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Research Article

On the Exponential Stabilization of Vibrations in Inhomogeneous Viscoelastic Materials With Heat Conduction

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Abstract

This article deals with the vibrations of a flexible structure incorporating Kelvin–Voigt-type viscoelasticity combined with heat conduction governed by a simple adaptation of Cattaneo's law. The well-posedness of the system is proved using semigroup techniques. The system achieves uniform stabilization via an explicit exponential energy decay estimate, which is constructed using an appropriate Lyapunov functional.

Keywords: Cattaneo's law; exponential stability; inhomogeneous flexible structure; existence of solutions.

2020 Mathematics Subject Classification: 35A02, 35B35, 35B40, 74K10.

1. Introduction

The one-dimensional wave equation is given by

$$m(x)u_{tt} - \big(p(x)u_x(x,t)\big)_x = 0 \quad \text{on} \quad (0,L)\times(0,\infty), \tag{1}$$

which describes the vibrations of an inhomogeneous flexible structure, where $u(x,t)$ is the displacement of a particle at position $(x,t) \in (0,L)\times(0,\infty)$, $m(x)$ denotes the mass density, and $p(x)$ represents the stiffness coefficient at a distance x from the left. To study the theoretical principles governing stabilization in flexible structural systems and to control their vibrations, one common approach to achieving energy dissipation is the incorporation of damping forces. Various types of damping mechanisms have been discussed in the literature, including boundary damping, distributed damping, internal damping, and localized damping (see [4, 15, 17, 18]).

The exponential decay of the problem associated with the equation

$$m(x)u_{tt} - \big(p(x)u_x + 2\delta(x)u_{xt}\big)_x = 0 \quad \text{on} \quad (0,L)\times(0,\infty), \tag{2}$$

where $\delta(x) > 0$ for all $x \in [0,L]$ is the coefficient of internal viscoelastic damping, known as Kelvin–Voigt damping (see [9]), has been studied extensively. To incorporate thermoelastic effects into the above model, it is coupled with a thermal diffusion mechanism. The temperature difference $\theta(x,t)$ over a fixed temperature satisfies the one-dimensional Fourier law [6]:

$$\theta_t(x,t) + \kappa q_x(x,t) = 0, \qquad q(x,t) + \kappa\theta_x(x,t) = 0, \quad \text{on} \quad (0,L)\times(0,\infty), \tag{3}$$

where $\kappa > 0$ is a coupling constant and $q = q(x,t)$ denotes the heat flux.

The thermo-diffusion model in (3) does not fully agree with physical reality. To address this issue, the Cattaneo–Vernotte law provides a modified version of the second equation in (3), namely

$$q(x,t) + \tau q_t(x,t) + \kappa\theta_x(x,t) = 0, \tag{4}$$

which is obtained by replacing $q(x,t)$ with $q(x,t+\tau)$ and retaining only the first-order term in τ in the Taylor expansion around t, where $\tau > 0$ represents the time lag in the response of the heat flux. Applying a similar modification to the first equation in (3) yields

$$\theta_t(x,t) + \kappa q_x(x,t) + \kappa\tau q_{xt}(x,t) = 0. \tag{5}$$

The modified system (4)–(5) therefore provides a more realistic model. In particular, we emphasize that both equations in Fourier's law (3) are consistently modified by replacing $q(x,t)$ with $q(x,t+\tau)$; the first equation is not neglected.

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Hence, we are concerned with the stability analysis of the following system of equations:

$$m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \alpha\theta_x = 0, \tag{6}$$

$$\theta_t + \kappa q_x + \kappa\tau q_{xt} + \alpha u_{xt} = 0, \tag{7}$$

$$\tau q_t + q + \kappa\theta_x = 0, \tag{8}$$

with the initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x),\ q(x,0) = q_0(x) \quad \forall\ x\ \in [0,L], \tag{9}$$

and the set of boundary conditions

$$u(0,t) = u(L,t) = 0 \quad \text{and} \quad \theta(0,t) = \theta(L,t) = 0,\ \forall\ t\ \geq 0, \tag{10}$$

or

$$u(0,t) = u(L,t) = 0 \quad \text{and} \quad q(0,t) = q(L,t) = 0,\ \forall\ t\ \geq 0. \tag{11}$$

The first set corresponds to a rigidly clamped structure with temperature held constant at both extremities, whereas the second corresponds to a rigidly clamped structure with zero heat flux on the boundary. In the sequel, we consider only the boundary conditions (10). Similar results are expected to hold for the boundary conditions (11).

