**Contributions to Mathematics**
www.shahindp.com/locate/cm

*Contrib. Math.* **13** (2026) 28–37
DOI: 10.47443/cm.2026.008

*Research Article*

# On Inequalities for the Euler Sombor Index

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(Received: 12 January 2026. Received in revised form: 5 February 2026. Accepted: 7 February 2026. Published online: 11 February 2026.)



**Abstract**

The Euler Sombor index of a graph is a recently introduced and intensely studied topological index. In this paper, relations between the Euler Sombor index and the first Zagreb index $M_1$ are established. A series of upper and lower bounds for the Euler Sombor index in terms of various graph parameters and other topological indices are also derived. Moreover, a fast algorithm is provided for estimating the Euler Sombor index in terms of $M_1$. Furthermore, new upper and lower bounds for the forgotten index and the sigma index are established in terms of the second Zagreb index and other graph parameters, which are stronger than previously reported ones.

**Keywords:** Euler Sombor index; first Zagreb index; topological index; chemical graph theory.

**2020 Mathematics Subject Classification:** 05C09, 05C92, 05C85.

## 1. Introduction

Using standard notations in graph theory [2], let $G = (V, E)$ be a simple, undirected, and connected graph, where $V$ is the set of its vertices and $E$ is the set of its edges. Denote by $n = n(G)$ its number of vertices and by $m = m(G)$ its number of edges. For a vertex $v \in V$, denote by $d_v$ its degree. Denote by $\delta$ and $\Delta$ the minimum degree and the maximum degree, respectively, of $G$. A fundamental result in graph theory, known as the handshake lemma, states that $\sum_{v\in V} d_v = 2m$. For an edge $e = xy \in E$, denote by $imb(e) = |d_x - d_y|$, the edge imbalance of $e$. Denote by $P_n$ the path graph having $n$ vertices and by $S_n$ the star graph having $n$ vertices. A graph $G$ is called regular if all its vertices have the same degree; $G$ is biregular if it is bipartite and the vertex degrees are constant on each part; and it is bidegreed if $\left|\{d_v : v \in V\}\right| = 2$. Notice that every biregular graph is bipartite and bidegreed, but not every bipartite bidegreed graph is necessarily biregular.

The Euler Sombor index of $G$ defined by

$$EU(G) \ = \sum_{uv\in E} \sqrt{d_u^2 + d_v^2 + d_u d_v},$$

and introduced by Gutman in [9], is a recently introduced topological index. The Euler Sombor index is a part of a family of indices collectively known as Sombor-type indices (Sombor index, elliptic Sombor index, Euler Sombor index, etc.), intensively studied in recent years [15].

Recent research established several mathematical properties of the Euler Sombor index [9, 12, 14, 18, 19]. Moreover, the Euler Sombor index was shown to be useful for predicting physicochemical properties of substances [19]. In this paper, we extend the theoretical study of the Euler Sombor index by providing its relations with the first Zagreb index and establishing its bounds in terms of multiple graph parameters.

We now recall some topological indices that will be linked to the Euler Sombor index via various bounds and inequalities in the subsequent sections. For a graph $G = (V, E)$, the first and second Zagreb indices [11] are defined, respectively, as

$$M_1(G) = \sum_{xy\in E} (d_x + d_y) = \sum_{v\in V} d_v^2 \ \text{ and } \ M_2(G) = \sum_{xy\in E} d_x d_y.$$

The forgotten topological index [7] is defined as

$$F(G) = \sum_{xy\in E} (d_x^2 + d_y^2) = \sum_{v\in V} d_v^3.$$

The Albertson irregularity index [1] and the $\sigma$ index [8, 10] are defined, respectively, as $irr(G) = \sum_{xy\in E} |d_x - d_y|$ and $\sigma(G) = \sum_{xy\in E}(d_x - d_y)^2$. It is easy to see that $\sigma(G) = F(G) - 2M_2(G)$. Lastly, the generalized sum-connectivity index [22] is defined as $\chi_\alpha = \sum_{xy\in E}(d_x + d_y)^\alpha$, where $\alpha \in \mathbb{R}$. Note that for $\alpha = 1$, $\chi_1 = M_1$.

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In theoretical computer science, particularly in algorithmic complexity theory, several notations are used to classify algorithms according to how their running time grows with the size of the input. We recall two such notations that will be used later. If $f$ and $g$ are two real functions, we say that $|f|$ is asymptotically bounded above by $g$ and denote it by $f = \mathcal{O}(g)$ if there exists a constant $k > 0$ and $n_0$ such that $|f(n)| \leq k \cdot g(n)$ for all $n > n_0$. In the same context, we say that $f$ and $g$ are of the same order, and write $f = \Theta(g)$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$. For further details, see [5], especially Chapter 3. Finally, throughout this paper, the abbreviation "iff" is used to mean "if and only if".

## 2. Preliminaries

In this section, we give a series of technical lemmas and recall some previously known results that are needed in the subsequent parts of this paper. We also define a simple concept that will be needed repeatedly throughout the paper.

**Lemma 2.1.**

**(a)** *The function* $f_1 : [1, \infty) \to (0, \infty)$*, defined as* $f_1(x) = x + \frac{1}{x}$ *, is strictly increasing.*

**(b)** *Let* $a > 0$ *and* $f_2 : [0, \infty) \to (0, \infty)$ *be defined by* $f_2(x) = 1 - \frac{a}{2a+x} + \left(\frac{a}{2a+x}\right)^2$*. Then,* $f_2$ *is strictly increasing.*

**(c)** *The function* $f_3 : [1, \infty) \to (0, \infty)$*, defined as* $f_3(x) = x\left(1 - \frac{\sqrt{x^2+x+1}}{x+1}\right)$ *, is strictly increasing and* $f_3(x) \geq 1 - \frac{\sqrt{3}}{2}$*, with equality iff* $x = 1$*.*

**Proof.** By direct computation $f_i^{'}(x) > 0$ for all $x \in (a_i, \infty)$, where $1 \leq i \leq 3$ and $[a_i, \infty)$ denotes the domain of $f_i$. □

**Lemma 2.2.** *Let* $0 < a \leq b$ *and* $F_1 : [a, b]^2 \to \mathbb{R}$ *be defined as*

$$F_1(x, y) = \frac{\sqrt{x^2 + xy + y^2}}{x + y}.$$

*Then,*

$$\frac{\sqrt{3}}{2} \leq F_1(x, y) \leq \frac{\sqrt{a^2 + ab + b^2}}{a + b}$$

*with equality in the first inequality iff* $x = y$ *and in the second inequality iff* $\{x, y\} = \{a, b\}$*.*

