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Research Article

Refined Oscillation Criteria for Half-Linear Emden–Fowler Equations

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Abstract

The main purpose of this article is to propose new criteria for the Emden–Fowler second-order neutral differential equation

$$\left(r(\varsigma)\left((\varkappa(\varsigma)+p(\varsigma)\varkappa(\eta(\varsigma)))'\right)^{\lambda}\right)'+q(\varsigma)f(\varkappa(\varrho(\varsigma)))=0, \quad \varsigma \geq \varsigma_0 > 0,$$

where $0 \leq p(\varsigma) \leq p_0 < \infty$. Using a couple of Riccati substitutions and fewer restrictions on $\varrho(\varsigma)$, new oscillation results are obtained, which improve and complement some known results in the literature. Some examples are also presented to illustrate the main results.

Keywords: oscillation; Emden–Fowler equation; half-linear differential equations; Riccati substitution.

2020 Mathematics Subject Classification: 34C10, 34K11.

1. Introduction

The famous Emden–Fowler equation is named after the Swiss astrophysicist Jacob Robert Emden (1862–1940) and the British astronomer Sir Ralph Howard Fowler (1889–1944). Fowler studied this equation to simulate various phenomena in fluid mechanics [10] .

The Emden–Fowler-type equations are widely applied in mathematical physics and astrophysics to describe various phenomena, including thermal explosions, the structure of stars, the behavior of isothermal gas spheres, and thermionic currents. These applications have been extensively studied in the literature; for instance, see the works of Chandrasekhar [6] and Davis [7]. In light of the aforementioned discussion, there has been a renewed interest in establishing new sufficient conditions for the oscillation of solutions of these equations.

The purpose of this study is to establish a new criterion for testing the oscillation of solutions of the second-order Emden–Fowler differential equation

$$\left(r\left(z'\right)^{\lambda}\right)'(\varsigma)+q(\varsigma)f(\varkappa(\varrho(\varsigma)))=0, \quad \varsigma \geq \varsigma_0 > 0, \tag{1}$$

where $z(\varsigma)=\varkappa(\varsigma)+p(\varsigma)\varkappa(\eta(\varsigma))$. Throughout this paper, we assume that

(H1) $\lambda > 0$ is a quotient of odd positive integers;

(H2) $r(\varsigma), p(\varsigma), q(\varsigma), \eta(\varsigma), \varrho(\varsigma) \in C\left([t_0,\infty)\right)$, where $r(\varsigma)$ and $q(\varsigma)$ are positive and $0 \leq p(\varsigma) \leq p_0 < \infty$;

(H3) $\eta(\varsigma) \leq \varsigma, \eta \circ \varrho = \varrho \circ \eta, \lim_{\varsigma\to\infty}\eta(\varsigma)=\lim_{\varsigma\to\infty}\varrho(\varsigma)=\infty$ and $\eta'(\varsigma) \geq \eta_0$, where η_0 is a positive constant;

(H4) $f \in \mathrm{C}(\mathbb{R},\mathbb{R})$ and there exists a positive constant k such that $f(u)/u^{\lambda} \geq k$ for all $u \neq 0$.

A solution $\varkappa(\varsigma)$ of Equation (1) is said to be oscillatory if it does not remain strictly positive or strictly negative for all sufficiently large values of ς; otherwise, it is classified as nonoscillatory. The equation itself is referred to as oscillatory if every solution exhibits oscillatory behavior.

Over the past three decades, the oscillation theory of functional differential equations has attracted significant interest due to its wide range of applications in engineering and the natural sciences. For contributions related to the oscillatory behavior of various classes of neutral delay functional differential equations, we refer the reader to [3, 4, 9, 11]. In what follows, we present some fundamental details that inspired our research.