The concept of stabilization encompasses various notions of stability, including bounded-input bounded-output stability [25], exponential stability [8, 14], and asymptotic stability [2, 21]. Several authors [5, 7, 12] investigated the stabilization of the wave equation in a bounded domain and obtained both asymptotic and uniform stability results. The asymptotic behavior of coupled systems describing elastic dynamics and Fourier's law of heat conduction has been analyzed in [11, 19]. Based on Cattaneo's law [3], the stability of thermoelastic systems was studied in [20, 23]. Houasni et al. [10] demonstrated the existence, uniqueness, and exponential stability of solutions for a flexible structure incorporating second sound and past history effects. The work in [16] considered a flexible system with second sound and time delay.

The rest of the paper is organized as follows. In Section 2, we study the well-posedness of the system using semigroup theory. In Section 3, we define the total energy of the system and demonstrate its dissipative behavior due to damping. Section 4 establishes the exponential stability of the system using a Lyapunov functional approach. Finally, Section 5 provides a concluding overview of the results and their potential applications.

2. Well-posedness

Using Sobolev spaces (see [1]) and semigroup theory [24] in the space $L^2(0,L)$, we define the inner product and norm as

$$\langle u, v\rangle_{L^2(0,L)} = \int_0^L u\bar{v}dx \tag{12}$$

and

$$\|u\|^2_{L^2(0,L)} = \int_0^L |u|^2 dx = \int_0^L u^2 dx. \tag{13}$$

Denoting $v = u_t$ and $U(t) = (u, v, \theta, q)^T$, the system (6)–(10) can be written as the following abstract Cauchy problem:

$$U'(t) = \mathcal{A}U(t), \quad U(0) = U_0 = (u_0, v_0, \theta_0, q_0)^T \quad \text{for all} \quad t > 0, \tag{14}$$

where the differential operator $\mathcal{A}$ is defined by

$$\mathcal{A}U = \begin{bmatrix} v \\ \dfrac{1}{m(x)}[(p(x)u_x + 2\delta(x)v_x)_x - \alpha\theta_x] \\ -[\kappa q_x + \alpha v_x + \kappa\tau q_{xt}] \\ -\dfrac{1}{\tau}[q + \kappa\theta_x] \end{bmatrix}. \tag{15}$$

Let $\mathcal{H} = H_0^1(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2(0,L)$ be the Hilbert space equipped with the inner product

$$\langle U, U^* \rangle_{\mathcal{H}} = \int_0^L p(x) u_x \overline{u_x^*} dx + \int_0^L m(x) v \overline{v^*} dx + \int_0^L \theta \overline{\theta^*} dx + \int_0^L \tau q \overline{q^*} dx, \tag{16}$$

with $U = (u, v, \theta, q)$ and $U^* = (u^*, v^*, \theta^*, q^*)$.