**Proof.** If $a = b$ the inequalities are trivial. Suppose $a < b$. Since $F_1(x, y) = F_1(y, x)$, we can assume $x \leq y$. Denoting by $t = \frac{x}{y}$, we have

$$F_1(x, y) = \sqrt{1 - \frac{xy}{x^2 + 2xy + y^2}} = \sqrt{1 - \frac{t}{(t+1)^2}},$$

where $t \in [\frac{a}{b}, 1]$. Define $f : [\frac{a}{b}, 1] \to \mathbb{R}$ by $f(t) = 1 - \frac{t}{(t+1)^2}$. Then,

$$f^{'}(t) = \frac{t - 1}{(t + 1)^3} < 0,$$

for all $t \in [\frac{a}{b}, 1)$. Hence, $f$ and $\sqrt{f}$ are strictly decreasing functions. Thus, $\sqrt{f(1)} \leq \sqrt{f(t)} \leq \sqrt{f(\frac{a}{b})}$, with left equality (right equality, respectively) iff $t = 1$ ($t = \frac{a}{b}$, respectively), which yields the conclusion. □

**Lemma 2.3.** *Let* $0 < a \leq b$ *and* $F_2 : [a, b]^2 \to \mathbb{R}$ *be defined as* $F_2(x, y) = x + y - \sqrt{x^2 + xy + y^2}$*. Then,*

$$\left(2 - \sqrt{3}\right) a \leq F_2(x, y) \leq \left(2 - \sqrt{3}\right) b$$

*with equality in the left inequality iff* $x = y = a$ *and in the right inequality iff* $x = y = b$*.*

**Proof.** If $a = b$ the inequalities are trivial. Suppose $a < b$.

The left inequality is equivalent to $\sqrt{x^2 + xy + y^2} \leq x + y - \alpha$, where we denote by $\alpha = (2 - \sqrt{3})a$. Since both sides are positive, by squaring and rearranging the inequality is equivalent to $\alpha^2 - 2\alpha x - 2\alpha y + xy \geq 0$, or $(x - 2\alpha)(y - 2\alpha) \geq 3\alpha^2$. Since $x \geq a = (2 + \sqrt{3})\alpha$, we have $x - 2\alpha \geq \sqrt{3}\alpha$, and similarly $y - 2\alpha \geq \sqrt{3}\alpha$. By multiplying, we obtain the desired result, with equality iff $x = y = a$.

The right inequality is equivalent to $\sqrt{x^2 + xy + y^2} \geq x + y - (2 - \sqrt{3})b$. Since, by Lemma 2.2, $\sqrt{x^2 + xy + y^2} \geq \frac{\sqrt{3}}{2}(x + y)$, it suffices to show that $\frac{\sqrt{3}}{2}(x + y) \geq x + y - (2 - \sqrt{3})b$. By rearranging, this is equivalent to $x + y \leq 2b$, with equality iff $x = y = b$. □

The result given in Lemma 2.2 suggests that the extremal values of $F_1$ may be directly related to the extremal values of the distance function $d : [a,b]^2 \to \mathbb{R}$, defined by $d(x,y) = |x-y|$. Concretely, $F_1$ and $d$ achieve their minimum, respectively, maximum under the same conditions: iff $x = y$, respectively, iff $\{x,y\} = \{a,b\}$. This observation suggests that, by considering $d$, we may obtain better estimates for $F_1$. This is indeed the case, as detailed in the next lemma.

**Lemma 2.4.** *Let $0 < a \le b$ and $F_1, d : [a,b]^2 \to \mathbb{R}$ be defined as above. For $i \in \{1,2\}$, denote by $\alpha_i : [0, b-a] \to \mathbb{R}$ the functions defined as*

$$\alpha_1(d) = \frac{\sqrt{3b^2 - 3bd + d^2}}{2b - d} \quad \text{and} \quad \alpha_2(d) = \frac{\sqrt{3a^2 + 3ad + d^2}}{2a + d}.$$

*Then, $\alpha_1(d(x,y)) \le F_1(x,y) \le \alpha_2(d(x,y))$, with equality in the first inequality iff $y = b$ and in the second inequality iff $x = a$. Moreover, the functions $\alpha_i$ are strictly increasing such that*

$$\frac{\sqrt{3}}{2} \le \alpha_i(d) \le \frac{\sqrt{a^2 + ab + b^2}}{a+b}$$

*for all $d \in [0, b-a]$ and $i \in \{1,2\}$, with equality in the left inequality iff $x = y$, and in the right inequality iff $\{x,y\} = \{a,b\}$.*

**Proof.** If $a = b$ or $x = y$, then the claims are trivial. Suppose $a < b$ and $x \ne y$. Since $F_1(x,y) = F_1(y,x)$ and $d(x,y) = d(y,x)$, we can assume $x < y$. Arbitrarily fix $x = x_0$ and $y = y_0$ such that $a \le x_0 < y_0 \le b$ and denote by $d_0 = d(x_0, y_0)$, $\tilde{\alpha} = \alpha_1(d_0)$ and $\tilde{\tilde{\alpha}} = \alpha_2(d_0)$. The stated inequalities are equivalent to

$$\tilde{\alpha}(x_0 + y_0) \le \sqrt{x_0^2 + x_0 y_0 + y_0^2} \le \tilde{\tilde{\alpha}}(x_0 + y_0).$$

We start with the left inequality. By squaring and rearranging, the left inequality becomes

$$(1 - \tilde{\alpha}^2)x_0^2 + (1 - 2\tilde{\alpha}^2)x_0 y_0 + (1 - \tilde{\alpha}^2)y_0^2 \ge 0,$$

or $(1 - \tilde{\alpha}^2)u_0^2 + (1 - 2\tilde{\alpha}^2)u_0 + (1 - \tilde{\alpha}^2) \ge 0$, where $u_0 = \frac{x_0}{y_0}$. Note that $u_0 = 1 - \frac{d_0}{y_0} \le 1 - \frac{d_0}{b} < 1$, with equality in the non-strict inequality iff $y_0 = b$. Consider now the quadratic function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(u) = (1 - \tilde{\alpha}^2)u^2 + (1 - 2\tilde{\alpha}^2)u + (1 - \tilde{\alpha}^2)$. By direct computation, its discriminant is

$$\Delta_u = 4\tilde{\alpha}^2 - 3 = \left(\frac{d_0}{2b - d_0}\right)^2$$

and its roots are $1 - \frac{d_0}{b} < 1$ and $\frac{b}{b - d_0} > 1$. Since the leading coefficient of $g_1$ is $1 - \tilde{\alpha}^2 > 0$, we have $g_1(u) \ge 0$ iff $u \in (-\infty, 1 - \frac{d_0}{b}] \cup [\frac{b}{b-d_0}, +\infty)$ with equality iff $u \in \left\{1 - \frac{d_0}{b}, \frac{b}{b-d_0}\right\}$. Since $u_0 \in (-\infty, 1 - \frac{d_0}{b}]$, the conclusion follows.

