \varphi_{jk}(g_j), \;\; k \geq i, j\}.$$

Proposition 3.2. *For every $i \leq j$, if $a_i \in G_\infty$ and $a_j = \varphi_{ij}(a_i)$, then $\widehat{a}_i = \widehat{a}_j$.*

Proof. Since, for $j \in I$, $a_j = \varphi_{jj}(a_j) = \varphi_{ij}(a_i)$, it holds that $\widehat{a}_i = \widehat{a}_j$. □

Proposition 3.3 (see [8]). *The relation $\odot$ is well-defined on $\widehat{G}$.*

Proposition 3.4 (see [8]). *The relation $\odot$ is associative and $(\widehat{G}, \odot)$ is a semihypergroup.*

Proposition 3.5 (see [8]). *Let (G_i, φ_{ij}) be a direct system over a directed set I. Then, the pair $(\widehat{G}, \odot)$, where $\widehat{G}$ is the set of equivalence classes defined as above and $\odot$ is an appropriately defined hyperoperation, forms a quasi-hypergroup.*

Theorem 3.1. *Let (G_i, φ_{ij}) be a direct system over a directed set I. Then, $(\widehat{G}, \odot)$ is a hypergroup.*

Proof. From Propositions 3.4 and 3.5, the proof follows. $\square$

Theorem 3.2 (see [8]). *Let (G_i, φ_{ij}) be a direct system of canonical hypergroups over a directed set I. Then, the pair $(\widehat{G}, \oplus)$ is a canonical hypergroup.*

Proposition 3.6. *Let (G_i, φ_{ij}) be a direct system of hypergroups and let $\{H_i\}_{i\in I}$ be a compatible family of subhypergroups. Then, $\lim_{i\in I} H_i$ is a subhypergroup of $\lim_{i\in I} G_i$.*

Proof. Let $\lim_{i\in I} H_i = \widehat{H}$. Suppose that $\widehat{x}, \widehat{y} \in \widehat{H}$, and let $\widehat{z} \in \widehat{x} \odot \widehat{y}$. By the definition of the direct limit, there exist indices $i, j \in I$ such that $x \in H_i$, $y \in H_j$.

Since $\widehat{z} \in \widehat{x} \odot \widehat{y}$, it follows from the definition of the hyperoperation in the direct limit that for some $k \geq i, j$,

$$z \in \varphi_{ik}(x) \circ \varphi_{jk}(y),$$

where z is a representative of $\widehat{z}$ in G_k. Because $\{H_i\}_{i\in I}$ is a compatible family, $\varphi_{ik}(x) \in H_k$ and $\varphi_{jk}(y) \in H_k$. Since H_k is a subhypergroup, $z \in H_k$, and thus $\widehat{z} \in \widehat{H}$. This establishes that $\widehat{H}$ is closed under the hyperoperation.

It remains to verify the reproductive axiom. Let $\widehat{x} \in \widehat{H}$. We must show that $\widehat{x} \odot \widehat{H} = \widehat{H}$. Suppose $\widehat{b} \in \widehat{H}$ with a representative $b \in H_k$. By the reproductive property in H_k, we have $b \in \varphi_{ik}(x) \circ H_k$ for the appropriate representative x of $\widehat{x}$. Hence, there exists an $h_k \in H_k$ such that $b \in \varphi_{ik}(x) \circ h_k$. Let $\widehat{h}$ be the element of $\widehat{H}$ represented by h_k. By the definition of the hyperoperation in the direct limit, this implies $\widehat{b} \in \widehat{x} \odot \widehat{h} \subseteq \widehat{x} \odot \widehat{H}$. Therefore, $\widehat{H} \subseteq \widehat{x} \odot \widehat{H}$. The reverse inclusion follows from the closure of $\widehat{H}$ under the hyperoperation, which has already been proven.

Consequently, we conclude that $(\widehat{H}, \odot)$ is a subhypergroup of $\lim_{i\in I} G_i$. $\square$

Definition 3.4. *Let (G_i, φ_{ij}) be a direct system over a directed set I. Then, $(\widehat{G}, \odot)$ is called a direct system hypergroup, or DSH for short.*

Proposition 3.7. *Let (G_i, φ_{ij}) be a direct system of hypergroups indexed by I. Then, $\widehat{G}_i$ is subhypergroup of $(\widehat{G}, \odot)$, where*

$$\widehat{G}_i = \{\widehat{a}_i : a_i \in G_i\}.$$

Proof. To show that $\widehat{G}_i$ is a subhypergroup, we must verify that it is closed under the hyperoperation and satisfies the reproductive axiom.