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For the canonical case, that is, when

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\lambda}(s)} ds = \infty, \tag{2}$$

Grace et al. [11] established the condition

$$\limsup_{t\to\infty} \int_T^t \left(\psi(s)Q(s)\hat{R}(s) - \frac{\left(\psi'_+(s)\right)^{\lambda+1} r(s)}{(\lambda+1)^{\lambda+1}\psi^{\lambda}(s)} \right) ds = \infty$$

that guarantees the oscillation of the half-linear neutral differential equation

$$\left(r(\varsigma) \left((\varkappa(\varsigma) + p(\varsigma)\varkappa(\eta(\varsigma)))' \right)^{\lambda} \right)' + q(\varsigma) x^{\lambda}(\varrho(\varsigma)) = 0, \quad 0 \le p(\varsigma) < 1, \tag{3}$$

where $\psi'_+(\varsigma) = \max\{0, \psi'(\varsigma)\}$,

$$\hat{R}(\varsigma) = \exp\left(-\lambda \int_{\varrho(\varsigma)}^t \frac{ds}{r^{1/\lambda}(s)[R(s) + \frac{1}{\lambda}\int_{t_1}^s R(u)R^{\lambda}(\varrho(u))Q(u)du]} \right), \quad Q(\varsigma) = (1 - p(\varrho(\varsigma)))^{\lambda} q(\varsigma) \quad \text{and} \quad R(\varsigma) = \int_{t_1}^t r^{-1/\lambda}(s)ds.$$

In [2], Baculíková et al. studied the oscillatory properties of (1) under the restrictive condition $\varrho(\varsigma) \le \eta(\varsigma) \le \varsigma$. Recently, Moaaz et al. [20] established new oscillation results for (1) with $\varrho(\varsigma) \le \eta(\varsigma) \le \varsigma$ and $\varrho'(\varsigma) > 0$.

For the non-canonical case, that is, when

$$\int_{t_0}^{\infty} \frac{ds}{r^{1/\lambda}(s)} < \infty, \tag{4}$$

Han et al. [12] studied the oscillation of the second-order nonlinear neutral delay differential equations

$$\left(r(\varsigma)\psi(\varkappa(\varsigma)) \left| Z'(\varsigma) \right|^{\lambda-1} Z'(\varsigma) \right)' + q(\varsigma) f(\varkappa(\varrho(\varsigma))) = 0, \quad \varsigma \ge \varsigma_0 > 0, \tag{5}$$

where $Z(\varsigma) = \varkappa(\varsigma) + p(\varsigma)\varkappa(\eta(\varsigma))$ for $\varsigma \ge \varsigma_0, \lambda > 0, \int_{\varsigma_0}^{\infty} \frac{ds}{r^{1/\lambda}(s)} < \infty$ and $0 \le p(\varsigma) \le 1$. Sun et al. [21] established the sufficient conditions for the oscillation of (1) in the case of $\lambda = 1$ under the condition (4). They introduced the oscillation results in the following cases: (i) $\varrho(\varsigma) \le \varsigma$, $\varrho(\varsigma) \le \eta(\varsigma)$ and $\varrho'(\varsigma) > 0$; (ii) $\varrho'(\varsigma) > 0$ and $\varrho(\varsigma) \le \eta(\varsigma) \le \varsigma$; (iii) $\eta(\varsigma) \le \varsigma$ and $\varrho(\varsigma) \ge \eta(\varsigma)$.

Equation (1) finds extensive applications in mathematical, theoretical, and chemical physics. Notably, it appears in diverse real-world problems, including the study of p-Laplace equations, non-Newtonian fluid dynamics, and turbulent flow of polytropic gases in porous media. Consequently, the oscillatory behavior of various differential equation classes has been a subject of significant research. For relevant studies, see [1–5, 9, 11–18, 20–23].

Throughout this paper, functional inequalities are assumed to hold asymptotically, meaning they are valid for sufficiently large values of t. Also, without loss of generality, we focus solely on positive solutions of Equation (1). In this paper, we establish a new oscillation criterion for Equation (1), which applies

- for $0 \le p(\varsigma) \le p_0 < \infty$, unlike [11, 12];
- to all cases of $\varrho(\varsigma)$, not affected by the function $\eta(\varsigma)$, unlike [2, 20];
- without imposing the condition $\varrho'(\varsigma) > 0$, unlike [20, 21];
- without imposing any restrictions on the function $\varrho(\varsigma)$; that is, $\varrho(\varsigma)$ may be delayed, advanced, or alternate between advanced and delayed arguments, unlike all the references cited above.