To study the system (6)–(8), we first consider (14) in $\mathcal{H}$ with the domain $\mathcal{D}(\mathcal{A})$ defined by

$$\mathcal{D}(\mathcal{A}) = \Big((u, v, \theta, q) \in \mathcal{H}, \;\; u \in H^2(0,L) \bigcap H_0^1(0,L), \;\; v \in H_0^1(0,L), \;\; \theta \in H_0^1(0,L), \;\; q \in H^1(0,L) \Big). \tag{17}$$

Lemma 2.1. *Let $U_0 \in \mathcal{H}$. Then, there exists a unique solution $U = (u, v, \theta, q)$ of the system (6)–(10) satisfying $U \in C([0,\infty); \mathcal{H})$. Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in C^1([0,\infty); \mathcal{D}(\mathcal{A})) \bigcap C([0,\infty); \mathcal{H}). \tag{18}$$

Proof. First, we show that $\mathcal{A}$ is a dissipative operator, and then we prove that the resolvent set of $\mathcal{A}$ contains 0. We note that

$$\begin{aligned}
\langle \mathcal{A}U, U \rangle_{\mathcal{H}} &= \int_0^L p(x) v_x \overline{u_x} dx + \int_0^L m(x) \frac{1}{m(x)} \left[(p(x) u_x + 2\delta(x) v_x)_x - \alpha \theta_x \right] \overline{v} dx \\
&- \int_0^L (k q_x + \alpha v_x + \kappa \tau q_{xt}) \overline{\theta} dx - \int_0^L \tau \frac{1}{\tau} (q + \kappa \theta_x) \overline{q} dx \\
&= \int_0^L p(x) v_x \overline{u_x} dx - \int_0^L p(x) u_x \overline{v_x} dx - 2 \int_0^L \delta(x) v_x^2 dx - \int_0^L \alpha \theta_x \overline{v} dx \\
&- \int_0^L k q_x \overline{\theta} dx - \int_0^L \alpha v_x \overline{\theta} dx - \int_0^L \kappa \tau q_{xt} \overline{\theta} dx - \int_0^L q^2 dx - \int_0^L \kappa \theta_x \overline{q} dx \\
&= 2iIm \int_0^L p(x) v_x \overline{u_x} dx - 2 \int_0^L \delta(x) v_x^2 dx + \int_0^L \alpha \theta \overline{v_x} dx - \int_0^L \alpha v_x \overline{\theta} dx \\
&+ \int_0^L k q \overline{\theta_x} dx - \int_0^L \kappa \theta_x \overline{q} dx + \int_0^L (-q - \kappa \theta_x) \kappa \overline{\theta_x} dx - \int_0^L q^2 dx \\
&= 2iIm \int_0^L \left[p(x) v_x \overline{u_x} + \alpha \theta \overline{v_x} + \kappa q \overline{\theta_x} \,\right] dx - 2 \int_0^L \delta(x) v_x^2 dx \\
&- \int_0^L q \kappa \overline{\theta_x} dx - \int_0^L \kappa^2 \theta_x^2 dx - \int_0^L q^2 dx \\
&\leq 2iIm \int_0^L \left[p(x) v_x \overline{u_x} + \alpha \theta \overline{v_x} + \kappa q \overline{\theta_x} \,\right] dx \\
&+ \frac{1}{2} \int_0^L q^2 dx + \frac{1}{2} \int_0^L \kappa^2 \theta_x^2 dx - \int_0^L \kappa^2 \theta_x^2 dx - \int_0^L q^2 dx - 2 \int_0^L \delta(x) v_x^2 dx \\
&\leq 2iIm \int_0^L \left[p(x) v_x \overline{u_x} + \alpha \theta \overline{v_x} + \kappa q \overline{\theta_x} \,\right] dx \\
&- \frac{1}{2} \int_0^L q^2 dx - \frac{1}{2} \int_0^L \kappa^2 \theta_x^2 dx - 2 \int_0^L \delta(x) v_x^2 dx.
\end{aligned}$$

In the real part, we have

$$Re \, \langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq -2 \int_0^L \delta(x) v_x^2 dx - \frac{1}{2} \int_0^L q^2 dx - \frac{1}{2} \int_0^L \kappa^2 \theta_x^2 dx \leq 0. \tag{19}$$

Hence, $\mathcal{A}$ is a dissipative operator.

