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Research Article

Asymptotically Deferred Weighted Statistical Equivalence of Order α in Probability and Its Applications to Approximation Theory

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Abstract

In this paper, we introduce a unified framework for studying the asymptotic behavior of random variables by combining deferred methods with weighted statistical convergence in probability. We define the notions of asymptotically deferred weighted statistical equivalence of order α in probability and asymptotically deferred weighted strong equivalence of order α in probability. We also establish the relationship between these two concepts and investigate the inclusion relations corresponding to different orders. In addition, we apply the proposed framework to approximation theory by establishing a Korovkin-type theorem for sequences of positive linear operators on $C[0,1]$. Moreover, we present a Voronovskaya-type result describing the asymptotic behavior of the approximation error and derive an estimate for the rate of convergence by means of the modulus of continuity.

Keywords: statistical convergence; deferred Cesàro mean; weighted statistical convergence; probability; asymptotic equivalence; Korovkin-type theorem.

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1. Introduction

Approximation theory is one of the central branches of mathematical analysis. Its main aim is to approximate complicated functions by means of simpler and more manageable operators. A major breakthrough in this direction was provided by Korovkin [19]. For a comprehensive account of Korovkin-type approximation theory and its applications, we refer to the monograph of Altomare and Campiti [4]. We also note that the modulus of continuity plays a fundamental role in estimating the rate of convergence in approximation processes (see [5]).

Sequences with oscillatory characteristics are frequently difficult for classical convergence techniques to manage. Fast [10] and Steinhaus [30] separately devised statistical convergence to overcome such constraints. By using statistical convergence to prove Korovkin-type theorems, Gadjiev and Orhan [11] offered one of the most important applications of this technique to approximation theory. Duman and Orhan [8] made important contributions to statistical approximation by positive linear operators after this development.

Statistical convergence has been studied in a variety of abstract frameworks due to its adaptability. For example, convergence properties were described in random n-normed spaces [31] and linear n-normed spaces [27]. The relationship between statistical convergence and operator dynamics was established by important contributions in operator theory and Fock spaces [13, 25, 33]. These ideas have lately been expanded to include 2-normed spaces [32], generalized metric spaces [16], and credibility spaces [28]. Furthermore, variants like Wijsman lacunary invariant statistical convergence were investigated by Huban and Gürdal [14].

To increase the applicability of these techniques, Karakaya and Chishti [15] presented the idea of weighted statistical convergence, which Mursaleen et al. [24] later modified. This formulation was recently refined by Ghosal [12] to incorporate the order of convergence α. Weighted statistical convergence was further extended to Korovkin and Voronovskaya type theorems by Braha et al. [6].

Agnew [1] introduced the concept of deferred summability concurrently with these advancements. This approach makes use of two sequences of non-negative integers a_n and b_n that fulfill $a_n < b_n$ for all $n \in \mathbb{N}$ and where the difference $b_n - a_n$ approaches to infinity. Subsequently, deferred statistical convergence was introduced by Küçükaslan and Yılmaztürk [21], who also established the relationship between deferred statistical convergence and strong deferred summability. Additionally,

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Et et al. [9] generalized deferred statistical convergence of order α in metric spaces, whereas Küçükaslan et al. [20] examined statistical convergence of metric-valued sequences. Recently, this concept has been further enriched by investigations in credibility spaces involving fuzzy variables [17, 29] and in neutrosophic normed spaces [18, 23].

In the present paper, we combine the weighted and deferred approaches. Let $p = (p_k)$ be a sequence of positive real numbers and let

$$P(D_n) = \sum_{k=a_n+1}^{b_n} p_k \to \infty \qquad (n \to \infty),$$

where $D_n = (a_n, b_n]$ denotes the deferred interval. Throughout the paper, $P(D_n)$ denotes the deferred weighted partial sum and should not be confused with the probability measure P on the underlying probability space.

Within probability theory, sequences of random variables exhibit behaviors that are substantially different from those of real sequences. Statistical convergence in probability was studied by Das et al. [7]. On the other hand, in many stochastic models one is interested not only in convergence to a limit, but also in whether two sequences become asymptotically equivalent. In this direction, Patterson [26] extended Marouf's notion [22] of asymptotic equivalence of real sequences to the statistical setting. More recently, Akbaş and Işık investigated weighted statistical convergence in probability [3] and asymptotically λ-statistical equivalent sequences of order α in probability [2].

Motivated by these developments, we introduce a deferred weighted probabilistic version of asymptotic equivalence. This allows us to combine deferred averaging, weighting, and probabilistic asymptotics in a single framework. After establishing the basic inclusion relations between the new notions, we apply the theory to approximation processes by proving a Korovkin-type theorem, and we further derive a Voronovskaya-type result together with a rate of convergence estimate.