2. Main Results

For the sake of brevity, we introduce the following notations:

$$Q(\varsigma) = \min\{q(\varsigma), q(\eta(\varsigma))\}, \ R(\varsigma) = \int_t^{\infty} \frac{ds}{r^{1/\lambda}(s)}, \ F(\varsigma) = \frac{1}{(\lambda+1)^{\lambda+1}} \frac{(\theta'(\varsigma))^{\lambda+1}}{\theta^{\lambda}(\varsigma)} \left(r(\varsigma) + \frac{p_0^{\lambda}}{\eta_0^{\lambda+1}} r(\eta(\varsigma)) \right), \ K_0 = \begin{cases} k, & 0 < \lambda < 1, \\ 2^{1-\lambda}k, & \lambda \ge 1, \end{cases}$$

$$A_1(\varsigma) = \frac{\theta'(\varsigma)}{\theta(\varsigma)} + \frac{1+\lambda}{r^{1/\lambda}(\varsigma)R(\varsigma)}, \quad B_1(\varsigma) = \frac{\theta'(\varsigma)}{\theta(\varsigma)} + \frac{(1+\lambda)\eta'(\varsigma)}{r^{1/\lambda}(\eta(\varsigma))R(\eta(\varsigma))},$$

$$C_1(\varsigma) = \frac{\theta(\varsigma)}{(\lambda+1)^{\lambda+1}} \left(r(\varsigma)(A_1(\varsigma))^{\lambda+1} + \frac{p_0^{\lambda}}{\eta_0^{\lambda+1}} r(\eta(\varsigma))(B_1(\varsigma))^{\lambda+1} \right) + (\lambda-1)\theta(\varsigma) \left(\frac{1}{r^{1/\lambda}(\varsigma)R^{\lambda+1}(\varsigma)} + \frac{p_0^{\lambda}}{\eta_0} \frac{\eta'(\varsigma)}{r^{1/\lambda}(\eta(\varsigma))R^{\lambda+1}(\eta(\varsigma))} \right), \text{ and}$$

$$C(\varsigma) = \lambda\theta(\varsigma) \left[\frac{1}{r^{\frac{1}{\lambda}}(\varsigma)R^{\lambda+1}(\varsigma)} + \frac{p_0^{\lambda}}{\eta_0} \frac{\eta'(\varsigma)}{r^{\frac{1}{\lambda}}(\eta(\varsigma))R^{\lambda+1}(\eta(\varsigma))} \right] + \theta'(\varsigma) \left[\frac{1}{R^{\lambda}(\varsigma)} + \frac{p_0^{\lambda}}{\eta_0} \frac{1}{R^{\lambda}(\eta(\varsigma))} \right] + \frac{(\theta'(\varsigma))^{\lambda+1}}{(\lambda+1)^{\lambda+1}(\theta(\varsigma))^{\lambda}} \left[r(\varsigma) + \frac{p_0^{\lambda}}{\eta_0} \frac{r(\eta(\varsigma))}{(\eta'(\varsigma))^{\lambda}} \right],$$

where $\theta(\varsigma)$ is a function to be specified later.

To prove our oscillation criterion, we need the following lemmas.

Lemma 2.1 (see [23]). *If $\chi \geq 0, \Upsilon \geq 0$, and $0 < \gamma \leq 1$, then $\chi^\gamma + \Upsilon^\gamma \geq (\chi + \Upsilon)^\gamma$.*

Lemma 2.2 (see [2]). *If $\chi \geq 0, \Upsilon \geq 0$, and $\gamma \geq 1$, then $\chi^\gamma + \Upsilon^\gamma \geq 2^{1-\gamma}(\chi + \Upsilon)^\gamma$.*

Lemma 2.3 (see [22]). *Let $G(U) = MU - N(U-R)^{(\lambda+1)/\lambda}$, where $N > 0$. If M and R are constants, λ is a ratio of odd positive integers, then G attains its maximum value on $\mathbb{R}$ at $U* = R + (\lambda M/((\lambda+1)N))^\lambda$ and*

$$\max_{U \in \mathbb{R}} G(U) = G(U*) = MR + \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}} \frac{M^{\lambda+1}}{N^\lambda}. \tag{6}$$

Firstly, we establish new oscillation criteria for (1) in the canonical form

Lemma 2.4. *Let $\varkappa(\varsigma)$ be a positive solution of (1). Assume that (2) holds. Then, for every t large enough,*

$$z(\varsigma) > 0, \quad z'(\varsigma) > 0 \ \text{ and } \ (r(\varsigma)(z'(\varsigma))^\lambda)' \leq 0.$$