We observe that there exists a unique solution $U = (u, v, \theta, q)^T \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = \mathbf{F}$, where $\mathbf{F} = (f_1, f_2, f_3, f_4)^T$, that is,

$$v = f_1 \quad \text{in} \quad H_0^1(0,L) \tag{20}$$

$$(p(x)u_x + 2\delta(x)v_x)_x - \alpha\theta_x = m(x)f_2 \quad \text{in} \quad L^2(0,L) \tag{21}$$

$$-\kappa q_x - \alpha v_x - \kappa\tau q_{xt} = f_3 \quad \text{in} \quad H_0^1(0,L) \tag{22}$$

$$-q - \kappa\theta_x = \tau f_4 \quad \text{in} \quad L^2(0,L) \tag{23}$$

with $f_1(0) = f_1(L) = 0$ and we used the boundary conditions (10).

Now, from equations (22) and (8), we have

$$-\kappa q_x - \alpha v_x - \kappa(-q_x - \kappa\theta_{xx}) = f_3(x),$$

$$\kappa^2\theta_{xx} = f_3(x) + \alpha v_x. \tag{24}$$

Integrating over $[0,x]$ in Equation (24), we obtain

$$\kappa^2\theta_x(x) = \kappa^2\theta_x(0) + \alpha v(x) + \int_0^x f_3(s)ds. \tag{25}$$

Once again integrating over $[0,x]$ in Equation (25), we obtain

$$\kappa^2\theta(x) = \kappa^2\theta_x(0)x + \alpha\int_0^x v(s)ds + \int_0^x\int_0^y f_3(s)dsdy. \tag{26}$$

Thus, we deduce that $\theta \in H^2(0,L)\bigcap H_0^1(0,L)$ is a unique solution satisfying Equation (26), where

$$\theta_x(0) = -\frac{1}{\kappa^2 L}\left[\alpha\int_0^L v(s)ds + \int_0^L\int_0^y f_3(s)dsdy\right]. \tag{27}$$

Using Equation (23) and putting the value of $\kappa\theta_x(x)$, we have

$$q = -\kappa\theta_x(0) - \frac{\alpha}{\kappa}v(x) - \frac{1}{\kappa}\int_0^x f_3(s)ds - \tau f_4(x). \tag{28}$$

Now, from Equation (21), we obtain

$$p(x)u_x(x) = -2\delta(x)f_{1x}(x) + (p(0)u_x(0) + 2\delta(0)f_{1x}(0)) + (\alpha\theta(x) - \alpha\theta(0)) + \int_0^x m(s)f_2(s)ds. \tag{29}$$

Again integrating in Equation (29) over $[0,x]$, we obtain

$$u(x) = -2\int_0^x \frac{\delta(s)}{p(s)}f_{1s}(s)ds + \Big(p(0)u_x(0) + 2\delta(0)f_{1x}(0)\Big)\int_0^x \frac{1}{p(s)}ds + \int_0^x \frac{1}{p(s)}\alpha\theta(s)ds + \int_0^x \frac{1}{p(y)}\int_0^y m(s)f_2(s)dsdy, \tag{30}$$

where

$$u_x(0)\int_0^L \frac{1}{p(s)}ds = \frac{1}{p(0)}\left[2\int_0^L \frac{\delta(s)}{p(s)}f_{1s}(s)ds - 2\delta(0)f_{1x}(0)\int_0^L \frac{1}{p(s)}ds - \int_0^L \frac{\alpha\theta(s)}{p(s)}ds - \int_0^L \frac{1}{p(y)}\int_0^y m(y)f_2(y)dydx\right]. \tag{31}$$

Consequently, from relations (20)–(23), we deduce that there exists a positive constant K such that

$$\|U\|_{\mathcal{H}} \le K\|F\|_{\mathcal{H}}.$$

This estimate implies that $0 \in \rho(\mathcal{A})$. By applying the Lumer–Phillips theorem [24], we conclude that $\mathcal{A}$ generates a contraction semigroup. This completes the proof of Lemma 2.1. □

3. Energy of the System

We define the energy functional $E(t)$ as

$$E(t) = \frac{1}{2}\left[\int_0^L p(x)u_x^2 dx + \int_0^L m(x)u_t^2 dx + \int_0^L \theta^2 dx + \tau \int_0^L q^2 dx\right] \quad \text{for} \quad t \geq 0. \tag{32}$$

The nature of this energy is described as follows.

