Complete Asymptotic Expansions for the Double Harmonic Sum and Related Constants

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Abstract

For $n \in \mathbb{N} := \{1, 2, \ldots\}$, let $L_n := \mathscr{H}_n - (2 \ln 2)n + \ln n$, where

$$\mathscr{H}_n := \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{i+j}.$$

It is known that $\lim_{n \to \infty} L_n = -2^{-1} - \gamma + \ln 2$, where γ denotes the Euler-Mascheroni constant. In this paper, we establish complete asymptotic expansions for the sequences $\mathscr{H}_n$ and L_n.

Keywords: Euler-Mascheroni constant; gamma function; psi function; asymptotic expansion.

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1. Introduction

The Euler-Mascheroni constant $\gamma = 0.577215664\ldots$ is defined as the limit of the sequence

$$D_n = H_n - \ln n, \tag{1}$$

where H_n denotes the nth harmonic number, defined for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{2}$$

The Euler-Mascheroni constant γ plays a significant role in many areas of mathematical analysis, including the theory of special functions in applied mathematics and mathematical physics. It also appears frequently in number theory, particularly in the study of the growth and order of magnitude of arithmetical functions (see [11]). Choi [6] provided a summary of several known representations of the Euler-Mascheroni constant γ. Mortici [15] introduced a new family of sequences that converge more rapidly to the Euler–Mascheroni constant. An extensive and impressive collection of integral representations of various types can be found in a recent paper by Choi and Srivastava [7]. Further information on γ is available in the survey article [9] as well as in the monographs [10, 12, 18].

The constant γ is closely related to the celebrated gamma function $\Gamma(z)$ by means of the familiar Weierstrass formula [1, p. 255, Equation (6.1.3)] (see also [17, Chapter 1, Section 1.1]):

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n} \right) e^{-z/n} \right] \quad (|z| < \infty).$$

Mortici [16] extended and refined an approximation formula for the gamma function due to Ramanujan. The logarithmic derivative of the gamma function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

is known as the psi (or digamma) function.

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The psi function has the following asymptotic expansion (see [14, p. 33]):

$$\psi(z) \sim \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k} z^{-k} \qquad (z \to \infty;\ |\arg z| < \pi) \tag{3}$$

and

$$\psi\left(z + \frac{1}{2}\right) \sim \ln z - \sum_{n=1}^{\infty} \frac{B_{2n}\left(\frac{1}{2}\right)}{2nz^{2n}} \qquad (z \to \infty;\ |\arg z| < \pi), \tag{4}$$

where Bernoulli polynomials $B_n(t)$ are defined by the generating function

$$\frac{ze^{tz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{z^n}{n!}, \tag{5}$$

and the Bernoulli numbers B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are defined by $B_n := B_n(0)$ in (5).

Lugo [13] considered the sequence $(L_n)_{n \in \mathbb{N}}$, which is essentially an interesting analogue of the sequence $(D_n)_{n \in \mathbb{N}}$ occurring in (1), defined by

$$L_n := \mathscr{H}_n - (2\ln 2)n + \ln n, \tag{6}$$

where

$$\mathscr{H}_n := \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{i+j}. \tag{7}$$

Remark 1.1. *Grouping the terms according to $k = i + j$, we observe that for each fixed k: there are $k - 1$ pairs when $k = 2, \ldots, n + 1$; there are $2n - k + 1$ pairs when $k = n + 2, \ldots, 2n$. These two ranges are also evident by arranging $\mathscr{H}_n := \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{i+j}$ in matrix form. Hence*

$$\mathscr{H}_n = \sum_{k=2}^{n+1} \frac{k-1}{k} + \sum_{k=n+2}^{2n} \frac{2n-k+1}{k} = 2 + (2n+1)H_{2n} - (2n+2)H_{n+1}.$$

Using the standard asymptotic formula for harmonic numbers,

$$H_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right), \quad n \to \infty,$$

where γ denotes the Euler-Mascheroni constant, we obtain

$$\mathscr{H}_n = 2n\ln 2 - \ln n + \ln 2 - \gamma - \frac{1}{2} + O(n^{-1}). \tag{8}$$

Moreover, employing the full asymptotic expansion

$$H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}} \quad (n \to \infty),$$

one can derive a more complete asymptotic expansion for H_n than the approximation given in (8).