Consider now the right inequality. By squaring and rearranging, we obtain $(1 - \tilde{\tilde{\alpha}}^2)x_0^2 + (1 - 2\tilde{\tilde{\alpha}}^2)x_0 y_0 + (1 - \tilde{\tilde{\alpha}}^2)y_0^2 \le 0$, or $(1 - \tilde{\tilde{\alpha}}^2)v_0^2 + (1 - 2\tilde{\tilde{\alpha}}^2)v_0 + (1 - \tilde{\tilde{\alpha}}^2) \le 0$, where $v_0 = \frac{y_0}{x_0}$. Note that $1 < v_0 = 1 + \frac{d_0}{x_0} \le 1 + \frac{d_0}{a}$, with equality in the non-strict inequality iff $x_0 = a$. Define the quadratic function $g_2 : \mathbb{R} \to \mathbb{R}$ by $g(v) = (1 - \tilde{\tilde{\alpha}}^2)v^2 + (1 - 2\tilde{\tilde{\alpha}}^2)v + (1 - \tilde{\tilde{\alpha}}^2)$. By direct computation, its discriminant is

$$\Delta_v = 4\tilde{\tilde{\alpha}}^2 - 3 = \left(\frac{d_0}{2a + d_0}\right)^2$$

and its roots are $\frac{a}{a+d_0}$ and $1 + \frac{d_0}{a}$. Since the leading coefficient of $g_2$ is $1 - \tilde{\tilde{\alpha}}^2 > 0$, we have $g_2(v) \le 0$ iff $v \in [\frac{a}{a+d_0}, 1 + \frac{d_0}{a}]$ with equality iff $v \in \{\frac{a}{a+d_0}, 1 + \frac{d_0}{a}\}$. Since $\frac{a}{a+d_0} < 1 < v_0 \le 1 + \frac{d_0}{a}$, the conclusion follows.

For the last part, observe that

$$\alpha_1(d) = \sqrt{1 - b\frac{b-d}{(2b-d)^2}} \quad \text{and} \quad \alpha_2(d) = \sqrt{1 - a\frac{a+d}{(2a+d)^2}}.$$

For $i \in \{1,2\}$, define $h_i : [0, b-a] \to \mathbb{R}$ by $h_1(d) = -\frac{b-d}{(2b-d)^2}$ and $h_2(d) = -\frac{a+d}{(2a+d)^2}$. Then

$$h_1'(d) = \frac{d}{(2b-d)^3} > 0 \quad \text{and} \quad h_2'(d) = \frac{d}{(2a+d)^3} > 0.$$

Thus, $h_i$ and $\alpha_i$ are strictly increasing functions. Since $\alpha_i(0) = \frac{\sqrt{3}}{2}$ and $\alpha_i(b-a) = \frac{\sqrt{a^2+ab+b^2}}{a+b}$, the conclusion follows. □

Next, we give a definition that will be used repeatedly in the subsequent sections.

**Definition 2.1.** *Let $G = (V,E)$ be a graph with at least one edge. Then the minimum edge imbalance and the maximum edge imbalance of $G$ are defined as $imb_m(G) := \min\{imb(e) \ : \ e \in E\}$ and $imb_M(G) := \max\{imb(e) \ : \ e \in E\}$. When there is no danger of confusion, we will use the simplified notations $imb_m$ and $imb_M$.*

Note that the inequalities given in the following proposition are direct consequences of Definition 2.1.

**Proposition 2.1.** *Let $G = (V, E)$ be a graph with at least one edge. Then, the following inequalities hold:*

**(a)** $imb_m(G) \leq imb(e) \leq imb_M(G)$ *for all* $e \in E$,

**(b)** $0 \leq imb_m(G) \leq imb_M(G) \leq \Delta - \delta$,

**(c)** $m \cdot imb_m(G) \leq irr(G) \leq m \cdot imb_M(G)$,

**(d)** $m \cdot imb_m^2(G) \leq \sigma(G) \leq m \cdot imb_M^2(G)$,

*with equality in the middle inequality of (b) and in any inequality of (c) and (d) iff* $imb(e)$ *is constant for all* $e \in E$.

Finally, we recall some previously obtained upper and lower bounds of the first Zagreb index.

**Theorem 2.1.** *Let $G$ be a connected graph with $n \geq 3$ vertices and $d_1 \geq d_2 \geq \cdots \geq d_n$ be the degrees of its vertices. Then, the following hold.*

**(a)** ([4, 13, 20]) $M_1(G) \geq \frac{4m^2}{n}$, *with equality iff $G$ is regular.*

**(b)** ([16, 17]) $M_1(G) \geq \frac{4m^2}{n} + \frac{1}{2}(\Delta - \delta)^2$, *with equality iff $d_2 = d_3 = \cdots = d_{n-1} = \frac{\Delta+\delta}{2}$, which includes the regular case. Moreover, this bound is stronger than the one given in part (a).*

**(c)** ([3]) $M_1(G) \geq \frac{4m^2+(n-1)(\Delta^2+\delta^2)-4m(\Delta+\delta)+2\Delta\delta}{n-2}$, *with equality iff $d_2 = d_3 = \cdots = d_{n-1}$. Moreover, this bound is stronger than the one given in part (b).*

**Theorem 2.2.** *Let $G$ be a connected graph with $m \geq 1$ edges. Then, the following hold.*

**(a)** ([6]) $M_1(G) \leq 2m(\Delta + \delta) - n\delta\Delta$, *with equality iff $G$ is regular or bidegreed.*

**(b)** ([21]) $M_1(G) \leq 2m(\Delta+\delta) - n\delta\Delta + (\delta-k)(\Delta-k)$, *where $k$ is an integer such that $\delta \leq k < \Delta$ and $2m - n\delta \equiv k - \delta (mod(\Delta - \delta))$, that is, $k = 2m - \delta(n-1) - (\Delta - \delta)\lfloor \frac{2m-n\delta}{\Delta-\delta} \rfloor$. Moreover, equality is achieved iff $G$ has at most one vertex of degree different from $\delta$ and $\Delta$, and this bound is stronger than the one given in part (a).*

## 3. Relations Between the Euler Sombor Index and the First Zagreb Index

We start this section with a theorem that provides multiple relations between the Euler Sombor and first Zagreb indices. This theorem has numerous consequences, as many of the subsequent results in this paper rely wholly or partially on it.