Differentiating (32) with respect to t, we obtain

$$E'(t) = \int_0^L p(x)u_x u_{xt} dx + \int_0^L m(x)u_t u_{tt} dx + \int_0^L \theta\theta_t dx + \tau \int_0^L qq_t dx. \tag{33}$$

Using Equation (6), we set

$$E'(t) = -2\int_0^L \delta(x)u_{xt}^2 dx - \int_0^L \alpha\theta_x u_t dx + \int_0^L \theta\theta_t dx + \int_0^L \tau qq_t dx. \tag{34}$$

Integrating by parts, we obtain

$$\begin{aligned} E'(t) &= -2\int_0^L \delta(x)u_{xt}^2 dx + \int_0^L \theta[-\kappa q_x - \kappa\tau q_{xt}]dx + \int_0^L q[-q - \kappa\theta_x]dx \\ &= -2\int_0^L \delta(x)u_{xt}^2 dx + \int_0^L \theta_x \kappa(-q - \kappa\theta_x)dx - \int_0^L q^2 dx \\ &\leq -2\int_0^L \delta(x)u_{xt}^2 dx - \frac{1}{2}\int_0^L q^2 dx - \frac{1}{2}\int_0^L \kappa^2\theta_x^2 dx. \end{aligned} \tag{35}$$

In (35), the right-hand side shows that the energy $E(t)$ of the system (6)–(10) is a decreasing function of time. Therefore, the energy of the system is dissipating. Hence,

$$E(t) \leq E(0) \quad \text{for} \quad t \geq 0, \tag{36}$$

where

$$E(0) = \frac{1}{2}\left[\int_0^L p(x)u_{0x}^2 dx + \int_0^L m(x)v_0^2 dx + \int_0^L \theta_0^2 dx + \tau \int_0^L q_0^2 dx\right]. \tag{37}$$

Our objective is to prove the uniform exponential decay of this energy system $E(t)$. Inequality (36) indicates that, if

$$u_0 \in H_0^1(0,L),\ v_0 \in L^2(0,L),\ \theta_0 \in L^2(0,L) \text{ and } q_0 \in L^2(0,L),$$

then $E(t) \leq E(0) < \infty$ for $t \geq 0$.

4. Exponential Stability

We recall the Schwartz inequality (see [22])

$$\phi.\psi \leq |\phi.\psi| \leq \frac{1}{2}\left(\alpha\phi^2 + \frac{\psi^2}{\alpha}\right) \quad \text{for} \quad \alpha > 0, \tag{38}$$

and the Poincaré inequality (see [22])

$$\int_0^L u^2 dx \leq \frac{L^2}{\pi^2}\int_0^L u_x^2 dx \leq \frac{L^4}{\pi^4}\int_0^L u_{xx}^2 dx \quad \text{and also} \quad \int_0^L \theta^2 dx \leq 4\frac{L^2}{\pi^2}\int_0^L \theta_x^2 dx. \tag{39}$$

Since $m(x)$, $p(x)$ and $\delta(x)$ are continuous functions on the interval $[0, L]$, by the mean value theorem of integral calculus, there exist $\xi_i \in [0, L]$, with $i = 1, 2, ..., 5$, such that

$$\int_0^L m(x)u^2 dx = m(\xi_1) \int_0^L u^2 dx. \tag{40}$$

$$\int_0^L m(x)u_t^2 dx = m(\xi_2) \int_0^L u_t^2 dx. \tag{41}$$

$$\int_0^L p(x)u_x^2 dx = p(\xi_3) \int_0^L u_x^2 dx. \tag{42}$$

$$\int_0^L \delta(x)u_x^2 dx = \delta(\xi_4) \int_0^L u_x^2 dx. \tag{43}$$

$$\int_0^L \delta(x)u_{xt}^2 dx = \delta(\xi_5) \int_0^L u_{xt}^2 dx. \tag{44}$$

It is clear that $m(\xi_1)$, $m(\xi_2)$, $p(\xi_3)$, $\delta(\xi_4)$, $\delta(\xi_5)$ are positive and bounded over $[0, L]$. Applying the above inequalities and (40), we obtain