Lugo [13] proved the following asymptotic formula:

$$L_n \sim -\frac{1}{2} - \gamma + \ln 2 - \frac{5}{8n} + \frac{7}{48n^2} + O(n^{-3}), \qquad n \to \infty. \tag{9}$$

Clearly, from (9), it follows that

$$L := \lim_{n \to \infty} L_n = -\frac{1}{2} - \gamma + \ln 2. \tag{10}$$

Recently, Chen and Srivastava [4] proved that

$$\gamma = -\mathscr{H}_n + \ln 2 - 1 + \left(n + \frac{1}{2}\right)\psi\left(n + \frac{1}{2}\right) - \left(n + \frac{3}{2}\right)\psi(n) + (2\ln 2)n - \frac{3}{2n}, \tag{11}$$

which provides new analytical representations for the Euler-Mascheroni constant γ in terms of the psi (or digamma) function $\psi(z)$. Also in [4], the authors obtained the following approximation formula:

$$\mathscr{H}_n \sim -\frac{1}{2} - \gamma + \ln 2 + (2\ln 2)n - \ln\left(n + \frac{5}{8} + \frac{19}{384n} - \frac{107}{3072n^2}\right) + O(n^{-4}) \tag{12}$$

as $n \to \infty$. Chen and Srivastava [5] derived the following asymptotic formula for L_n:

$$L_n \sim L - \frac{5}{8n} + \frac{7}{48n^2} - \frac{1}{64n^3} - \frac{31}{1920n^4} + \frac{1}{128n^5} + \frac{127}{16128n^6} - \cdots, \qquad n \to \infty. \tag{13}$$

However, the authurs did not give the general formula for the coefficient of $\frac{1}{n^j}$ ($j \in \mathbb{N}$) in (13). Some inequalities for $L - L_n$ were established in [2, 4].

In this paper, we develop the approximation formula (12) to produce a complete asymptotic expansion (Theorems 3.1 and 3.2), and we derive a formula for determining the coefficients of the expansion (13) (Remark 3.2). Also, we present new asymptotic expansion for $\mathscr{H}_n$ (Theorem 3.3).

2. Lemmas

The following lemmas are required in the sequel.

Lemma 2.1. *For* $n \in \mathbb{N}$*, let*

$$F(n) = \frac{3}{2n} + \frac{1}{2} - \left(n + \frac{1}{2}\right)\psi\left(n + \frac{1}{2}\right) + \left(n + \frac{3}{2}\right)\psi(n). \tag{14}$$

Then, it holds that

$$F(n) \sim \ln n + \sum_{j=1}^{\infty} \frac{a_j}{n^j}, \qquad n \to \infty \tag{15}$$

with the coefficients a_j *given by*

$$a_1 = \frac{5}{8}, \quad a_j = \frac{(-1)^j - 1 + 2^{-j}}{j+1}B_{j+1} - \frac{3(-1)^j + 1 - 2^{1-j}}{2j}B_j, \quad j \geq 2, \tag{16}$$

where B_n *are the Bernoulli numbers. Namely,*

$$F(n) \sim \ln n + \frac{5}{8n} - \frac{7}{48n^2} + \frac{1}{64n^3} + \frac{31}{1920n^4} - \frac{1}{128n^5} - \frac{127}{16128n^6} + \cdots, \quad n \to \infty. \tag{17}$$

Proof. Using (3), we find that

$$\left(n + \frac{3}{2}\right)\psi(n) \sim \left(n + \frac{3}{2}\right)\ln n - \frac{1}{2} + \sum_{j=1}^{\infty}(-1)^j\left(\frac{B_{j+1}}{j+1} - \frac{3B_j}{2j}\right)\frac{1}{n^j}. \tag{18}$$

Noting that

$$B_n(\tfrac{1}{2}) = -(1 - 2^{1-n})B_n, \qquad n \in \mathbb{N}_0$$

and $B_{2n+1} = 0$, $n \in \mathbb{N}$ (see [1, p. 805]), we can write (4) as

$$\psi\left(z + \frac{1}{2}\right) \sim \ln z + \sum_{k=2}^{\infty} \frac{(1 - 2^{1-k})B_k}{k}\frac{1}{z^k}. \tag{19}$$

Using (19), we obtain

$$\left(n + \frac{1}{2}\right)\psi\left(n + \frac{1}{2}\right) \sim \left(n + \frac{1}{2}\right)\ln n + \sum_{j=1}^{\infty}\left(\frac{(1 - 2^{-j})B_{j+1}}{j+1} + \frac{(1 - 2^{1-j})B_j}{2j}\right)\frac{1}{n^j}. \tag{20}$$