**Theorem 3.1.** *Let $G$ be a connected graph with $m \geq 1$ edges. Let*

$$\tilde{\alpha}(G) = \frac{\sqrt{3\Delta^2 - 3\Delta \cdot imb_m(G) + imb_m(G)^2}}{2\Delta - imb_m(G)}, \quad \tilde{\tilde{\alpha}}(G) = \frac{\sqrt{3\delta^2 + 3\delta \cdot imb_M(G) + imb_M(G)^2}}{2\delta + imb_M(G)} \text{ and } \beta = \frac{\sqrt{\Delta^2 + \Delta\delta + \delta^2}}{\Delta + \delta}.$$

*Then, the following statements hold true.*

**(a)** $\frac{\sqrt{3}}{2} M_1(G) \leq \tilde{\alpha}(G) M_1(G) \leq EU(G) \leq \tilde{\tilde{\alpha}}(G) M_1(G) \leq \beta M_1(G)$, *with equality in the first inequality iff $imb_m(G) = 0$, in the second and third inequalities iff $G$ is regular or biregular, and in the fourth inequality iff $imb_M(G) = \Delta - \delta$.*

**(b)** $M_1(G) - \left(2 - \sqrt{3}\right) m\Delta \leq EU(G) \leq M_1(G) - \left(2 - \sqrt{3}\right) m\delta$, *with equality in either inequality iff $G$ is regular.*

**(c)** $0 \leq EU(G) - \frac{\sqrt{3}}{2} M_1(G) \leq \left(1 - \frac{\sqrt{3}}{2}\right) irr(G)$, *with equality in either inequality iff $G$ is regular.*

**Proof.** For part (a), choose $e \in E$ and denote it by $e = xy$ such that $d_x \leq d_y$. Clearly, $\delta \leq d_x \leq d_y \leq \Delta$. Hence, by Lemma 2.4 with $a = \delta$ and $b = \Delta$, we obtain

$$\frac{\sqrt{3}}{2} \leq \alpha_1(imb(e)) \leq F_1(d_x, d_y) \leq \alpha_2(imb(e)) \leq \frac{\sqrt{\Delta^2 + \Delta\delta + \delta^2}}{\Delta + \delta}.$$

Since $imb_m(G) \leq imb(e) \leq imb_M(G)$ and the functions $\alpha_i$ are strictly increasing such that $\frac{\sqrt{3}}{2} \leq \alpha_i(d) \leq \frac{\sqrt{\Delta^2+\Delta\delta+\delta^2}}{\Delta+\delta}$ for all $d \in [0, \Delta - \delta]$ and $i \in \{1, 2\}$, we have

$$\frac{\sqrt{3}}{2} \leq \alpha_1(imb_m(G)) \leq F_1(d_x, d_y) \leq \alpha_2(imb_M(G)) \leq \frac{\sqrt{\Delta^2 + \Delta\delta + \delta^2}}{\Delta + \delta}.$$

By multiplying with $d_x + d_y$ and summing over all edges, we obtain the desired result. The equality cases for the first and fourth inequalities are a direct consequence of Lemma 2.4. The equality occurs in the second inequality iff $imb_m(G) = imb(e)$ and $\alpha_1(imb(e)) = F_1(d_x, d_y)$ for all $e \in E$, where $e = xy$ such that $d_x \le d_y$. By Lemma 2.4, this is equivalent to $d_y = \Delta$ and $d_x = \Delta - imb_m(G)$, or $e$ is a $(\Delta - imb_m(G), \Delta)$-edge for all $e \in E$, which implies that $G$ is regular or biregular. A similar analysis yields the equality case of the third inequality.

By applying Lemma 2.3 with $a = \delta$ and $b = \Delta$ and summing over all edges $e \in E$, part (b) follows.

For part (c), note that the first inequality is a consequence of part (a). Let $e \in E$ with $e = xy$. By Lemma 2.3 with $a = \min\{d_x, d_y\}$, we have

$$\sqrt{d_x^2 + d_x d_y + d_y^2} \le d_x + d_y - (2 - \sqrt{3})a.$$

Since $\min\{d_x, d_y\} = \frac{1}{2}(d_x + d_y - |d_x - d_y|)$, this is equivalent to

$$\sqrt{d_x^2 + d_x d_y + d_y^2} \le \frac{\sqrt{3}}{2}(d_x + d_y) + (1 - \frac{\sqrt{3}}{2})|d_x - d_y|.$$

By summing over all $e \in E$, the result follows. The equality cases are direct consequences of part (a) and Lemma 2.3. □

The following result is an immediate consequence of Theorem 3.1 and Lemma 2.2.

**Corollary 3.1.** *Let $G$ be a connected graph with $m \ge 1$ edges. Then*

$$\frac{\sqrt{3}}{2} M_1(G) \le EU(G) \le \frac{\sqrt{\Delta^2 + \Delta\delta + \delta^2}}{\Delta + \delta} M_1(G)$$

*with equality in the first inequality iff $G$ is regular and in the second inequality iff $G$ is regular or biregular.*