2. Main Results

In this section, we introduce the notions of asymptotically deferred weighted statistical equivalence and asymptotically deferred weighted strong equivalence of order α in probability. We then study their basic properties and establish the corresponding inclusion relations.

Let $\alpha \in (0, 1]$ be a fixed real number, let $(S, \mathcal{F}, P)$ be a probability space, and let the definitions of D_n and $P(D_n)$ be as stated in the introduction. Additionally, we assume that the sequence of random variables Y_n is non-zero almost surely, i.e., $P(Y_k = 0) = 0$ for all $k \in \mathbb{N}$, in order to guarantee that the ratios in the following formulations are well-defined.

Definition 2.1. *Two sequences of random variables $\{X_n\}$ and $\{Y_n\}$ are said to be asymptotically deferred weighted statistically equivalent of order α in probability, denoted by*

$$X \overset{PDS^{\alpha}_{N_p}}{\sim} Y,$$

provided that for every $\varepsilon > 0$ and $\delta > 0$ the following limit holds

$$\lim_{n\to\infty} \frac{1}{[P(D_n)]^{\alpha}} \left| \left\{ k \in D_n : P\left(\left| \frac{X_k}{Y_k} - 1 \right| \geq \varepsilon \right) \geq \delta \right\} \right|_p = 0,$$

where $|\cdot|_p$ denotes the weighted cardinality, i.e., $|K|_p = \sum_{k\in K} p_k$.

Definition 2.2. *Under the same assumptions, $\{X_n\}$ and $\{Y_n\}$ are said to be asymptotically deferred weighted strongly equivalent of order α in probability, denoted by*

$$X \overset{PDW^{\alpha}_{N_p}(r)}{\sim} Y,$$

if for every $\varepsilon > 0$ and $r > 0$,

$$\lim_{n\to\infty} \frac{1}{[P(D_n)]^{\alpha}} \sum_{k=a_n+1}^{b_n} p_k \left[P\left(\left| \frac{X_k}{Y_k} - 1 \right| \geq \varepsilon \right) \right]^r = 0.$$

Definition 2.3. *A sequence of random variables $\{Z_n\}$ is said to be deferred weighted statistically convergent to zero of order α in probability, denoted by $Z_n \xrightarrow{PDS^{\alpha}_{N_p}(P)} 0$, if for every $\varepsilon > 0$ and $\delta > 0$,*

$$\lim_{n\to\infty} \frac{1}{[P(D_n)]^{\alpha}} \left| \{ k \in D_n : P(|Z_k| \geq \varepsilon) \geq \delta \} \right|_p = 0.$$

This definition naturally extends to sequences of functions $\{U_n\}$ in $C[0, 1]$ via the supremum norm, i.e., $\|U_n\|_\infty \xrightarrow{PDS^{\alpha}_{N_p}(P)} 0$, where $\|U_n\|_\infty = \sup_{x\in[0,1]} |U_n(x)|$.

We now establish the inclusion relations between different orders of convergence.

Theorem 2.1. *Let* α *and* β *be fixed real numbers such that* $0 < \alpha \le \beta \le 1$. *Let* $\{X_n\}$ *and* $\{Y_n\}$ *be sequences of random variables. Then the following implications hold.*

(i) $X \overset{PDS^{\alpha}_{N_p}}{\sim} Y \implies X \overset{PDS^{\beta}_{N_p}}{\sim} Y.$

(ii) $X \overset{PDW^{\alpha}_{N_p}(r)}{\sim} Y \implies X \overset{PDW^{\beta}_{N_p}(r)}{\sim} Y.$

Proof. We provide the proof for (i); the proof for (ii) follows a similar logic. Suppose $X \overset{PDS^{\alpha}_{N_p}}{\sim} Y$. Let $\varepsilon > 0$ and $\delta > 0$ be given. Define the set of indices K_{D_n} as follows

$$K_{D_n} = \left\{ k \in (a_n, b_n] : P\left(\left| \frac{X_k}{Y_k} - 1 \right| \ge \varepsilon \right) \ge \delta \right\}.$$