$$\left|\int_0^L m(x)uu_t dx\right| \leq \frac{1}{2}\left[b_1 \int_0^L m(x)u^2 + \frac{1}{b_1}\int_0^L m(x)u_t^2 dx\right] \tag{45}$$

$$\leq \frac{1}{2}\left[\frac{l^2}{\pi^2} b_1 m(\xi_1) \int_0^L u_x^2 dx + \frac{1}{b_1}\int_0^L m(x)u_t^2 dx\right] \tag{46}$$

$$\leq \frac{1}{2}\frac{l^2}{\pi^2}\left[b_1 m(\xi_1) \int_0^L u_x^2 dx + \frac{m(\xi_2)}{b_1}\int_0^L u_{xt}^2 dx\right], \tag{47}$$

$$\left|\int_0^L \theta_x(q + \beta u_{xt})dx\right| \leq \frac{1}{2}\left[b_2 \int_0^L \theta_x^2 dx + \frac{2}{b_2}\int_0^L (q^2 + \beta^2 u_{xt}^2)dx\right] \tag{48}$$

and

$$\left|\int_0^L u_x(\alpha\theta + \beta q)dx\right| \leq \frac{1}{2}\left[b_3 \int_0^L u_x^2 dx + \frac{2}{b_3}\int_0^L (\alpha^2\theta^2 + \beta^2 q^2)dx\right]. \tag{49}$$

In the above inequalities, we have chosen constants $b_1, b_2, b_3 > 0$.

Lemma 4.1. *Consider the functional given as follows:*

$$G(t) = \int_0^L m(x)uu_t dx + \int_0^L \delta(x)u_x^2 dx \ \textit{ for } \ t \geq 0. \tag{50}$$

Then, its time derivative is given by

$$G'(t) = \int_0^L m(x)u_t^2 dx - \int_0^L p(x)u_x^2 dx + \alpha \int_0^L \theta u_x dx. \tag{51}$$

Proof. Differentiating (50) with respect to t, we obtain

$$G'(t) = \int_0^L m(x)u_t^2 dx + \int_0^L m(x)uu_{tt} dx + 2\int_0^L \delta(x)u_{xt}u_x dx. \tag{52}$$

With the help of Equation (6), we obtain

$$G'(t) = \int_0^L m(x)u_t^2 dx + \int_0^L u\left[(p(x)u_x + 2\delta u_{xt} - \alpha\theta_x)\right]_x dx + 2\int_0^L \delta(x)u_{xt}u_x dx. \tag{53}$$

Integrating by parts and using the boundary conditions (10), we obtain

$$G'(t) = \int_0^L m(x)u_t^2 dx - \int_0^L p(x)u_x^2 dx + \alpha \int_0^L \theta u_x dx. \tag{54}$$

Hence, the lemma follows. □

Lemma 4.2. *For the functional $G(t)$ given in (50), it holds that*

$$-\mu_0 E(t) \leq G(t) \leq (\mu_0 + \mu_1)E(t) \ \ \textit{for} \ \ t \geq 0, \tag{55}$$

where

$$\mu_0 := \frac{L}{\pi}\sqrt{\frac{m(\xi_1)}{p(\xi_3)}}, \qquad \mu_1 := 2\frac{\delta(\xi_4)}{p(\xi_3)}. \tag{56}$$