First, let $\widehat{a}_i, \widehat{b}_i \in \widehat{G}_i$ and consider an element $\widehat{b} \in \widehat{a}_i \odot \widehat{b}_i$. By the definition of the hyperoperation in the direct limit, there exists an index $k \geq i, j$ such that $b \in a_k \circ b_k$, where $a_k = \varphi_{ik}(a_i)$ and $b_k = \varphi_{ik}(b_i)$. This implies that $b \in \varphi_{ik}(a_i \circ b_i)$, and consequently, $b = \varphi_{ik}(c)$ for some $c \in a_i \circ b_i$. Therefore, $\widehat{b} = \widehat{c} \in \widehat{G}_i$. This establishes that $\widehat{a}_i \odot \widehat{b}_i \subseteq \widehat{G}_i$, proving that $(\widehat{G}_i, \odot)$ is a subsemihypergroup.

It remains to show that $\widehat{G}_i$ satisfies the reproductive axiom. Let $\widehat{a}_i \in \widehat{G}_i$. We will prove that $\widehat{a}_i \odot \widehat{G}_i = \widehat{G}_i$. By symmetry, the proof for $\widehat{G}_i \odot \widehat{a}_i = \widehat{G}_i$ is analogous.

Let $\widehat{b} \in \widehat{G}_i$. Then $\widehat{b} = \widehat{b}_i$ for some $b_i \in G_i$. Since G_i is itself a hypergroup, it satisfies the reproductive axiom: $G_i = a_i \circ G_i$. Thus, $b_i \in a_i \circ x$ for some $x \in G_i$. This implies that $\widehat{b}_i \in \widehat{a}_i \odot \widehat{x} \subseteq \widehat{a}_i \odot \widehat{G}_i$. Therefore, $\widehat{G}_i \subseteq \widehat{a}_i \odot \widehat{G}_i$. The reverse inclusion, $\widehat{a}_i \odot \widehat{G}_i \subseteq \widehat{G}_i$, follows from the first part of the proof, because $\widehat{G}_i$ is a subsemihypergroup. Therefore, we conclude that $\widehat{a}_i \odot \widehat{G}_i = \widehat{G}_i$. $\square$

4. Quotient Direct Limits of Hypergroups

In this section, we define a regular relation on direct limits and construct quotient direct systems by compatible subhypergroups.

Theorem 4.1. *Let (G_i, φ_{ij}) be a direct system of canonical hypergroups and $H \subseteq G_\infty$ such that $\widehat{H}$ is a subhypergroup of $\widehat{G}$. Then,*

$$\widehat{H} = \bigcup_{i\in I} \widehat{H}_i,$$

where each H_i is a subhypergroup of G_i.

Proof. Suppose that $\widehat{H}$ is a subhypergroup of $\widehat{G}$, and define $H_i = \{x \in G_i : \widehat{x} \in \widehat{H}\}$. This set H_i is non-empty. Now, let $x, y \in H_i$. By definition, $\widehat{x}, \widehat{y} \in \widehat{H}$, and since $\widehat{H}$ is a subhypergroup, it follows that $\widehat{x} \odot \widehat{y} \subseteq \widehat{H}$. The hyperoperation in the direct limit is given as follows:

$$\widehat{x} \odot \widehat{y} = \{\widehat{z} : \text{for some } k \geq i,\ z \in \varphi_{ik}(x) \circ \varphi_{ik}(y)\}.$$

Because the maps φ_{ik} are homomorphisms, $\varphi_{ik}(x) \circ \varphi_{ik}(y) = \varphi_{ik}(x \circ y)$. Therefore,

$$\widehat{x} \odot \widehat{y} = \{\widehat{z} : \text{for some } k \geq i,\ z \in \varphi_{ik}(x \circ y)\} \subseteq \widehat{H}.$$