We are given that

$$\lim_{n\to\infty} \frac{|K_{D_n}|_p}{[P(D_n)]^{\alpha}} = 0,$$

where

$$|K_{D_n}|_p = \sum_{k \in K_{D_n}} p_k.$$

Since $P(D_n) \to \infty$ as $n \to \infty$, we can assume $P(D_n) \ge 1$ for sufficiently large n. The condition $0 < \alpha \le \beta$ implies $[P(D_n)]^{\alpha} \le [P(D_n)]^{\beta}$, which leads to the inequality

$$\frac{1}{[P(D_n)]^{\beta}} \le \frac{1}{[P(D_n)]^{\alpha}}.$$

Multiplying both sides by the non-negative weighted cardinality $|K_{D_n}|_p$, we obtain

$$\frac{|K_{D_n}|_p}{[P(D_n)]^{\beta}} \le \frac{|K_{D_n}|_p}{[P(D_n)]^{\alpha}}.$$

Taking the limit as $n \to \infty$, the right-hand side approaches zero by the hypothesis. Therefore, the left-hand side must also approach zero. Hence, we conclude $X \overset{PDS^{\beta}_{N_p}}{\sim} Y$. □

Theorem 2.2. *Let* $\alpha \in (0, 1]$ *and* $r > 0$. *If* $X \overset{PDW^{\alpha}_{N_p}(r)}{\sim} Y$, *then* $X \overset{PDS^{\alpha}_{N_p}}{\sim} Y$.

Proof. Suppose $X \overset{PDW^{\alpha}_{N_p}(r)}{\sim} Y$. Let $\varepsilon > 0$ and $\delta > 0$ be given. We define the set of indices where the probability of the difference is significant as

$$K_{D_n}(\varepsilon, \delta) = \left\{ k \in D_n : P\left(\left| \frac{X_k}{Y_k} - 1 \right| \ge \varepsilon \right) \ge \delta \right\}.$$

For any $k \in K_{D_n}(\varepsilon, \delta)$, we have the inequality

$$\left[P\left(\left| \frac{X_k}{Y_k} - 1 \right| \ge \varepsilon \right) \right]^r \ge \delta^r.$$

Multiplying both sides by the non-negative weight p_k, we get

$$p_k \left[P\left(\left| \frac{X_k}{Y_k} - 1 \right| \ge \varepsilon \right) \right]^r \ge p_k \delta^r.$$

Now, consider the weighted sum over the deferred interval D_n. The sum over the whole interval is clearly greater than or equal to the sum over the subset $K_{D_n}(\varepsilon, \delta)$:

$$\begin{aligned} \sum_{k \in D_n} p_k \left[P\left(\left| \frac{X_k}{Y_k} - 1 \right| \ge \varepsilon \right) \right]^r &\ge \sum_{k \in K_{D_n}(\varepsilon,\delta)} p_k \left[P\left(\left| \frac{X_k}{Y_k} - 1 \right| \ge \varepsilon \right) \right]^r \\ &\ge \sum_{k \in K_{D_n}(\varepsilon,\delta)} p_k \delta^r \\ &= \delta^r |K_{D_n}(\varepsilon, \delta)|_p. \end{aligned}$$

Dividing both sides by $[P(D_n)]^\alpha$, we obtain

$$\frac{1}{[P(D_n)]^\alpha}\sum_{k\in D_n} p_k\left[P\left(\left|\frac{X_k}{Y_k}-1\right|\geq\varepsilon\right)\right]^r\geq\delta^r\frac{|K_{D_n}(\varepsilon,\delta)|_p}{[P(D_n)]^\alpha}.$$

Taking the limit as $n\to\infty$, the left-hand side tends to zero by the hypothesis of deferred weighted strong equivalence. Consequently, the right-hand side must also tend to zero. Since $\delta>0$ is fixed, we conclude that

$$\lim_{n\to\infty}\frac{|K_{D_n}(\varepsilon,\delta)|_p}{[P(D_n)]^\alpha}=0,$$

which implies $X\overset{PDS^\alpha_{N_p}}{\sim}Y$. □

Example 2.1. *Let* $(S,\mathcal{F},P)$ *be a probability space. Consider the sequence of random variables* $\{Y_n\}$ *defined as* $Y_n(\omega)=1$ *almost surely for all* $n\in\mathbb{N}$. *We will construct a sequence* $\{X_n\}$ *to demonstrate the strict inclusion.*

Let the deferred interval be defined by $a_n=2^{n-1}$ *and* $b_n=2^n$. *Let the weight sequence be* $p_k=1$ *for all* k. *Consequently, the weighted length of the interval is*

$$P(D_n)=\sum_{k=a_n+1}^{b_n}p_k=b_n-a_n=2^n-2^{n-1}=2^{n-1}.$$

Let α *and* β *be real numbers such that* $0<\alpha<\beta\leq1$. *Choose a parameter* γ *satisfying* $\alpha<\gamma<\beta$. *We define a subset of indices* $K_n\subset D_n$ *for each* n, *such that the cardinality is given by* $|K_n|=\lfloor(2^{n-1})^\gamma\rfloor$. *Note that since* $\gamma<1$, K_n *is a proper subset of* D_n *for large* n.