Proof. We note that

$$\begin{aligned}
\left|\int_0^L m(x)uu_t dx\right| &= \left|\int_0^L \sqrt{m}(x)u\sqrt{m(x)}u_t dx\right| \\
&\leq \frac{1}{2}\left[\frac{\pi}{L}\sqrt{\frac{p(\xi_3)}{m(\xi_1)}}\int_0^L m(x)u^2 dx + \frac{L}{\pi}\sqrt{\frac{m(\xi_1)}{p(\xi_3)}}\int_0^L m(x)u_t^2 dx\right] \\
&\leq \frac{1}{2}\frac{L}{\pi}\sqrt{\frac{m(\xi_1)}{p(\xi_3)}}\left[\int_0^L p(x)u_x^2 dx + \int_0^L m(x)u_t^2 dx\right] \\
&\leq \mu_0 E(t) \ \ \text{for} \ \ t \geq 0.
\end{aligned} \tag{57}$$

Also, we have

$$\begin{aligned}
0 \leq \int_0^L \delta(x)u_x^2 dx = \delta(\xi_4)\int_0^L u_x^2 dx \\
&\leq 2\frac{\delta(\xi_4)}{p(\xi_3)}E(t) \\
&\leq \mu_1 E(t) \ \ \text{for} \ \ t \geq 0.
\end{aligned} \tag{58}$$

Thus, from (57) and (58), it follows that

$$-\mu_0 E(t) \leq G(t) \leq (\mu_0 + \mu_1)E(t) \ \ \text{for} \ \ t \geq 0. \tag{59}$$

Therefore, the lemma follows. □

Theorem 4.1. *For any initial values $U_0 = (u_0, v_0, \theta_0, q_0) \in \mathcal{H}$ associated with the system (6)–(8) under the boundary conditions specified in (10), the energy functional E defined in (32) satisfies the inequality*

$$E(t) \leq Me^{-2\mu\epsilon t}E(0), \quad t \geq 0, \tag{60}$$

for some positive constants μ, ϵ and M.

Proof. Proceeding as in [13], we introduce a Lyapunov functional $V(t)$ defined by

$$V(t) := E(t) + \epsilon G(t), \quad \forall \ t \geq 0, \tag{61}$$

where $\epsilon > 0$ is a small constant. By Lemma 4.2, the functional $V(t)$ satisfies the estimates

$$(1 - \mu_0\epsilon)E(t) \leq V(t) \leq [1 + (\mu_0 + \mu_1)\epsilon]E(t), \ \ \forall \ t \geq 0. \tag{62}$$

We assume

$$\mu_0\epsilon < 1, \tag{63}$$

for which $V(t) \geq 0, \ \ \forall \ t \geq 0$. Thus, it follows that $V(t) \sim E(t)$. Let μ be a fixed constant satisfying the following inequality

$$0 < \mu < 1. \tag{64}$$

Differentiating (61), we obtain

$$\begin{aligned} V'(t) + 2\mu\epsilon V(t) &= E'(t) + \epsilon G'(t) + \mu\epsilon E(t) + 2\mu\epsilon^2 G(t) \\ &= -2\int_0^L \delta(x) u_{xt}^2 dx - \frac{1}{2}\int_0^L q^2 dx - \frac{1}{2}\int_0^L \theta_x^2 \kappa^2 dx \\ &+ \epsilon \left[\int_0^L m(x) u_t^2 dx - \int_0^L p(x) u_x^2 dx + \int_0^L u_x \alpha\theta dx \right] \\ &+ \mu\epsilon \left[\int_0^L p(x) u_x^2 dx + \int_0^L m(x) u_t^2 dx + \int_0^L \theta^2 dx + \tau \int_0^L q^2 dx \right] \\ &+ 2\mu\epsilon^2 \left[\int_0^L m(x) u u_t dx + \int_0^L \delta(x) u_x^2 dx \right]. \end{aligned}$$