Now, consider any $z \in x \circ y$ in G_i. Taking $k = i$, we have $z \in \varphi_{ii}(x \circ y) = x \circ y$. Consequently, $\widehat{z} \in \widehat{x} \odot \widehat{y} \subseteq \widehat{H}$, which implies $z \in H_i$. This proves that $x \circ y \subseteq H_i$. For $x \in H_i$, we have $x \circ H_i \subseteq H_i$. Let $b \in H_i$. Then, $\widehat{b} \in \widehat{H} = \widehat{x} \odot \widehat{H}$ and for some $k \geq i, j$, $b \in \varphi_{ik}(x) \circ \varphi_{jk}(y)$, where $\widehat{y} \in \widehat{H}$. Since, $b \in H_i$, then $k = i$ and

$$b \in \varphi_{ii}(x) \circ \varphi_{ji}(y) = x \circ \varphi_{ji}(y) \subseteq x \circ H_i.$$

Therefore, $H_i \subseteq x \circ H_i$ and this implies that $x \circ H_i = H_i$. Similarly, we have $H_i \circ x = H_i$ for every $x \in H_i$. Since H_i is non-empty and satisfies the reproductive axiom, and the hyperoperation $\circ$ is associative (as it is inherited from G_i), it follows that $(H_i, \circ)$ is a hypergroup.

If $\widehat{b} \in \widehat{H}$ and $b \in G_i$ is a representative, then $b \in H_i$, which implies that $\widehat{b} \in \widehat{H}_i$. Hence,

$$\widehat{H} \subseteq \bigcup_{i \in I} \widehat{H}_i.$$

Also, for every $b \in \bigcup_{i \in I} \widehat{H}_i$, we have $\widehat{b} \in H_i$ for some $i \in I$. Then, $\widehat{b} \in \widehat{H}$ and $\bigcup_{i \in I} \widehat{H}_i \subseteq \widehat{H}$. Therefore, $\widehat{H} = \bigcup_{i \in I} \widehat{H}_i$ □

Theorem 4.2. *Let (G_i, φ_{ij}) be a direct system of hypergroups and the family $\{H_i\}_{i \in I}$ is compatible with the direct system. Then, $T = \bigcup_{i \in I} \widehat{H}_i$ is a subhypergroup of $\widehat{G}$.*

Proof. Suppose $\{H_i\}$ is a family of compatible with the direct system and $\widehat{x}, \widehat{y} \in T$. Then, for some $i, j \in I$ and $x_1 \in H_i, y \in H_j$, we have $\widehat{x} = \widehat{x}_1$ and $\widehat{y} = \widehat{y}_1$. This implies that

$$\widehat{x} \odot \widehat{y} = \widehat{x}_1 \odot \widehat{y}_1 = \{\widehat{z} : k \geq i, j,\ z \in \varphi_{ik}(x_1) \circ \varphi_{jk}(y_1)\}.$$

Also,

$$\varphi_{ik}(x_1) \in \varphi_{ik}(H_i) \subseteq H_k, \quad \varphi_{jk}(y_1) \in \varphi_{jk}(H_j) \subseteq H_k.$$

This implies that $z \in H_k$ and $\widehat{z} \in \widehat{H}_k$. Hence, $\widehat{x} \odot \widehat{y} \subseteq T$. For every $\widehat{x} \in T$, we have $\widehat{x} \odot T \subseteq T$, where $x \in H_i$. Also, $\widehat{b} \in T$ implies that, for some $k \in I$, we have $\widehat{b} \in \widehat{H}_k$ and $\widehat{b} = \widehat{b}_1$ for some $b_1 \in H_k$. Since, H_k is a hypergroup, it holds that

$$H_k = \varphi_{ik}(x) \circ H_k.$$

This implies that $b_1 \in \varphi_{ik}(x) \circ a$ for some $a \in H_k$. Therefore,

$$\widehat{b} = \widehat{b}_1 \in \widehat{\varphi_{ik}(x)} \odot \widehat{a},$$

and $T \subseteq \widehat{x} \odot T$. □

Theorem 4.3. *Let (G_i, φ_{ij}) be a direct system of canonical hypergroups, and let $\{H_i\}_{i \in I}$ be a family of compatible subhypergroups with the direct system, i.e., $\varphi_{ij}(H_i) \subseteq H_j$ for all $i \leq j$. Let $\widehat{G} = \lim_{i \in I} G_i$ be the direct limit with canonical maps and $\widehat{H} = \bigcup_{i \in I} \widehat{H}_i$. Then, the following relation is an equivalence relation on $\widehat{G}$:*

$$\widehat{x} \equiv \widehat{y} \iff \exists\, k \geq i, j \text{ such that } \varphi_{ik}(x) \in \varphi_{jk}(y) + H_k,$$

where $\widehat{x}, \widehat{y} \in \widehat{G}$.