Proof. The proof is similar to that of Lemma 3 in [2], and hence, the details are omitted. □

Theorem 2.1. *Assume that conditions (H1)−(H4) and (2) are satisfied. Moreover, suppose that there exists a function $L(\varsigma) \in C([\varsigma_0, \infty))$ such that*

$$L(\varsigma) \leq \inf\{\varsigma, \varrho(\varsigma)\}, \quad \lim_{\varsigma \to \infty} L(\varsigma) = \infty. \tag{7}$$

If there exists a positive function $\theta \in C^1([\varsigma_0, \infty))$ such that

$$\limsup_{\varsigma \to \infty} \int_{\varsigma_*}^{\varsigma} \left[K_0 \theta(s) Q(s) \left(\frac{\int_{\varsigma_1}^{L(s)} r^{-1/\lambda}(u) du}{\int_{\varsigma_1}^{s} r^{-1/\lambda}(u) du} \right)^\lambda - F(s) \right] ds = \infty, \tag{8}$$

for a sufficiently large $\varsigma_1 \geq \varsigma_0$ and for some $\varsigma_* > \varsigma_1$, then every solution $\varkappa(\varsigma)$ of Equation (1) is oscillatory.

Proof. Assume that $\varkappa(\varsigma)$ is a nonoscillatory solution of Equation (1). Then, without loss of generality, $\varkappa(\varsigma)$ is an eventually positive satisfying $\varkappa(\varsigma) > 0, \ \varkappa(\eta(\varsigma)) > 0$, and $\varkappa(\varrho(\varsigma)) > 0$ for $\varsigma \geq \varsigma_1 \geq \varsigma_0$. From (1) and (H4), we have

$$\left(r\left(z'\right)^\lambda\right)'(\varsigma) \leq -kq(\varsigma)x^\lambda(\varrho(\varsigma)) \leq 0. \tag{9}$$

From Lemma 2.4, we have $\left(r\left(z'\right)^\lambda\right)'(\varsigma) \leq 0$, and consequently, $\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^\lambda\right)' = \left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^\lambda\right)' \eta'(\varsigma)$. It follows that there exists a $\varsigma_2 \geq \varsigma_1$ such that

$$p_0^\lambda \frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^\lambda\right)'}{\eta'(\varsigma)} \leq -kp_0^\lambda q(\eta(\varsigma))x^\lambda(\varrho(\eta(\varsigma))), \quad \varsigma \geq \varsigma_2.$$

Now, in view of (H3), we obtain

$$\frac{p_0^\lambda}{\eta_0}\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^\lambda\right)' \leq -kp_0^\lambda q(\eta(\varsigma))x^\lambda(\varrho(\eta(\varsigma))), \quad \text{for} \quad \varsigma \geq \varsigma_2. \tag{10}$$

Combining (9) with (10) and using the assumption $\varrho \circ \eta = \eta \circ \varrho$, we have

$$\begin{aligned} \left(r(\varsigma)\left(z'(\varsigma)\right)^\lambda\right)' + \frac{p_0^\lambda}{\eta_0}\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^\lambda\right)' &\leq -k\left(q(\varsigma)x^\lambda(\varrho(\varsigma)) + p_0^\lambda q(\eta(\varsigma))x^\lambda(\varrho(\eta(\varsigma)))\right) \\ &\leq -kQ(\varsigma)\left(x^\lambda(\varrho(\varsigma)) + p_0^\lambda x^\lambda(\eta(\varrho(\varsigma)))\right). \end{aligned} \tag{11}$$

In the following, we have two different cases $0 < \lambda < 1$ and $\lambda \geq 1$. Firstly, consider the case when $0 < \lambda < 1$. Using Lemma 2.1 with (11), we obtain

$$\left(r(\varsigma)\left(z'(\varsigma)\right)^\lambda\right)' + \frac{p_0^\lambda}{\eta_0}\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^\lambda\right)' \leq -kQ(\varsigma)\left(x(\varrho(\varsigma)) + p_0\varkappa(\eta(\varrho(\varsigma)))\right)^\lambda \leq -kQ(\varsigma)z^\lambda(\varrho(\varsigma)). \tag{12}$$