Now, define the sequence of random variables $\{X_k\}$ *as follows:*

$$X_k(\omega)=\begin{cases}1+Z_k(\omega), & \text{if } k\in\bigcup_{n=1}^\infty K_n\\ 1, & \text{if } k\notin\bigcup_{n=1}^\infty K_n\end{cases}$$

where Z_k *are independent random variables following a Bernoulli distribution with parameter* $q=0.5$ *to distinguish from the weight sequence* p_k. *That is,* $P(Z_k=1)=0.5$ *and* $P(Z_k=0)=0.5$.

For any $\varepsilon\in(0,1)$, *consider the probability* $P(|X_k/Y_k-1|\geq\varepsilon)$. *Since* $Y_k=1$, *we imply*

$$\left|\frac{X_k}{Y_k}-1\right|=|X_k-1|.$$

We analyze the probability in two cases:

- *If* $k\notin\bigcup_{n=1}^\infty K_n$, *then* $X_k=1$, *so* $|X_k-1|=0$. *Thus,* $P(|X_k-1|\geq\varepsilon)=0$.
- *If* $k\in\bigcup_{n=1}^\infty K_n$, *then* $X_k=1+Z_k$. *Consequently,* $|X_k-1|=|Z_k|$. *Since* $\varepsilon\in(0,1)$, *the inequality* $|Z_k|\geq\varepsilon$ *holds if and only if* $Z_k=1$. *Therefore,*
$$P\left(\left|\frac{X_k}{Y_k}-1\right|\geq\varepsilon\right)=P(Z_k=1)=q=0.5.$$

Let $\delta\in(0,0.5)$. *The set of indices where the probability is at least* δ *corresponds exactly to the set* $\bigcup K_n$. *Now, we verify the limit for order* β

$$\lim_{n\to\infty}\frac{|K_n|}{[P(D_n)]^\beta}=\lim_{n\to\infty}\frac{\lfloor(2^{n-1})^\gamma\rfloor}{(2^{n-1})^\beta}\approx\lim_{n\to\infty}(2^{n-1})^{\gamma-\beta}=0,$$

since $\gamma-\beta<0$. *Thus,* $X\overset{PDS^\beta_{N_p}}{\sim}Y$. *However, checking for order* α

$$\lim_{n\to\infty}\frac{|K_n|}{[P(D_n)]^\alpha}=\lim_{n\to\infty}\frac{\lfloor(2^{n-1})^\gamma\rfloor}{(2^{n-1})^\alpha}\approx\lim_{n\to\infty}(2^{n-1})^{\gamma-\alpha}=\infty,$$

since $\gamma-\alpha>0$. *Consequently,* X *is not asymptotically deferred weighted statistically equivalent of order* α *to* Y. *This proves the strict inclusion.*

3. Applications to Korovkin-Type Approximation Theorem

In this section, we apply the concept of deferred weighted statistical convergence of order α in probability to approximation theory. We establish a Korovkin-type theorem for sequences of positive linear operators.

Let $C[0,1]$ denote the space of all continuous real-valued functions on $[0,1]$ equipped with the supremum norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

We consider the standard test functions $e_i(x) = x^i$ for $i = 0, 1, 2$.

Lemma 3.1. *Let $\{L_n\}$ be a sequence of positive linear operators from $C[0,1]$ into $\mathcal{M}(S)$. If the sequence converges to the identity operator on the test functions in the deferred weighted statistical sense, i.e.,*

$$\|L_n(e_i) - e_i\|_\infty \xrightarrow{PDS^{\alpha}_{N_p}(P)} 0, \quad \textit{for } i = 0, 1, 2,$$

then for any function $f \in C[0,1]$,

$$\|L_n(f) - f\|_\infty \xrightarrow{PDS^{\alpha}_{N_p}(P)} 0.$$

Proof. Let $f \in C[0,1]$ and let $M = \|f\|_\infty$. Since f is continuous on $[0,1]$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ whenever $|t - x| < \delta$. Furthermore, for all $t, x \in [0,1]$, the following inequality holds:

$$|f(t) - f(x)| \le \varepsilon + \frac{2M}{\delta^2}(t - x)^2.$$

Using the linearity and positivity of L_n, we have

$$\begin{aligned}
|L_n(f;x) - f(x)| &= |L_n(f(t) - f(x);x) + f(x)(L_n(e_0;x) - 1)| \\
&\le L_n(|f(t) - f(x)|;x) + |f(x)||L_n(e_0;x) - 1| \\
&\le L_n\left(\varepsilon + \frac{2M}{\delta^2}(t - x)^2;x\right) + M|L_n(e_0;x) - 1| \\
&= \varepsilon L_n(e_0;x) + \frac{2M}{\delta^2}L_n((t - x)^2;x) + M|L_n(e_0;x) - 1|.
\end{aligned}$$