Proof. Suppose that $\widehat{x} \in \widehat{G}$, where $x \in G_i$. Hence, $\varphi_{ii}(x) \in \varphi_{ii}(x) + 0_{H_i}$. This implies that $\widehat{x} \equiv \widehat{x}$. Also, $\widehat{x} \equiv \widehat{y}$, implies that $\varphi_{ik}(x) \in \varphi_{jk}(y) + H_k$, for some $k \geq i, j$, where $x \in G_i$ and $y \in G_j$. Hence, $\varphi_{jk}(y) \in \varphi_{ik}(x) + (-h_k)$ for some $h_k \in H_k$. This implies that $y \equiv x$. Let $\widehat{x} \equiv \widehat{y}$ and $\widehat{y} \equiv \widehat{z}$. Then,

$$\varphi_{ik_1}(x) \in \varphi_{jk_1}(y) + H_{k_1}, \quad \varphi_{jk_2}(y) \in \varphi_{rk_2}(z) + H_{k_2}, \text{ for some } k_1 \geq i, j \text{ and } k_2 \geq j, r.$$

Hence, for $k = \max\{k_1, k_2\}$, we have

$$\begin{aligned}\varphi_{ik}(x) = \varphi_{k_1k}(\varphi_{ik_1}(x)) \in \varphi_{k_1k}(\varphi_{jk_1}(y) + H_{k_1}) &= \varphi_{k_1k}(\varphi_{jk_1}(y)) + \varphi_{k_1k}(H_{k_1})\\ &\subseteq \varphi_{jk}(y) + H_k.\end{aligned}$$

and

$$\begin{aligned}\varphi_{jk}(y) = \varphi_{k_2k}(\varphi_{jk_2}(y)) \in \varphi_{k_2k}(\varphi_{rk_2}(z) + H_{k_2}) &= \varphi_{k_2k}(\varphi_{rk_2}(z)) + \varphi_{k_2k}(H_{k_2})\\ &\subseteq \varphi_{jk}(z) + H_k.\end{aligned}$$

Therefore, $\varphi_{ik}(x) \in \varphi_{jk}(z) + H_k$. Consequently, $x \equiv z$, and hence, this relation is equivalence. ☐

Theorem 4.4. *Let* (G_i, φ_{ij}) *be a direct system of canonical hypergroups, and let* $\{H_i\}_{i\in I}$ *be a family of subhypergroups compatible with the direct system. Then, the relation* $\equiv$ *is regular on* $\widehat{G}$.

Proof. Assume that $\widehat{x} \equiv \widehat{y}$, where $x \in G_i$, $y \in G_j$, and $z \in G_r$. By the definition of $\equiv$, there exists an index $k_1 \geq i, j$ such that $\varphi_{ik_1}(x) \in \varphi_{jk_1}(y) + H_{k_1}$. Let $\widehat{a} \in \widehat{x} \oplus \widehat{z}$. Then, by the definition of the hyperoperation $\oplus$, there exists an index $k_2 \geq i, r$ and an element $a \in G_{k_2}$ such that $a \in \varphi_{ik_2}(x) + \varphi_{rk_2}(z)$. Choose $k \geq \max\{k_1, k_2\}$. Applying the morphism φ_{k_1k} to the first inclusion, we obtain

$$\varphi_{ik}(x) = \varphi_{k_1k}(\varphi_{ik_1}(x)) \in \varphi_{k_1k}(\varphi_{jk_1}(y) + H_{k_1}) = \varphi_{jk}(y) + H_k.$$

Similarly, applying φ_{k_2k} to the second relation gives $\varphi_{k_2k}(a) \in \varphi_{ik}(x) + \varphi_{rk}(z)$. Substituting the expression for $\varphi_{ik}(x)$, we obtain

$$\varphi_{k_2k}(a) \in (\varphi_{jk}(y) + H_k) + \varphi_{rk}(z) = \varphi_{jk}(y) + \varphi_{rk}(z) + H_k.$$

Thus, there exists an element $b \in \varphi_{jk}(y) + \varphi_{rk}(z)$ such that $\varphi_{k_2k}(a) \in b + H_k$, which implies $\widehat{a} \equiv \widehat{b}$. Since $b \in \varphi_{jk}(y) + \varphi_{rk}(z)$, it follows that $\widehat{b} \in \widehat{y} \oplus \widehat{z}$. Therefore, for every $\widehat{a} \in \widehat{x} \oplus \widehat{z}$, there exists $\widehat{b} \in \widehat{y} \oplus \widehat{z}$ such that $\widehat{a} \equiv \widehat{b}$.