**Remark 3.1.** *Since $\frac{\sqrt{3}}{2} \approx 0.866$ and $\frac{\sqrt{\Delta^2+\Delta\delta+\delta^2}}{\Delta+\delta} < 1$, the bounds given in Corollary 3.1 are remarkably close. The following discussion gives a precise meaning to this remark. Note that, in general, if $u_1 \le t \le u_2$, the estimate $\hat{t} = \frac{u_1+u_2}{2}$ of $t$ has an absolute error $|t - \hat{t}| \le \frac{u_2-u_1}{2} = \frac{u_2-u_1}{u_2+u_1}\hat{t}$. Said otherwise, $|t - \hat{t}| \le \epsilon\hat{t}$, where $\epsilon = \frac{u_2-u_1}{u_2+u_1}$. Since $\frac{\sqrt{3}}{2}M_1 \le EU \le \beta M_1 < M_1$, we can apply the above result with $u_1 = \frac{\sqrt{3}}{2}M_1$ and $u_2 = M_1$. Thus, $\hat{EU} \approx 93.3\% \cdot M_1$ is an estimate of $EU$ such that $|EU - \hat{EU}| \le 7.2\% \cdot \hat{EU}$. We can do better by using $u_2 = \beta$, denoting by $r = \frac{\Delta}{\delta}$ and observing that $\frac{\sqrt{\Delta^2+\Delta\delta+\delta^2}}{\Delta+\delta} = \frac{\sqrt{r^2+r+1}}{r+1}$ is strictly increasing as a function of $r$. Some examples are given in Table 3.1.*

| $\frac{\Delta}{\delta}$ | $u_2$ | $\hat{EU}$ | Error (as a percentage of $\hat{EU}$) |
|---|---|---|---|
| $\le 10$ | 0.9577 | $\approx 91.19\% \cdot M_1$ | $< 5.04\%$ |
| $\le 5$ | 0.9279 | $\approx 89.70\% \cdot M_1$ | $< 3.46\%$ |
| $\le 4$ | 0.9165 | $\approx 89.13\% \cdot M_1$ | $< 2.84\%$ |
| $\le 3$ | 0.9013 | $\approx 88.37\% \cdot M_1$ | $< 2.01\%$ |
| $\le 2$ | 0.8819 | $\approx 87.40\% \cdot M_1$ | $< 0.91\%$ |
| $\le 1.5$ | 0.8717 | $\approx 86.89\% \cdot M_1$ | $< 0.34\%$ |
| $\le 1.33$ | 0.8690 | $\approx 86.75\% \cdot M_1$ | $< 0.18\%$ |
| $\le 1.25$ | 0.8678 | $\approx 86.69\% \cdot M_1$ | $< 0.11\%$ |

**Table 3.1:** Estimates of $EU$ in terms of $M_1$ depending on $\frac{\Delta}{\delta}$.

The last entries in Table 3.1 are particular cases of the following result.

**Theorem 3.2.** *Let $G$ be a connected graph with at least one edge and suppose there exists $k \in \mathbb{N}^*$ with $\frac{\Delta}{\delta} \le 1 + \frac{1}{k}$. Furthermore, let $\beta : [1, \infty) \to [\frac{\sqrt{3}}{2}, 1)$ be the function defined by $\beta(r) = \frac{\sqrt{r^2+r+1}}{r+1}$ and let $\hat{EU}(G) = \frac{\beta(1)+\beta(1+\frac{1}{k})}{2}M_1$. Then, $\hat{EU}(G)$ approximates $EU(G)$ such that*

$$|EU(G) - \hat{EU}(G)| < \frac{1}{12(2k+1)^2}\hat{EU}(G)$$

*and*

$$\frac{\sqrt{3}}{2}M_1(G) < \hat{EU}(G) < \left(\frac{\sqrt{3}}{2} + \frac{1}{8\sqrt{3}(2k+1)^2}\right) M_1(G).$$

**Proof.** We use the notations given in Remark 3.1. By direct computation, $u_1 = \beta(1) = \frac{\sqrt{3}}{2}$ and

$$u_2 = \beta\left(1 + \frac{1}{k}\right) = \frac{\sqrt{3k^2 + 3k + 1}}{2k + 1}.$$

Since $\beta$ is strictly increasing, $u_1 < u_2$, we have

$$\epsilon = \frac{u_2 - u_1}{u_2 + u_1} = \frac{u_2^2 - u_1^2}{(u_2 + u_1)^2} < \frac{u_2^2 - u_1^2}{4u_1^2} = \frac{1}{12(2k+1)^2}.$$

For the second part, note that

$$u_2 = \sqrt{\frac{3}{4} + \frac{1}{4(2k+1)^2}}$$

and use the chain of inequalities $x < \sqrt{x^2 + a} < x + \frac{a}{2x}$ for $u_2$ with $x = \frac{\sqrt{3}}{2}$ and $a = \frac{1}{4(2k+1)^2}$. Thus,

$$\frac{\sqrt{3}}{2} < \frac{u_1 + u_2}{2} < \frac{\sqrt{3}}{2} + \frac{a}{4x},$$

and hence, the conclusion follows. □

For small values of $\frac{\Delta}{\delta}$, the estimate $\hat{EU}$ has a remarkably low error. Since $M_1$ is a vertex-degree-based index, we note that the estimate $\hat{EU}$ can be computed in $\Theta(n)$, whereas the exact value of $EU$ requires $\Theta(m)$. Hence, in general, $\hat{EU}$ is considerably faster to compute than the index $EU$, since $\Theta(n) \le \Theta(m) \le \Theta(n^2)$. Given that a certain level of error can be tolerated in determining $EU$, we therefore suggest the following algorithm, which for a graph $G$ with $n$ vertices decides in $\Theta(1)$ whether the estimate $\hat{EU}(G)$ of $EU(G)$ is within a predefined tolerable error $tol$ and computes this estimate in $\Theta(n)$. Here, the maximum admissible error $tol$ is expressed as a percentage of $\hat{EU}$ and $err(\hat{EU})$ represents the absolute error of the estimate.

**Algorithm 1:** Estimating $EU$

```
Data: tol, δ, Δ
Result: decision on admissibility of EU^,
        computation of EU^ and its error err(EU^)
u1 ← √3/2;
u2 ← √(Δ²+Δδ+δ²)/(Δ+δ);
ε ← (u2−u1)/(u2+u1);
if ε < tol then
    Data: d_v for all v ∈ V
    M1 ← Σ_{v∈V} d_v²;
    t^ ← (u1+u2)/2;
    EU^ ← t^ · M1;
    err(EU^) ← ε · M1;
    Output: EU^, err(EU^)
else
    Output: ”estimate is inadmissible”
end
```

We end this section by noting that, in general, for a graph $G$, an even better estimate $\hat{\hat{EU}}(G)$ of $EU(G)$ can be obtained by setting $u_1 = \tilde{\alpha}(G)$ and $u_2 = \tilde{\tilde{\alpha}}(G)$. Algorithm 1 can be easily modified to suit this case.