Substituting (18) and (20) into (14) yields

$$F(n) \sim \ln n + \frac{3}{2n} + \sum_{j=1}^{\infty} \left[\frac{(-1)^j - 1 + 2^{-j}}{j+1} B_{j+1} - \frac{3(-1)^j + 1 - 2^{1-j}}{2j} B_j \right] \frac{1}{n^j}$$

$$= \ln n + \frac{5}{8n} + \sum_{j=2}^{\infty} \left[\frac{(-1)^j - 1 + 2^{-j}}{j+1} B_{j+1} - \frac{3(-1)^j + 1 - 2^{1-j}}{2j} B_j \right] \frac{1}{n^j}.$$

The proof of Lemma 2.1 is complete. □

Lemma 2.2 (see [8]). *The following relations hold:*

$$\exp\left(\sum_{j=1}^{\infty} \frac{p_j}{x^j} \right) = 1 + \sum_{j=1}^{\infty} \frac{q_j}{x^j}, \qquad x \to \infty,$$

$$\ln\left(1 + \sum_{j=1}^{\infty} \frac{q_j}{x^j} \right) = \sum_{j=1}^{\infty} \frac{p_j}{x^j}, \qquad x \to \infty,$$

where the coefficients p_j and q_j have the following relations:

$$q_j = \sum_{k_1 + 2k_2 + \cdots + jk_j = j} \frac{p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}}{k_1! k_2! \cdots k_j!},$$

summed over all nonnegative integers k_j satisfying the equation

$$k_1 + 2k_2 + \cdots + jk_j = j,$$

and

$$p_j = \sum_{\substack{k_1 + 2k_2 + \cdots + jk_j = j \\ k_1 + k_2 + \cdots + k_j = k \\ 1 \le k \le j}} (-1)^{k-1} (k-1)! \frac{q_1^{k_1} q_2^{k_2} \cdots q_j^{k_j}}{k_1! k_2! \cdots k_j!},$$

where the summation is over all nonnegative integral solutions $(k_1, k_2, \ldots, k_j)$ of the equations

$$k_1 + 2k_2 + \cdots + jk_j = j, \quad k_1 + k_2 + \cdots + k_j = k, \quad k = 1, 2, \ldots, j.$$

Lemma 2.3 (see [3]). *Let $A(x)$ be a function with asymptotic expansion*

$$A(x) \sim \sum_{n=1}^{\infty} a_n x^{-n}, \qquad x \to \infty.$$

Then, the composition $\exp(A(x))$ has an asymptotic expansion of the form

$$\exp(A(x)) \sim \sum_{n=0}^{\infty} b_n x^{-n}, \qquad x \to \infty,$$

where

$$b_0 = 1 \quad and \quad b_n = \frac{1}{n} \sum_{k=1}^{n} k a_k b_{n-k}, \qquad n \ge 1.$$

Lemma 2.4 (see [3])**.** *Let* $q_0 = 1$ *and let*

$$Q(x) \sim \sum_{j=0}^{\infty} q_j x^{-j}, \qquad x \to \infty, \tag{21}$$

be an asymptotic expansion. Then, the following hold:

(i). *The composition* $R(x) = \ln(Q(x))$ *has asymptotic expansion of the form*

$$R(x) \sim \sum_{j=1}^{\infty} r_j x^{-j}, \qquad x \to \infty,$$

where

$$r_j = q_j - \frac{1}{j} \sum_{k=1}^{j-1} k r_k q_{j-k}, \qquad j \in \mathbb{N}. \tag{22}$$

(ii). *For every real number* r*, it holds that*

$$[Q(x)]^r \sim \sum_{j=0}^{\infty} P_j(r) x^{-j}, \qquad x \to \infty,$$

where

$$P_0(r) = 1, \quad P_j(r) = \frac{1}{j} \sum_{k=1}^{j} [k(1+r) - j] q_k P_{j-k}(r), \qquad j \in \mathbb{N}. \tag{23}$$

3. Main Results

The following result develops the approximation formula (12) to produce a complete asymptotic expansion.