Conversely, a symmetric argument shows that for every $\widehat{c} \in \widehat{y} \oplus \widehat{z}$, there exists $\widehat{d} \in \widehat{x} \oplus \widehat{z}$ such that $\widehat{c} \equiv \widehat{d}$.

Therefore, the relation $\equiv$ is right regular. An analogous proof establishes that $\equiv$ is also left regular. ☐

Proposition 4.1. *Let* (G_i, φ_{ij}) *be a direct system of canonical hypergroups, and let* $\{H_i\}_{i\in I}$ *be a family of compatible subhypergroups with the direct system. Then,* $(G_i/H_i^*, \overline{\varphi}_{ij})$ *is a direct system, where* $\overline{\varphi}_{ij} : G_i/H_i^* \longrightarrow G_j/H_j^*$ *and is defined by*

$$\overline{\varphi}_{ij}(H_i^*(x)) = H_j^*(\varphi_{ij}(x)).$$

Proof. We first verify that the induced maps are well-defined. Suppose that $H_i^*(x) = H_i^*(y)$ for $x, y \in G_i$. By the definition of a coset, this implies $x - y \in H_i$, or equivalently, $x \in y + H_i$. Thus, there exists an element $h \in H_i$ such that $x \in y + h$.

Now, consider the image under the homomorphism φ_{ij}. Then, we have $\varphi_{ij}(x) \in \varphi_{ij}(y + h) = \varphi_{ij}(y) + \varphi_{ij}(h)$. Since $\varphi_{ij}(h) \in H_j$ by the property of the direct system of subgroups, it follows that $\varphi_{ij}(x) \in \varphi_{ij}(y) + H_j$. Consequently,

$$H_j^*(\varphi_{ij}(x)) = H_j^*(\varphi_{ij}(y))$$

in the quotient group G_j/H_j^*. This establishes that the induced map φ_{ij} is well-defined.

We now verify that $(G_i/H_i^*, \overline{\varphi}_{ij})$ forms a direct system. For any indices $i \leq j \leq k$ and any element $H_i^*(x) \in G_i/H_i^*$, the following hold:

1. **Identity Mapping:** The map φ_{ii} acts as the identity on the coset because

$$\overline{\varphi}_{ii}(H_i^*(x)) = H_i^*(\varphi_{ii}(x)) = H_i^*(x).$$

2. **Compatibility Condition:** The maps satisfy the composition property because

$$\begin{aligned}(\overline{\varphi}_{jk} \circ \overline{\varphi}_{ij})(H_i^*(x)) &= \overline{\varphi}_{jk}(\overline{\varphi}_{ij}(H_i^*(x)))\\ &= \overline{\varphi}_{jk}(H_j^*(\overline{\varphi}_{ij}(x)))\\ &= H_k^*(\overline{\varphi}_{jk}(\overline{\varphi}_{ij}(x)))\\ &= H_k^*(\overline{\varphi}_{ik}(x))\\ &= \overline{\varphi}_{ik}(H_i^*(x)).\end{aligned}$$

Therefore, $\overline{\varphi}_{jk} \circ \overline{\varphi}_{ij} = \overline{\varphi}_{ik}$.
Since both conditions for a direct system are satisfied, the proof is complete. ☐

Proposition 4.2. *Let (G_i, φ_{ij}) be a direct system of canonical hypergroups, and let $\{H_i\}_{i\in I}$ be a family of compatible subhypergroups with the direct system and $\widehat{H} = \bigcup_{i\in I} \widehat{H}_i$. Then,*

$$\lim_{i\in I} G_i/H_i^* \cong \widehat{G}/\widehat{H}^*,$$

where $\lim_{i\in I} G_i = \widehat{G}$.

Proof. Define a map $\psi : \lim_{i\in I} G_i/H_i^* \longrightarrow \widehat{G}/\widehat{H}^*$ by

$$\psi(\widehat{H_i^*(x)}) = \widehat{H}^*(\widehat{x}),$$

where $\widehat{H_i^*(x)}$ denotes the equivalence class of x in $\lim_{i\in I} G_i/H_i^*$, and $\widehat{H}^*(\widehat{x})$ denotes the equivalence class of $\widehat{x}$ in $\widehat{G}/\widehat{H}$. First, we show that ψ is well-defined.