Now, we explicitly bound the term $L_n((t - x)^2;x)$. Using linearity, we can write

$$L_n((t - x)^2;x) = L_n(e_2;x) - 2xL_n(e_1;x) + x^2L_n(e_0;x).$$

To relate this to the approximation errors, we rearrange the terms by adding and subtracting $x^2 = x^2 \cdot 1$, $2x^2 = 2x \cdot x$, and $x^2 \cdot e_0(x)$ appropriately since $e_2(x) - 2xe_1(x) + x^2e_0(x) = 0$,

$$L_n((t - x)^2;x) = [L_n(e_2;x) - e_2(x)] - 2x[L_n(e_1;x) - e_1(x)] + x^2[L_n(e_0;x) - e_0(x)].$$

Since $x \in [0,1]$, we have $|x| \le 1$ and $|x^2| \le 1$. Taking the supremum norm, we obtain

$$\|L_n((t - x)^2)\|_\infty \le \|L_n(e_2) - e_2\|_\infty + 2\|L_n(e_1) - e_1\|_\infty + \|L_n(e_0) - e_0\|_\infty.$$

Also, note that $L_n(e_0;x) \le 1 + |L_n(e_0;x) - 1|$, so $\|L_n(e_0)\|_\infty \le 1 + \|L_n(e_0) - e_0\|_\infty$. Substituting these bounds back into the main inequality

$$\|L_n(f) - f\|_\infty \le \varepsilon(1 + \|L_n(e_0) - e_0\|_\infty) + M\|L_n(e_0) - e_0\|_\infty + \frac{2M}{\delta^2}\Big(\|L_n(e_2) - e_2\|_\infty + 2\|L_n(e_1) - e_1\|_\infty + \|L_n(e_0) - e_0\|_\infty\Big).$$

Grouping the error terms, we can find a constant $C > 0$ (specifically, $C = \max\{\varepsilon + M + \frac{2M}{\delta^2}, \frac{4M}{\delta^2}\}$) such that

$$\|L_n(f) - f\|_\infty \le \varepsilon + C\sum_{i=0}^{2} \|L_n(e_i) - e_i\|_\infty.$$

Since $\varepsilon > 0$ is arbitrary and each term $\|L_n(e_i) - e_i\|_\infty$ converges to 0 in the deferred weighted statistical probability sense, the linear combination also converges to 0. Specifically, for any $\eta > \varepsilon$, the set of indices where the error exceeds η is contained in the union of sets where test function errors are large. Thus,

$$\|L_n(f) - f\|_\infty \xrightarrow{PDS^{\alpha}_{N_p}(P)} 0.$$

□

Theorem 3.1. *Let* $(S, \mathcal{F}, P)$ *be a probability space and let* $T_n, S_n : C[0,1] \to \mathcal{M}(S)$ *be sequences of positive linear operators. Assume that the sequence* $\{S_n\}$ *acts as a Korovkin approximation process, i.e.,*

$$\|S_n(e_i) - e_i\|_\infty \overset{PDS^{\alpha}_{N_p}(P)}{\longrightarrow} 0, \quad \text{for } i = 0, 1, 2.$$

If, in addition, the sequences satisfy the asymptotic equivalence condition on the test functions

$$\|T_n(e_i) - S_n(e_i)\|_\infty \overset{PDS^{\alpha}_{N_p}(P)}{\longrightarrow} 0, \quad \text{for } i = 0, 1, 2, \tag{1}$$

then for any function $f \in C[0,1]$*, we have*

$$\|T_n(f) - S_n(f)\|_\infty \overset{PDS^{\alpha}_{N_p}(P)}{\longrightarrow} 0.$$

Proof. Let $f \in C[0,1]$. By the triangle inequality, we can decompose the difference as

$$\|T_n(f) - S_n(f)\|_\infty \leq \|T_n(f) - f\|_\infty + \|f - S_n(f)\|_\infty.$$

First, since $\{S_n\}$ satisfies the conditions of Lemma 3.1, we immediately have

$$\|S_n(f) - f\|_\infty \overset{PDS^{\alpha}_{N_p}(P)}{\longrightarrow} 0.$$

Next, we examine $\{T_n\}$. Using the triangle inequality on the test functions

$$\|T_n(e_i) - e_i\|_\infty \leq \|T_n(e_i) - S_n(e_i)\|_\infty + \|S_n(e_i) - e_i\|_\infty.$$