Secondly, consider the case when $\lambda \geq 1$. Using Lemma 2.2 with (11), we obtain

$$\left(r(\varsigma)\left(z'(\varsigma)\right)^\lambda\right)' + \frac{p_0^\lambda}{\eta_0}\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^\lambda\right)' \leq -2^{1-\lambda}kQ(\varsigma)\left(x(\varrho(\varsigma)) + p_0\varkappa(\eta(\varrho(\varsigma)))\right)^\lambda \leq -2^{1-\lambda}kQ(\varsigma)z^\lambda(\varrho(\varsigma)). \tag{13}$$

Combining (12) and (13), we have

$$\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)' + \frac{p_0^{\lambda}}{\eta_0}\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)' \leq -K_0 Q(\varsigma) z^{\lambda}(\varrho(\varsigma)). \tag{14}$$

Define the following Riccati substitution

$$w(\varsigma) = \theta(\varsigma)\frac{r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}}{z^{\lambda}(\varsigma)}, \quad \varsigma \geq \varsigma_2. \tag{15}$$

It is clear that $w(\varsigma) > 0$, for $\varsigma \geq \varsigma_2$, satisfies

$$w'(\varsigma) = \theta'(\varsigma)\frac{r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}}{z^{\lambda}(\varsigma)} + \theta(\varsigma)\frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)'}{z^{\lambda}(\varsigma)} - \lambda\theta(\varsigma)r(\varsigma)\left(\frac{z'(\varsigma)}{z(\varsigma)}\right)^{\lambda+1}.$$

By (15), we have $\left(\frac{z'(\varsigma)}{z(\varsigma)}\right)^{\lambda+1} = \left(\frac{w(\varsigma)}{\theta(\varsigma)r(\varsigma)}\right)^{\frac{\lambda+1}{\lambda}}$, and consequently,

$$w'(\varsigma) = \frac{\theta'(\varsigma)}{\theta(\varsigma)}w(\varsigma) + \theta(\varsigma)\frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)'}{z^{\lambda}(\varsigma)} - \frac{\lambda}{\theta^{\frac{1}{\lambda}}(\varsigma)r^{\frac{1}{\lambda}}(\varsigma)}w^{\frac{\lambda+1}{\lambda}}(\varsigma). \tag{16}$$

Applying the inequality (6) of Lemma 2.3 with $M = \frac{\theta'(\varsigma)}{\theta(\varsigma)}$, $U - R = w(\varsigma)$, and $N = \frac{\lambda}{\theta^{1/\lambda}(\varsigma)r^{1/\lambda}(\varsigma)}$, we obtain

$$w'(\varsigma) \leq \theta(\varsigma)\frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)'}{z^{\lambda}(\varsigma)} + \frac{1}{(1+\lambda)^{\lambda+1}}\frac{(\theta'(\varsigma))^{\lambda+1}r(\varsigma)}{\theta^{\lambda}(\varsigma)}. \tag{17}$$

Now, define another function

$$v(\varsigma) = \theta(\varsigma)\frac{r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}}{z^{\lambda}(\eta(\varsigma))}, \quad \varsigma \geq \varsigma_2. \tag{18}$$

It is clear that $v(\varsigma) > 0$, for $\varsigma \geq \varsigma_2$, satisfies

$$\begin{aligned} v'(\varsigma) &= \theta'(\varsigma)\frac{r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}}{z^{\lambda}(\eta(\varsigma))} + \theta(\varsigma)\frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)'}{z^{\lambda}(\eta(\varsigma))} - \lambda\theta(\varsigma)\eta'(\varsigma)r(\eta(\varsigma))\left(\frac{z'(\eta(\varsigma))}{z(\eta(\varsigma))}\right)^{\lambda+1} \\ &= \frac{\theta'(\varsigma)}{\theta(\varsigma)}v(\varsigma) + \theta(\varsigma)\frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)'}{z^{\lambda}(\eta(\varsigma))} - \frac{\lambda\eta'(\varsigma)}{\theta^{\frac{1}{\lambda}}(\varsigma)r^{\frac{1}{\lambda}}(\eta(\varsigma))}v^{\frac{\lambda+1}{\lambda}}(\varsigma) \\ &\leq \frac{\theta'(\varsigma)}{\theta(\varsigma)}v(\varsigma) + \theta(\varsigma)\frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)'}{z^{\lambda}(\eta(\varsigma))} - \frac{\lambda\eta_0}{\theta^{1/\lambda}(\varsigma)r^{1/\lambda}(\eta(\varsigma))}v^{\frac{\lambda+1}{\lambda}}(\varsigma). \end{aligned} \tag{19}$$