Let $\widehat{H_i^*(x)} = \widehat{H_j^*(y)}$. Then there exists $k \geq i, j$, such that

$$\overline{\varphi}_{ik}(H_i^*(x)) = \overline{\varphi}_{jk}(H_j^*(y)).$$

This implies that

$$H_k^*(\varphi_{ik}(x)) = H_k^*(\varphi_{ik}(y)).$$

Hence, $\varphi_{ik}(x) \in \varphi_{jk}(y) + H_k$, and for some $h_k \in H_k$, we have $\varphi_{ik}(x) \in \varphi_{jk}(y) + \varphi_{kk}(h_k)$. Then,

$$\widehat{H}^*(\widehat{x}) = \widehat{H}^*(\widehat{\varphi_{ik}(x)}) = \widehat{H}^*(\widehat{\varphi_{jk}(y)}) = \widehat{H}^*(\widehat{y}),$$

and ψ is well-defined. Next, we show that ψ is a homomorphism. Let $\widehat{H_i^*(x)}, \widehat{H_j^*(y)} \in \lim_{i\in I} G_i/H_i^*$. Then,

$$\begin{aligned}
\psi(\widehat{H_i^*(x)} \oplus \widehat{H_j^*(y)}) &= \psi(\widehat{H_k^*(z)}),\ \ H_k^*(z) \in \overline{\varphi}_{ik}(H_i^*(x)) + \overline{\varphi}_{jk}(H_j^*(y)) \\
&= \psi(\widehat{H_k^*(z)}),\ \ H_k^*(z) \in H_k^*(\varphi_{ik}(x)) + H_k^*(\varphi_{jk}(y)) \\
&= \widehat{H}^*(\widehat{z}),\ \ z \in \varphi_{ik}(x) + \varphi_{jk}(y) \\
&= \widehat{H}^*(\widehat{\varphi_{ik}(x)}) \oplus \widehat{H}^*(\widehat{\varphi_{jk}(y)}) \\
&= \widehat{H}^*(\widehat{x}) \oplus \widehat{H}^*(\widehat{y}) \\
&= \psi(\widehat{H_i^*(x)}) \oplus \psi(\widehat{H_j^*(y)}).
\end{aligned}$$

To show that ψ is injective, suppose $\psi(\widehat{H_i^*(x)}) = \psi(\widehat{H_j^*(y)})$. Then, $\widehat{H}^*(\widehat{x}) = \widehat{H}^*(\widehat{y})$ in $\widehat{G}/\widehat{H}$, which implies $\widehat{x} \equiv \widehat{y}$. By the earlier characterization of the equivalence relation $\equiv$, there exists $k \geq i, j$ such that $\varphi_{ik}(x) \in \varphi_{jk}(y) + H_k$ in G_k/H_k^*. This means that

$$H_k^*(\varphi_{ik}(x)) = H_k^*(\varphi_{jk}(y)),$$

and consequently,

$$\overline{\varphi}_{ik}(H_i^*(x)) = \overline{\varphi}_{jk}(H_j^*(y)).$$

Therefore, $\widehat{H_i^*(x)} = \widehat{H_j^*(y)}$ in $\lim_{i\in I} G_i/H_i^*$, proving that ψ is injective.

Since ψ is a well-defined homomorphism that is both surjective and injective, it is an isomorphism. This completes the proof of the proposition. □

5. Conclusion

We have shown that for a direct system of canonical hypergroups (G_i, ϕ_{ij}) with a compatible family of subhypergroups $\{H_i\}$, there is a natural isomorphism $\lim_{i\in I}(G_i/H_i^*) \cong \widehat{G}/\widehat{H}$, where $\lim_{i\in I} G_i = \widehat{G}$ and $\widehat{H} = \bigcup_{i\in I} \widehat{H}_i$. This proves that direct limits commute with quotients in the category of canonical hypergroups, extending classical results from group theory. Future work includes extending these results to other hyperstructures (semihypergroups, hyperrings, H_v-groups), studying fuzzy and soft hypergroup limits, developing homological algebra for hypergroups, and applying the isomorphism to concrete examples of finite canonical hypergroups.

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