## 4. Bounds for the Euler Sombor Index

In this section, we give a series of lower and upper bounds for the Euler Sombor index depending on various graph parameters such as $n$, $m$, $\delta$ and $\Delta$. We start with lower bounds.

**Proposition 4.1.** *Let $G$ be a connected graph with $m \ge 1$ edges. Then, the following lower bounds for $EU(G)$ hold, and the equality in each case holds iff $G$ is regular:*

**(a)** $EU(G) \ge \frac{\sqrt{3}}{2}\delta^2 n$,

**(b)** $EU(G) \ge \sqrt{3}\delta m$,

**(c)** $EU(G) \ge 2\sqrt{3}\frac{m^2}{n}$.

**Proof.** The bounds are direct consequences of the inequality $EU(G) \geq \frac{\sqrt{3}}{2}M_1(G)$ obtained in Corollary 3.1, combined with appropriate lower bounds for $M_1$. For (a), use

$$M_1(G) = \sum_{v \in V} d_v^2 \geq \sum_{v \in V} \delta^2 = \delta^2 n.$$

For (b), apply

$$M_1(G) = \sum_{uv \in E} (d_u + d_v) \geq \sum_{uv \in E} 2\delta = 2\delta m.$$

Finally, for (c), use Theorem 2.1(a). In all the inequalities used in this proof, equality occurs iff $G$ is regular. □

Some remarks are needed here. First, the bound given in Proposition 4.1(b) was obtained in [19] using a different argument. Second, the bounds given in Proposition 4.1 are listed from weakest to strongest, since

$$2\sqrt{3}\frac{m^2}{n} \geq \sqrt{3}\delta m \geq \frac{\sqrt{3}}{2}\delta^2 n.$$

To see this, note that both inequalities are equivalent to $2m \geq \delta n$ which is immediate by the handshake lemma. Third, by using tighter lower bounds for $EU$ in terms of $M_1$ and/or for $M_1$ in terms of $n$, $m$, $\delta$ and $\Delta$, we can obtain stronger lower bounds for $EU$ at the cost of having more complex expressions for these bounds. This is the subject of the next result.

**Theorem 4.1.** *Let $G$ be a connected graph with $n \geq 3$ vertices. Then, the following lower bounds for $EU(G)$ hold:*

**(a)** $EU(G) \geq 2\sqrt{3}\frac{m^2}{n} + \frac{\sqrt{3}}{4}(\Delta - \delta)^2$, *with equality iff $G$ is regular,*

**(b)** $EU(G) \geq \frac{\sqrt{3\Delta^2 - 3\Delta \cdot imb_m + imb_m^2}\,\left(4m^2 + (n-1)(\Delta^2 + \delta^2) - 4m(\Delta + \delta) + 2\Delta\delta\right)}{(2\Delta - imb_m)(n-2)}$, *with equality iff $G = S_n$ or $G$ is a regular bipartite graph.*

*Also, both bounds are stronger than those given in Proposition 4.1. Moreover, bound (b) is stronger than bound (a).*

**Proof.** For (a) use $EU(G) \geq \frac{\sqrt{3}}{2}M_1(G)$ together with Theorem 2.1(b). For (b), use the second inequality of Theorem 3.1(a) together with Theorem 2.1(c). For the equality cases, combine the equality cases of Theorems 2.1 and 3.1. For the last part of the theorem, note that the bounds of Theorem 2.1 are listed from weakest to strongest; using the second remark given immediately preceding this theorem and applying the first inequality of Theorem 3.1(a) yields the desired conclusion. □

Next, we give upper bounds for the Euler Sombor index.

**Theorem 4.2.** *Let $G$ be a graph with $m \geq 1$ edges. Then, the following upper bounds for $EU(G)$ hold:*

**(a)** $EU(G) \leq \frac{\Delta^2\sqrt{\Delta^2 + \Delta\delta + \delta^2}}{\Delta + \delta}n$,

**(b)** $EU(G) \leq 2m\left[\Delta - (1 - \frac{\sqrt{3}}{2})\delta\right]$,

**(c)** $EU(G) \leq \frac{2\Delta\sqrt{\Delta^2 + \Delta\delta + \delta^2}}{\Delta + \delta}m$.

*The equality holds in (a) and (c) iff $G$ is a regular bipartite graph and in (b) iff $G$ is regular. Moreover, bound (c) is the strongest in the sense that*

$$\frac{2\Delta\sqrt{\Delta^2 + \Delta\delta + \delta^2}}{\Delta + \delta}m \leq \min\left\{2m\left[\Delta - \left(1 - \frac{\sqrt{3}}{2}\right)\delta\right], \frac{\Delta^2\sqrt{\Delta^2 + \Delta\delta + \delta^2}}{\Delta + \delta}n\right\}$$

*with equality iff $G$ is regular, and bounds (a) and (b) are incomparable for $n \geq 19$.*

**Proof.** The bounds in (a) and (c) are direct consequences of the inequality

$$EU(G) \leq \frac{\sqrt{\Delta^2 + \Delta\delta + \delta^2}}{\Delta + \delta}M_1(G)$$

obtained in Corollary 3.1, combined with appropriate upper bounds for $M_1$. For (a), use $M_1(G) = \sum_{v \in V} d_v^2 \leq \sum_{v \in V} \Delta^2 = \Delta^2 n$, and for (c), apply $M_1(G) = \sum_{uv \in E}(d_u + d_v) \leq \sum_{uv \in E} 2\Delta = 2\Delta m$. For (b), combine the second inequality of Theorem 3.1(b) with $M_1(G) \leq 2\Delta m$. The equality cases are obtained by combining the equality cases in Theorem 3.1 with those in the upper bounds used above.

For the last part of the theorem, note that the inequality

$$\frac{2\Delta\sqrt{\Delta^2+\Delta\delta+\delta^2}}{\Delta+\delta}m \le \frac{\Delta^2\sqrt{\Delta^2+\Delta\delta+\delta^2}}{\Delta+\delta}n$$

becomes $2m \le \Delta n$, which is immediate by the handshake lemma. Observe that, by denoting $\frac{\Delta}{\delta} = r$, the inequality

$$\frac{2\Delta\sqrt{\Delta^2+\Delta\delta+\delta^2}}{\Delta+\delta}m \le 2m\left[\Delta - (1-\frac{\sqrt{3}}{2})\delta\right]$$

becomes

$$r\left(1-\frac{\sqrt{r^2+r+1}}{r+1}\right) \ge 1-\frac{\sqrt{3}}{2},$$

which is true by Lemma 2.1(c). Also, note that in both inequalities of this paragraph, equality is achieved iff $G$ is regular. Finally, the incomparability of the bounds (a) and (b) for $n \ge 19$ follows from the computation and comparison of their values for $P_n$ and $S_n$. □

As in the case of the lower bounds, we can improve the upper bounds given in Theorem 4.2 by considering tighter upper bounds for $EU$ in terms of $M_1$ and/or for $M_1$ in terms of $n$, $m$, $\delta$ and $\Delta$.