Now, using (32)–(33), (45)–(49) and (50)–(51), we obtain

$$\begin{aligned} V'(t) + 2\mu\epsilon V(t) \le& -2\int_0^L \delta(x) u_{xt}^2 dx - \frac{1}{2}\int_0^L q^2 dx - \frac{1}{2}\int_0^L \theta_x^2 \kappa^2 dx \\ &+ \epsilon \left[\int_0^L m(x) u_t^2 dx - \int_0^L p(x) u_x^2 dx + \frac{1}{2}\left(b_1 \int_0^L u_x^2 dx + \frac{1}{b_1}\int_0^L (\alpha^2\theta^2 dx) \right) \right] \\ &+ \mu\epsilon \left[\int_0^L p(x) u_x^2 dx + \int_0^L m(x) u_t^2 dx + \int_0^L \theta^2 dx + \tau \int_0^L q^2 dx \right] \\ &+ 2\mu\epsilon^2 \left[\frac{1}{2}\frac{L^2}{\pi^2}\left(b_2 m(\xi_1) \int_0^L u_x^2 dx + \frac{m(\xi_2)}{b_2}\int_0^L u_{xt}^2 dx \right) + \int_0^L \delta(x) u_x^2 dx \right] \\ &\le A\int_0^L p(x) u_x^2 dx + B\int_0^L \delta(x) u_{xt}^2 dx + C\int_0^L \theta_x^2 dx + D\int_0^L q^2 dx, \end{aligned}$$

where

$$A = \epsilon\left[-(1-\mu) + \frac{b_1\epsilon}{2p(\xi_3)} + \frac{\mu\epsilon L^2 b_2 m(\xi_1)}{\pi^2 p(\xi_3)} + \frac{2\mu\epsilon\delta(\xi_4)}{p(\xi_3)} \right],$$

$$B = -2 + \epsilon\left[1 + \mu + \frac{\mu\epsilon}{b_2} \right] \frac{L^2 m(\xi_2)}{\pi_2 \delta(\xi_5)},$$

$$C = -\frac{\kappa^2}{2} + 2\epsilon\left[\frac{\alpha^2}{2b_1} + 2\mu \right] \frac{L^2}{\pi^2},$$

$$D = -\frac{1}{2} + \mu\epsilon\tau.$$

Now, by a suitable choice of the constants b_1, b_2 and b_3, it can be shown that

$$A \le 0, \quad B \le 0, \quad C \le 0, \quad \text{and} \quad D \le 0 \quad \text{for} \quad 0 < \epsilon \le \epsilon_0,$$

where

$$\epsilon_0 := \min\left[\frac{1}{\mu_0}, \frac{1-\mu}{2(\mu_0+\mu_1)\mu}, \frac{2}{[1+2\mu]\frac{L^2}{\pi^2}\frac{m(\xi_2)}{\delta(\xi_5)}}, \frac{\kappa^2}{4(\frac{\alpha^2}{2b_1}+2\mu)\frac{L^2}{\pi^2}}, \frac{1}{2\mu\tau} \right].$$

Thus,

$$V'(t) + 2\mu\epsilon V(t) \le 0, \quad t \ge 0. \tag{65}$$

Now, multiplying by $e^{2\mu\epsilon t}$ in Equation (65) and integrating over $[0, t]$ yields

$$e^{2\mu\epsilon t} V(t) \le V(0).$$

Consequently, by using (62), we have

$$E(t) \le Me^{-2\mu\epsilon t}E(0) \quad \text{for } t \ge 0,$$

where

$$M = \frac{1 + (\mu_0 + \mu_1)\epsilon}{1 - \mu_0\epsilon} > 1.$$

This completes the proof. □

5. Conclusion

This study focuses on the stabilization of vibrations in a flexible structure described by the system (6)–(10). The result presented in Theorem 4.1 shows that the system energy decays exponentially at the rate $2\mu\epsilon$. Moreover, the considered system's solution $U(t) = (u, v, \theta, q)$ approaches 0 as $t \to \infty$ for any initial value

$$U_0 = (u_0, v_0, \theta_0, q_0) \in H_0^2(0, L) \times L^2(0, L) \times L^2(0, L) \times L^2(0, L),$$

which ensures that the system is uniformly stable.

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