By the hypotheses of the theorem, both terms on the right-hand side converge to 0 in the PDS sense for $i = 0, 1, 2$. Therefore,

$$\|T_n(e_i) - e_i\|_\infty \overset{PDS^{\alpha}_{N_p}(P)}{\longrightarrow} 0, \quad \text{for } i = 0, 1, 2.$$

Thus, by applying Lemma 3.1 to the sequence $\{T_n\}$, we obtain

$$\|T_n(f) - f\|_\infty \overset{PDS^{\alpha}_{N_p}(P)}{\longrightarrow} 0.$$

Finally, since both $\|T_n(f) - f\|_\infty$ and $\|S_n(f) - f\|_\infty$ converge to 0 in the deferred weighted statistical probability sense, their sum also converges to 0. This completes the proof. □

Theorem 3.2. *Let conditions of Theorem 3.1 hold. Additionally, assume that there exists a constant* $M > 0$ *such that* $\|T_n(e_0)\|_\infty \leq M$ *and* $\|S_n(e_0)\|_\infty \leq M$ *almost surely. Let* $f \in C[0,1]$ *and* $\omega(f, \delta)$ *be the modulus of continuity of* f*. Then, for any* $x \in [0,1]$ *and any* $\delta > 0$*,*

$$|T_n(f;x) - S_n(f;x)| \leq |f(x)| \cdot |T_n(e_0;x) - S_n(e_0;x)| + \omega(f,\delta)\left[T_n(e_0;x) + \frac{1}{\delta}\sqrt{T_n(e_0;x)\mu_{n,T}(x)}\right]$$

$$+ \omega(f,\delta)\left[S_n(e_0;x) + \frac{1}{\delta}\sqrt{S_n(e_0;x)\mu_{n,S}(x)}\right],$$

where $\mu_{n,T}(x) = T_n((t-x)^2; x)$ *and* $\mu_{n,S}(x) = S_n((t-x)^2; x)$*.*

Proof. Let $f \in C[0,1]$ and $x \in [0,1]$. We decompose the difference involving the operators as follows

$$T_n(f;x) - S_n(f;x) = T_n(f;x) - f(x)T_n(e_0;x) + f(x)(T_n(e_0;x) - S_n(e_0;x)) + f(x)S_n(e_0;x) - S_n(f;x).$$

Applying the triangle inequality, we obtain

$$|T_n(f;x) - S_n(f;x)| \leq |T_n(f;x) - f(x)T_n(e_0;x)| + |f(x)| \cdot |T_n(e_0;x) - S_n(e_0;x)| + |S_n(f;x) - f(x)S_n(e_0;x)|. \tag{2}$$

Using the property of the modulus of continuity, $|f(t) - f(x)| \leq \omega(f,\delta)(1 + \frac{|t-x|}{\delta})$, and the linearity and positivity of T_n, we have

$$\begin{aligned} |T_n(f;x) - f(x)T_n(e_0;x)| &\leq T_n(|f(t) - f(x)|; x) \\ &\leq \omega(f,\delta)\left(T_n(e_0;x) + \frac{1}{\delta}T_n(|t-x|;x)\right). \end{aligned}$$

Now, we apply the Cauchy-Schwarz inequality for positive linear operators. Specifically, since

$$T_n(|t-x|;x) = T_n(|t-x| \cdot 1; x),$$

we have

$$T_n(|t-x|;x) \leq \sqrt{T_n((t-x)^2;x) \cdot T_n(1^2;x)} = \sqrt{\mu_{n,T}(x) T_n(e_0;x)}.$$

Substituting this back into the inequality for T_n, we get

$$|T_n(f;x) - f(x)T_n(e_0;x)| \leq \omega(f,\delta) \left(T_n(e_0;x) + \frac{1}{\delta}\sqrt{T_n(e_0;x)\mu_{n,T}(x)} \right). \tag{3}$$

Following strictly similar steps for the operator S_n, we obtain

$$|S_n(f;x) - f(x)S_n(e_0;x)| \leq \omega(f,\delta) \left(S_n(e_0;x) + \frac{1}{\delta}\sqrt{S_n(e_0;x)\mu_{n,S}(x)} \right). \tag{4}$$

Finally, substituting the bounds (3) and (4) into (2) yields the stated inequality. □

We present a Voronovskaya-type theorem to define the asymptotic behavior of the approximation error. This theorem provides an asymptotic formula for the difference between the operators T_n and S_n.

Let $C^2[0,1]$ denote the space of functions $f \in C[0,1]$ such that the first and second derivatives f', f'' exist and are continuous on $[0,1]$.