**Theorem 4.3.** *Let $G$ be a graph with $m \ge 1$ edges. Then, the following upper bounds for $EU(G)$ hold, with equality iff $G$ is a regular or biregular graph:*

- **(a)** $EU(G) \le \sqrt{\Delta^2+\Delta\delta+\delta^2}\left(2m - n\frac{\delta\Delta}{\delta+\Delta}\right)$,
- **(b)** $EU(G) \le \dfrac{\sqrt{3\delta^2+3\delta\cdot imb_M+imb_M^2}}{2\delta+imb_M}\left[2m(\Delta+\delta)-n\delta\Delta+(\delta-k)(\Delta-k)\right]$, *where $k$ is defined in Theorem 2.2(b).*

*Moreover, both bounds are stronger than those of Theorem 4.2 and bound (b) is stronger than bound (a).*

**Proof.** For (a), use

$$EU(G) \le \frac{\sqrt{\Delta^2+\Delta\delta+\delta^2}}{\Delta+\delta}M_1(G)$$

together with Theorem 2.2(a). For (b), use the third inequality of Theorem 3.1(a) together with Theorem 2.2(b). For the equality cases, combine the equality cases of Theorems 2.2 and 3.1. For the last part, Theorem 2.2(b) combined with the fourth inequality of Theorem 3.1(a) shows that bound (b) is stronger than bound (a). Finally, to show that both bounds are stronger than those of Theorem 4.2, it suffices to establish that

$$\sqrt{\Delta^2+\Delta\delta+\delta^2}\left(2m-n\frac{\delta\Delta}{\delta+\Delta}\right) \le \frac{2\Delta\sqrt{\Delta^2+\Delta\delta+\delta^2}}{\Delta+\delta}m,$$

or $2m(\Delta+\delta)-n\delta\Delta \le 2\Delta m$. But this is equivalent to $2m \le \Delta n$, which is immediate by the handshake lemma. □

## 5. Further Bounds and Inequalities

In [19], it was showed that, for any graph $G$ with $m \ge 1$ edges, the inequality

$$EU(G) \le \sqrt{m\,(F(G)+M_2(G))} \tag{1}$$

holds. Here, we provide a series of new inequalities relating the Euler Sombor index to the forgotten index $F$ and the second Zagreb index $M_2$ via various graph parameters, in a manner similar to (1).

**Theorem 5.1.** *Let $G$ be a graph without isolated vertices. Then, the following inequalities hold:*

- **(a)** $EU(G) \le \sqrt{m\left[1-\frac{\delta}{2\delta+imb_M}+\left(\frac{\delta}{2\delta+imb_M}\right)^2\right](F(G)+2M_2(G))}$,
- **(b)** $EU(G) \le \sqrt{\left(\frac{3}{4}m+\frac{\sigma(G)}{16\delta^2}\right)(F(G)+2M_2(G))}$,
- **(c)** $EU(G) \le \sqrt{\left(\frac{3}{4}m+\frac{1}{4}imb_M^2(G)\chi_{-2}(G)\right)(F(G)+2M_2(G))}$.

*Moreover, the equality is achieved in (b) iff $G$ is regular and in (a) and (c) iff $G$ is regular or biregular.*

**Proof.** Let $e = xy$ be an edge of $G$ and $a_e, b_e \in (0, \infty)$ such that $a_e \le \min\{d_x, d_y\} \le \max\{d_x, d_y\} \le b_e$. By Lemma 2.4 with $F_1 : [a_e, b_e]^2 \to \mathbb{R}$, we have $F_1(d_x, d_y) \le \alpha_2(imb(e))$, or $\sqrt{d_x^2 + d_x d_y + d_y^2} \le \alpha_2(imb(e))(d_x + d_y)$, with equality iff $\min\{d_x, d_y\} = a_e$. By summing over all edges $e$ of $G$ and applying the Cauchy-Schwartz inequality, we obtain

$$EU(G) \le \sqrt{\sum_{e=xy\in E} \alpha_2^2(imb(e)) \sum_{e=xy\in E} (d_x + d_y)^2},$$

with equality iff $\frac{\alpha_2(imb(e))}{d_x+d_y}$ is constant for all $e = xy \in E$. Clearly, $\sum_{xy\in E}(d_x + d_y)^2 = F(G) + 2M_2(G)$. Note also that

$$\alpha_2^2(imb(e)) = 1 - \frac{a_e}{2a_e + imb(e)} + \left[\frac{a_e}{2a_e + imb(e)}\right]^2 = \frac{3}{4} + \frac{1}{4}\left[\frac{imb(e)}{2a_e + imb(e)}\right]^2. \tag{2}$$

For (a), choose $a_e = \delta$ for all $e \in E$. Note that by Lemma 2.1(b) and Proposition 2.1, $\alpha_2^2(e) \le f_{2,\delta}(imb_M)$ with equality iff $imb(e) = imb_M$. By summing over all edges $e$, the inequality follows, with equality iff each of $\min\{d_x, d_y\}$, $\frac{\alpha_2(imb(e))}{d_x+d_y}$ and $imb(e)$ is constant for all $e = xy \in E$, i.e. $G$ is regular or biregular.

For the remaining inequalities of this theorem, choose $a_e = \min\{d_x, d_y\}$ for all $e = xy \in E$. Then, by (2), we have

$$\alpha_2^2(e) = \frac{3}{4} + \frac{1}{4}\left(\frac{d_x - d_y}{d_x + d_y}\right)^2.$$

For (b), note that $\alpha_2^2(e) \le \frac{3}{4} + \frac{(d_x-d_y)^2}{16\delta^2}$ with equality iff $d_x = d_y = \delta$ and sum over all edges $e$. Clearly, equality is achieved iff $G$ is regular.