Theorem 3.1. *The sequence* $\mathscr{H}_n$ *has the asymptotic expansion*

$$\mathscr{H}_n \sim -\frac{1}{2} - \gamma + \ln 2 + (2 \ln 2)n - \ln\left(n\left(1 + \sum_{j=1}^{\infty} \frac{b_j}{n^j}\right)\right), \qquad n \to \infty, \tag{24}$$

with the coefficients b_j *given by*

$$b_j = \sum_{k_1 + 2k_2 + \cdots + jk_j = j} \frac{a_1^{k_1} a_2^{k_2} \cdots a_j^{k_j}}{k_1! k_2! \cdots k_j!}, \qquad j \in \mathbb{N}, \tag{25}$$

where a_j *are given in* (16)*, summed over all nonnegative integers* k_j *satisfying the following equation:*

$$k_1 + 2k_2 + \cdots + jk_j = j.$$

Proof. The formula (11) is equivalent to the following result [4]:

$$-\mathscr{H}_n + \left(-\frac{1}{2} - \gamma + \ln 2\right) + (2 \ln 2)n = F(n), \tag{26}$$

where $F(n)$ is given in (14). In view of (12), we can let

$$\mathscr{H}_n \sim -\frac{1}{2} - \gamma + \ln 2 + (2 \ln 2)n - \ln\left(n\left(1 + \sum_{j=1}^{\infty} \frac{b_j}{n^j}\right)\right), \qquad n \to \infty,$$

where b_j ($j \in \mathbb{N}$) are real numbers to be determined.

Thus, we obtain

$$F(n) \sim \ln\left(n\left(1 + \sum_{j=1}^{\infty} \frac{b_j}{n^j}\right)\right), \qquad n \to \infty,$$

which implies, by (15), that

$$\ln\left(1 + \sum_{j=1}^{\infty} \frac{b_j}{n^j}\right) \sim \sum_{j=1}^{\infty} \frac{a_j}{n^j}, \qquad n \to \infty, \tag{27}$$

where a_j are given in (16). By Lemma 2.2, we obtain

$$b_j = \sum_{k_1 + 2k_2 + \cdots + jk_j = j} \frac{a_1^{k_1} a_2^{k_2} \cdots a_j^{k_j}}{k_1! k_2! \cdots k_j!},$$

summed over all nonnegative integers k_j satisfying the equation

$$k_1 + 2k_2 + \cdots + jk_j = j.$$

This completes the proof of Theorem 3.1. $\qquad\square$

Using Lemma 2.3, we can provide an alternative representation formula to calculate the coefficients b_j ($j \in \mathbb{N}$) in (24) as in Theorem 3.1. The representation using a recursive algorithm for the coefficients b_j in (29) is more practical for numerical evaluation than the expression in (25).

Theorem 3.2. *The sequence $\mathscr{H}_n$ has the asymptotic expansion*

$$\mathscr{H}_n \sim -\frac{1}{2} - \gamma + \ln 2 + (2\ln 2)n - \ln\left(n\left(\sum_{j=0}^{\infty} \frac{b_j}{n^j}\right)\right), \qquad n \to \infty, \tag{28}$$

with the coefficients b_j ($j \in \mathbb{N}_0$) given by

$$b_0 = 1 \quad \text{and} \quad b_j = \frac{1}{j} \sum_{k=1}^{j} k a_k b_{j-k}, \qquad j \geq 1, \tag{29}$$

where a_j are given in (16).

Proof. We write (27) as

$$\exp\left(\sum_{j=1}^{\infty} \frac{a_j}{n^j}\right) \sim \sum_{j=0}^{\infty} \frac{b_j}{n^j},$$

where a_j are given in (16). By Lemma 2.3, we have

$$b_0 = 1 \quad \text{and} \quad b_j = \frac{1}{j} \sum_{k=1}^{j} k a_k b_{j-k}, \qquad j \geq 1.$$

The proof of Theorem 3.2 is complete. $\qquad\square$

Remark 3.1. *The representation using the recursive algorithm is better for numerical evaluations. The first few coefficients b_j are given as follows:*

$$b_0 = 1,$$
$$b_1 = a_1 b_0 = \frac{5}{8},$$
$$b_2 = \frac{1}{2} a_1 b_1 + a_2 b_0 = \frac{1}{2} \cdot \frac{5}{8} \cdot \frac{5}{8} + \left(-\frac{7}{48}\right) \cdot 1 = \frac{19}{384},$$
$$b_3 = \frac{1}{3} a_1 b_2 + \frac{2}{3} a_2 b_1 + a_3 b_0 = \frac{1}{3} \cdot \frac{5}{8} \cdot \frac{19}{384} + \frac{2}{3} \cdot \left(-\frac{7}{48}\right) \cdot \frac{5}{8} + \frac{1}{64} \cdot 1 = -\frac{107}{3072}.$$

Continuing this procedure, we find that

$$b_4 = \frac{21263}{1474560}, \quad b_5 = \frac{1195}{262144}, \quad b_6 = -\frac{141610033}{11890851840}.$$