Theorem 3.3. *Let the conditions of Theorem 3.1 hold, and let $f \in C^2[0,1]$. Assume that*

$$[P(D_n)]^\alpha \big(T_n(1;x) - S_n(1;x)\big) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} 0.$$

Define

$$\alpha_n(x) = T_n((t-x);x) - S_n((t-x);x), \qquad \beta_n(x) = T_n((t-x)^2;x) - S_n((t-x)^2;x).$$

Assume that there exist random variables $\Theta_1(x)$, $\Theta_2(x)$, $\Gamma_T(x)$, and $\Gamma_S(x)$ such that

$$[P(D_n)]^\alpha \alpha_n(x) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} \Theta_1(x),$$

$$[P(D_n)]^\alpha \beta_n(x) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} \Theta_2(x),$$

$$[P(D_n)]^\alpha T_n((t-x)^2;x) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} \Gamma_T(x),$$

and

$$[P(D_n)]^\alpha S_n((t-x)^2;x) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} \Gamma_S(x).$$

Furthermore, assume that

$$[P(D_n)]^\alpha T_n((t-x)^4;x) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} 0 \quad \textit{and} \quad [P(D_n)]^\alpha S_n((t-x)^4;x) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} 0. \tag{5}$$

Then

$$[P(D_n)]^\alpha \big(T_n(f;x) - S_n(f;x)\big) - \left(f'(x)\Theta_1(x) + \frac{1}{2} f''(x)\Theta_2(x) \right) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} 0.$$

Proof. Fix $x \in [0,1]$. Since $f \in C^2[0,1]$, Taylor's formula yields

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + \psi(t,x)(t-x)^2,$$

where

$$\psi(t,x) = \frac{f(t) - f(x) - f'(x)(t-x) - \frac{1}{2}f''(x)(t-x)^2}{(t-x)^2} \quad (t \neq x),$$

and $\psi(x,x) = 0$. The function $\psi(\cdot,x)$ is continuous on $[0,1]$ and satisfies $\psi(t,x) \to 0$ as $t \to x$.

Applying T_n and S_n to the above expansion with respect to t, subtracting the resulting identities, and multiplying by $[P(D_n)]^\alpha$, we obtain

$$[P(D_n)]^\alpha(T_n(f;x)-S_n(f;x)) = f(x)[P(D_n)]^\alpha(T_n(1;x)-S_n(1;x)) + f'(x)[P(D_n)]^\alpha\alpha_n(x) + \frac{1}{2}f''(x)[P(D_n)]^\alpha\beta_n(x)$$
$$+ [P(D_n)]^\alpha(R_n(T;x)-R_n(S;x)), \tag{6}$$

where

$$R_n(T;x) = T_n(\psi(t,x)(t-x)^2;x), \qquad R_n(S;x) = S_n(\psi(t,x)(t-x)^2;x).$$

By the assumptions, the first three terms on the right-hand side of (6) converge, in the deferred weighted statistical sense in probability, to

$$0, \qquad f'(x)\Theta_1(x), \qquad \frac{1}{2}f''(x)\Theta_2(x),$$

respectively. It remains to prove that

$$[P(D_n)]^\alpha R_n(T;x) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} 0 \quad \text{and} \quad [P(D_n)]^\alpha R_n(S;x) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} 0.$$

We first consider the term involving T_n. Let $\eta > 0$ and $\rho > 0$ be arbitrary. Since $\psi(t,x) \to 0$ as $t \to x$, for every $\sigma > 0$ there exists $h > 0$ such that

$$|\psi(t,x)| < \sigma \qquad \text{whenever } |t-x| < h.$$

Since $\psi(\cdot,x)$ is bounded on $[0,1]$, there exists a constant $K > 0$ such that $|\psi(t,x)| \le K$ for all $t \in [0,1]$. Hence, for every $t \in [0,1]$,

$$|\psi(t,x)(t-x)^2| \le \sigma(t-x)^2 + \frac{K}{h^2}(t-x)^4.$$

Applying the positivity of T_n, we obtain

$$|R_n(T;x)| \le \sigma\, T_n((t-x)^2;x) + \frac{K}{h^2}T_n((t-x)^4;x).$$

Thus,

$$[P(D_n)]^\alpha|R_n(T;x)| \le \sigma B_n^T(x) + \frac{K}{h^2}C_n^T(x), \tag{7}$$

where

$$B_n^T(x) := [P(D_n)]^\alpha T_n((t-x)^2;x), \qquad C_n^T(x) := [P(D_n)]^\alpha T_n((t-x)^4;x).$$