For (c), observe that $\alpha_2^2(e) \le \frac{3}{4} + \frac{imb_M^2}{4(d_x+d_y)^2}$ with equality iff $imb(e) = imb_M$ and sum over all edges $e$. The equality is attained iff each of $\frac{\alpha_2(imb(e))}{d_x+d_y}$ and $imb(e)$ is constant for all $e = xy \in E$. We study this condition in more detail. Denote by $imb(e) = k$, the common value of all edge imbalances of $G$. If $k = 0$, the condition is trivially met and $G$ is regular. Suppose now that $k > 0$. Then $\frac{\alpha_2(imb(e))}{d_x+d_y}$ is constant iff

$$\frac{4\alpha_2^2(imb(e))}{(d_x + d_y)^2} = \frac{3}{(2a_e + k)^2} + \frac{k^2}{(2a_e + k)^4}$$

is constant. Since the last expression is strictly decreasing as a function of $a_e$, it is constant iff $a_e = \min\{d_x, d_y\}$ is constant, implying $\min\{d_x, d_y\} = \delta$ for all $e = xy \in E$. Thus, for any edge $e = xy$ of $G$, $\{d_x, d_y\} = \{\delta, \delta + k\}$, implying $G$ is biregular. □

**Remark 5.1.** *All statements below refer to Theorem 5.1.*

**(i)** *By using $\alpha_2^2(e) \le f_{2,\delta}(\Delta - \delta)$ in the proof of (a), we obtain $EU(G) \le (\Delta + \delta)^{-1}\sqrt{m(\Delta^2 + \Delta\delta + \delta^2)(F(G) + 2M_2(G))}$, which is an inequality less complex than (a) at the cost of being weaker, since $f_{2,\delta}(imb_M) \le f_{2,\delta}(\Delta - \delta)$.*

**(ii)** *A simpler proof for (a) can be obtained by combining (1) with the third inequality of Theorem 3.1(a). Note that, although more direct, this argument does not lead to (b) and (c).*

**(iii)** *By combining the inequality (b) with Proposition 2.1(d), we obtain*

$$EU(G) \le \sqrt{m\left[\frac{3}{4} + \left(\frac{imb_M}{4\delta}\right)^2\right](F(G) + 2M_2(G))}.$$

*This inequality is less complex than (b) at the cost of being weaker.*

We now provide bounds for the forgotten and sigma indices in terms of the second Zagreb index and some graph parameters, which, although unrelated to the Euler Sombor index, arose during the course of our investigation of the main topics of this paper.

**Theorem 5.2.** *Let $G$ be a graph without isolated vertices. Then, the following inequalities hold:*

**(a)** $\left(\frac{\Delta}{\Delta - imb_m} + \frac{\Delta - imb_m}{\Delta}\right)M_2(G) \le F(G) \le \left(\frac{\delta + imb_M}{\delta} + \frac{\delta}{\delta + imb_M}\right)M_2(G)$, *with equality in either inequality iff $G$ is regular or biregular.*

**(b)** $\frac{imb_m^2}{\Delta(\Delta - imb_m)}M_2(G) \le \sigma(G) \le \frac{imb_M^2}{\delta(\delta + imb_M)}M_2(G)$, *with equality in either inequality iff $G$ is regular or biregular.*

**Proof.** First, we prove (a). If $G$ is regular, then the inequalities are trivial. If $G$ is not regular, then $\delta < \Delta$ and $imb_M > 0$. Let $e = xy$ be an edge of $G$ such that $d_x \le d_y$ and denote by $t = \frac{d_y}{d_x} \ge 1$. Then,

$$\frac{d_y}{d_x} = 1 + \frac{imb(e)}{d_x} \le 1 + \frac{imb_M}{\delta} = \frac{\delta + imb_M}{\delta}$$

with equality iff $d_x = \delta$ and $imb(e) = imb_M$. Similarly,

$$\frac{d_y}{d_x} = \frac{d_y}{d_y - imb(e)} \ge \frac{d_y}{d_y - imb_m} \ge \frac{\Delta}{\Delta - imb_m}$$

with equality iff $d_y = \Delta$ and $imb_e = imb_m$. By Lemma 2.1(a), we have

$$f_1\left(\frac{\Delta}{\Delta - imb_m}\right) \le f_1(t) \le f_1\left(\frac{\delta + imb_M}{\delta}\right),$$

or

$$\frac{\Delta}{\Delta - imb_m} + \frac{\Delta - imb_m}{\Delta} \le \frac{d_x}{d_y} + \frac{d_y}{d_x} \le \frac{\delta + imb_M}{\delta} + \frac{\delta}{\delta + imb_M}. \tag{3}$$

Multiplying (3) by $d_x d_y$ and summing over all edges of $G$, we obtain the desired inequalities of (a). The equality cases are obvious from the above analysis.

To establish (b), subtract $2M_2(G)$ from (a) and use $\sigma(G) = F(G) - 2M_2(G)$. □

**Corollary 5.1.** *Let $G$ be a graph without isolated vertices. Then, the following inequalities hold:*

*(a)* $2M_2(G) \le F(G) \le \left(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}\right) M_2(G)$, *with equality in the first inequality iff $G$ is regular and in the second inequality iff $G$ is regular or biregular.*

*(b)* $0 \le \sigma(G) \le \frac{(\Delta-\delta)^2}{\Delta\delta} M_2(G)$, *with equality in the first inequality iff $G$ is regular and in the second inequality iff $G$ is regular or biregular.*

**Proof.** For (a), note that

$$1 \le \frac{\Delta}{\Delta - imb_m} \quad \text{and} \quad \frac{\delta + imb_M}{\delta} \le \frac{\Delta}{\delta}.$$

Then, by Lemma 2.1(a), we have

$$2 \le \frac{\Delta}{\Delta - imb_m} + \frac{\Delta - imb_m}{\Delta} \quad \text{and} \quad \frac{\delta + imb_M}{\delta} + \frac{\delta}{\delta + imb_M} \le \frac{\Delta}{\delta} + \frac{\delta}{\Delta}.$$

This, combined with Theorem 5.2(a), proves the result.

For (b), subtract $2M_2(G)$ from (a) and use $\sigma(G) = F(G) - 2M_2(G)$. □

**Remark 5.2.** *The left inequalities of Corollary 5.1 are obvious since $F(G) - 2M_2(G) = \sigma(G) \ge 0$. The left inequalities of Theorem 5.2 improve these bounds.*

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