Hence, we obtain the following explicit asymptotic expansion:

$$\mathscr{H}_n \sim -\frac{1}{2} - \gamma + \ln 2 + (2\ln 2)n - \ln\left(n + \frac{5}{8} + \frac{19}{384n} - \frac{107}{3072n^2} + \frac{21263}{1474560n^3} + \frac{1195}{262144n^4} - \cdots\right). \tag{30}$$

Remark 3.2. *It follows from* (24) *that*

$$L_n \sim L - \ln\left(1 + \sum_{j=1}^{\infty} \frac{b_j}{n^j}\right). \tag{31}$$

Using (27), *we have*

$$L_n \sim L - \sum_{j=1}^{\infty} \frac{a_j}{n^j}, \tag{32}$$

with the coefficients a_j given in (16). *That is,*

$$L_n \sim L - \frac{5}{8n} - \sum_{j=2}^{\infty}\left[\frac{(-1)^j - 1 + 2^{-j}}{j+1}B_{j+1} - \frac{3(-1)^j + 1 - 2^{1-j}}{2j}B_j\right]\frac{1}{n^j}, \tag{33}$$

which is the expansion (13).

From (32), *it follows that*

$$\mathscr{H}_n \sim -\frac{1}{2} - \gamma + \ln 2 + (2\ln 2)n - \ln n - \sum_{j=1}^{\infty} \frac{a_j}{n^j}, \tag{34}$$

with the coefficients a_j given in (16).

The following result presents a new asymptotic expansion for $\mathscr{H}_n$.

Theorem 3.3. *The sequence $\mathscr{H}_n$ has the asymptotic expansion*

$$\mathscr{H}_n \sim -\frac{1}{2} - \gamma + \ln 2 + (2\ln 2)n - \psi\left(\sum_{j=0}^{\infty} q_j n^{-j+1}\right), \qquad n \to \infty \tag{35}$$

with the coefficients q_j $(j \in \mathbb{N}_0)$ given by

$$q_0 = 1, \quad q_j = \frac{1}{j}\sum_{k=1}^{j-1} kr_k q_{j-k} + \sum_{k=1}^{j} \frac{(-1)^k B_k}{k}P_{j-k}(-k) + a_j. \tag{36}$$

Here, B_n are Bernoulli numbers, a_j are given in (16), *r_j can be calculated using* (22), *and $P_j(r)$ can be calculated using* (23).

Proof. We can let

$$-\frac{1}{2} - \gamma + \ln 2 + (2\ln 2)n - \mathscr{H}_n \sim \psi\left(n\sum_{j=0}^{\infty} q_j n^{-j}\right), \qquad n \to \infty,$$

where q_j $(j \in \mathbb{N}_0)$ are real numbers to be determined. Using (3), we obtain

$$\psi\left(n\sum_{j=0}^{\infty} q_j n^{-j}\right) \sim \ln n + \ln\left(\sum_{j=0}^{\infty} q_j n^{-j}\right) - \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k}\left(n\sum_{j=0}^{\infty} q_j n^{-j}\right)^{-k}.$$

On the other hand, it follows from (34) that, as $n \to \infty$,

$$-\frac{1}{2} - \gamma + \ln 2 + (2\ln 2)n - \mathscr{H}_n \sim \ln n + \sum_{j=1}^{\infty} a_j n^{-j},$$

where a_j are given in (16).

Hence, we have

$$\ln\left(\sum_{j=0}^{\infty} q_j n^{-j}\right) - \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k}\left(n\sum_{j=0}^{\infty} q_j n^{-j}\right)^{-k} \sim \sum_{j=1}^{\infty} a_j n^{-j}. \tag{37}$$

Extracting the coefficients of the power n^0, it follows from here that $\ln q_0 = 0$, and hence $q_0 = 1$. Using Lemma 2.4, the left side of (37) can be written as

$$\sum_{k=1}^{\infty} r_k n^{-k} - \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k} n^{-k} \sum_{j=0}^{\infty} P_j(-k) n^{-j} = \sum_{j=1}^{\infty}\left(r_j - \sum_{k=1}^{j} \frac{B_k(1)}{k} P_{j-k}(-k)\right) n^{-j}. \tag{38}$$

The coefficient q_j which should be determined from here is hidden in the calculation of r_j. Using (22), we can write

$$r_j = q_j - \frac{1}{j}\sum_{k=1}^{j-1} k r_k q_{j-k}. \tag{39}$$

Linking together (37), (38), and (39) immediately follows (36), which proves Theorem 3.3. $\qquad\square$