Since $B_n^T(x) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} \Gamma_T(x)$, the sequence $\{B_n^T(x)\}$ is deferred weighted statistically bounded in probability. Indeed, choose $M > 0$ such that

$$P(|\Gamma_T(x)| \ge M) < \frac{\rho}{4}.$$

Then, by the assumed convergence of $B_n^T(x)$ to $\Gamma_T(x)$,

$$\frac{1}{[P(D_n)]^\alpha}\left|\left\{k \in D_n : P\big(|B_k^T(x) - \Gamma_T(x)| \ge M\big) \ge \frac{\rho}{4}\right\}\right|_p \to 0.$$

For each k outside the exceptional set above, we have

$$P\big(B_k^T(x) \ge 2M\big) \le P\big(|B_k^T(x) - \Gamma_T(x)| \ge M\big) + P\big(|\Gamma_T(x)| \ge M\big) < \frac{\rho}{2}.$$

Now choose $\sigma > 0$ so small that

$$2M\sigma < \frac{\eta}{2}.$$

Since $C_n^T(x) \overset{PDS^\alpha_{N_p}(P)}{\longrightarrow} 0$, we also have

$$\frac{1}{[P(D_n)]^\alpha}\left|\left\{k \in D_n : P\left(\frac{K}{h^2}C_k^T(x) \ge \frac{\eta}{2}\right) \ge \frac{\rho}{2}\right\}\right|_p \to 0.$$

Therefore, outside a set of deferred weighted density of order α equal to zero, we have simultaneously

$$P\big(B_k^T(x) \ge 2M\big) < \frac{\rho}{2} \quad \text{and} \quad P\left(\frac{K}{h^2}C_k^T(x) \ge \frac{\eta}{2}\right) < \frac{\rho}{2}.$$

For such k, by (7),

$$P\left([P(D_k)]^{\alpha}|R_k(T;x)|\geq\eta\right)\leq P\left(\sigma B_k^T(x)\geq\frac{\eta}{2}\right)+P\left(\frac{K}{h^2}C_k^T(x)\geq\frac{\eta}{2}\right)$$

$$[3mm]\leq P\big(B_k^T(x)\geq 2M\big)+P\left(\frac{K}{h^2}C_k^T(x)\geq\frac{\eta}{2}\right)<\rho.$$

This proves that

$$[P(D_n)]^{\alpha}R_n(T;x)\overset{PDS^{\alpha}_{N_p}(P)}{\longrightarrow}0.$$

Exactly the same argument, with

$$B_n^S(x):=[P(D_n)]^{\alpha}S_n((t-x)^2;x),\qquad C_n^S(x):=[P(D_n)]^{\alpha}S_n((t-x)^4;x),$$

shows that

$$[P(D_n)]^{\alpha}R_n(S;x)\overset{PDS^{\alpha}_{N_p}(P)}{\longrightarrow}0.$$

Substituting these conclusions into (6), we obtain

$$[P(D_n)]^{\alpha}\big(T_n(f;x)-S_n(f;x)\big)-\left(f'(x)\Theta_1(x)+\frac{1}{2}f''(x)\Theta_2(x)\right)\overset{PDS^{\alpha}_{N_p}(P)}{\longrightarrow}0.$$

This completes the proof. □

Remark 3.1. *Theorem 3.1 applies to the whole space $C[0,1]$, whereas Theorem 3.3 is restricted to $C^2[0,1]$ because of the differentiability requirements of the Taylor expansion. The latter theorem gives a more precise asymptotic description of the approximation error. In particular, if $\Theta_1(x)=0$ and $\Theta_2(x)=0$ almost surely, then the difference $T_n(f;x)-S_n(f;x)$ is of smaller order than $[P(D_n)]^{-\alpha}$ in the deferred weighted statistical sense in probability.*

4. Conclusion and Future Work

In this paper, we introduced and studied the notion of asymptotically deferred weighted statistical equivalence of order α in probability. By combining deferred averaging, weighting, and probabilistic asymptotic behavior, we obtained a framework that extends several existing approaches in the literature. We established the basic inclusion relations between the statistical and strong forms of equivalence and illustrated the strictness of these relations by means of a counterexample. We then applied the theory to approximation processes by establishing a Korovkin-type theorem for sequences of positive linear operators. In addition, we derived a Voronovskaya-type result describing the asymptotic behavior of the approximation error and obtained an estimate for the rate of convergence in terms of the modulus of continuity.

Possible directions for future research include the study of analogous notions in ideal or lacunary settings, extensions to different classes of function spaces such as weighted spaces or L_p spaces, and multidimensional versions for functions of several variables. It would also be of interest to investigate concrete approximation processes under the present mode of convergence.

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