Remark 3.3. *Here, we provide explicit numerical values for the first few terms of q_j by using* (36). *This shows how easily we can determine the coefficients q_j in* (35). *By using* (22) *and* (23), *we obtain*

$$r_1 = q_1, \quad r_2 = q_2 - \frac{1}{2} q_1^2$$

and

$$P_0(-1) = 1, \quad P_1(-1) = -q_1, \quad P_2(-1) = -q_2 + q_1^2, \quad P_0(-2) = 1, \quad P_1(-2) = -2q_1,$$

respectively. Using (36), *we have*

$$q_1 = -B_1 P_0(-1) + a_1 = \frac{1}{2} + \frac{5}{8} = \frac{9}{8},$$

$$q_2 = \frac{1}{2} r_1 q_1 - B_1 P_1(-1) + \frac{1}{2} B_2 P_0(-2) + a_2$$

$$= \frac{1}{2}\cdot\frac{9}{8}\cdot\frac{9}{8} - \left(-\frac{1}{2}\right)\cdot\left(-\frac{9}{8}\right) + \frac{1}{2}\cdot\frac{1}{6}\cdot 1 + \left(-\frac{7}{48}\right) = \frac{1}{128},$$

$$q_3 = \frac{1}{3}\left(r_1 q_2 + 2 r_2 q_1\right) - B_1 P_2(-1) + \frac{1}{2} B_2 P_1(-2) + a_3$$

$$= \frac{1}{3}\left[\frac{9}{8}\cdot\frac{1}{128} + 2\left(\frac{1}{128} - \frac{1}{2}\cdot\left(\frac{9}{8}\right)^2\right)\cdot\frac{9}{8}\right] - \left(-\frac{1}{2}\right)\left(-\frac{1}{128} + \left(\frac{9}{8}\right)^2\right) + \frac{1}{2}\cdot\frac{1}{6}\cdot\left((-2)\cdot\frac{9}{8}\right) + \frac{1}{64} = -\frac{9}{1024}.$$

Likewise, using (36), *we obtain*

$$q_4 = \frac{481}{98304} \quad and \quad q_5 = \frac{1505}{786432}.$$

Consequently, we obtain the following explicit asymptotic expansions:

$$\mathscr{H}_n \sim -\frac{1}{2} - \gamma + \ln 2 + (2\ln 2)n - \psi\left(n + \frac{9}{8} + \frac{1}{128n} - \frac{9}{1024n^2} + \frac{481}{98304n^3} + \cdots\right), \quad n \to \infty.$$

Remark 3.4. *The following approximation formulas hold:*

$$\mathscr{H}_n \sim -\frac{1}{2} - \gamma + \ln 2 + (2\ln 2)n - \ln n - \frac{5}{8n} + \frac{7}{48n^2} - \frac{1}{64n^3} = \lambda_n, \tag{40}$$

$$\mathscr{H}_n \sim -\frac{1}{2} - \gamma + \ln 2 + (2\ln 2)n - \ln\left(n + \frac{5}{8} + \frac{19}{384n} - \frac{107}{3072n^2}\right) = \mu_n \tag{41}$$

and

$$\mathscr{H}_n \sim -\frac{1}{2} - \gamma + \ln 2 + (2\ln 2)n - \psi\left(n + \frac{9}{8} + \frac{1}{128n} - \frac{9}{1024n^2}\right) = \nu_n. \tag{42}$$

Moreover, we have

$$\mathscr{H}_n = \lambda_n + O(n^{-4}), \quad \mathscr{H}_n = \mu_n + O(n^{-4}), \quad \textit{and} \quad \mathscr{H}_n = \nu_n + O(n^{-4}).$$

The numerical computations presented in Table 1 indicate that, for $n \in \mathbb{N}$, the formula (42) *provides the best approximation among the formulas* (40)−(42).

Table 1. Comparison of approximation formulas (40)−(42).

n	$|\mathscr{H}_n - \lambda_n|$	$|\mathscr{H}_n - \mu_n|$	$|\mathscr{H}_n - \nu_n|$
1	7.434×10^{-3}	7.744×10^{-3}	2.413×10^{-3}
10	1.529×10^{-6}	1.388×10^{-6}	4.728×10^{-7}
100	1.606×10^{-10}	1.437×10^{-10}	4.88×10^{-11}
500	2.58×10^{-13}	2.305×10^{-13}	7.825×10^{-14}

Acknowledgment

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