Using the inequality (6) of Lemma 2.3, we obtain

$$v'(\varsigma) \leq \theta(\varsigma)\frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)'}{z^{\lambda}(\eta(\varsigma))} + \frac{1}{(\lambda+1)^{\lambda+1}}\frac{(\theta'(\varsigma))^{\lambda+1}r(\eta(\varsigma))}{\eta_0^{\lambda}\theta^{\lambda}(\varsigma)}. \tag{20}$$

Since $z'(\varsigma) > 0$ and $\eta(\varsigma) \leq \varsigma$, we have $z(\eta(\varsigma)) \leq z(\varsigma)$, and consequently, (20) takes the following form:

$$v'(\varsigma) \leq \theta(\varsigma)\frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)'}{z^{\lambda}(\varsigma)} + \frac{1}{(\lambda+1)^{\lambda+1}}\frac{(\theta'(\varsigma))^{\lambda+1}r(\eta(\varsigma))}{\eta_0^{\lambda}\theta^{\lambda}(\varsigma)}. \tag{21}$$

Combining (17) and (21) with (14), we obtain

$$\begin{aligned} w'(\varsigma) + \frac{p_0^{\lambda}}{\eta_0}v'(\varsigma) &\leq \theta(\varsigma)\frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)' + \frac{p_0^{\lambda}}{\eta_0}\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)'}{z^{\lambda}(\varsigma)} + \frac{1}{(\lambda+1)^{\lambda+1}}\frac{(\theta'(\varsigma))^{\lambda+1}}{\theta^{\lambda}(\varsigma)}\left(r(\varsigma) + \frac{p_0^{\lambda}}{\eta_0^{\lambda+1}}r(\eta(\varsigma))\right) \\ &\leq -K_0\theta(\varsigma)Q(\varsigma)\frac{z^{\lambda}(\varrho(\varsigma))}{z^{\lambda}(\varsigma)} + F(\varsigma). \end{aligned} \tag{22}$$

Since $z'(\varsigma) > 0$ and $\varrho(\varsigma) \geq L(\varsigma)$, it holds that $z(\varrho(\varsigma)) \geq z(L(\varsigma))$. It follows from (22) that

$$w'(\varsigma) + \frac{p_0^{\lambda}}{\eta_0}v'(\varsigma) \leq -K_0\theta(\varsigma)Q(\varsigma)\frac{z^{\lambda}(L(\varsigma))}{z^{\lambda}(\varsigma)} + F(\varsigma). \tag{23}$$

Since $r(\varsigma)(z'(\varsigma))^{\lambda}$ is positive and nonincreasing, we have

$$z(\varsigma)=z(\varsigma_2)+\int_{\varsigma_2}^{\varsigma}z'(s)ds=z(\varsigma_2)+\int_{\varsigma_2}^{\varsigma}\frac{(r(s)(z'(s))^{\lambda})^{1/\lambda}}{r^{1/\lambda}(s)}ds\geq z(\varsigma_2)+r^{1/\lambda}(\varsigma)z'(\varsigma)\int_{\varsigma_2}^{\varsigma}r^{-1/\lambda}(s)ds. \tag{24}$$

It follows that

$$r^{-1/\lambda}(\varsigma)z(\varsigma)-z'(\varsigma)\int_{\varsigma_2}^{\varsigma}r^{-1/\lambda}(s)ds\geq 0,$$

which implies that

$$\left(\frac{z(\varsigma)}{\int_{\varsigma_2}^{\varsigma}r^{-1/\lambda}(s)ds}\right)'\leq 0. \tag{25}$$

This with the fact that $L(\varsigma)\leq\varsigma$ yields

$$\frac{z(L(\varsigma))}{z(\varsigma)}\geq\frac{\int_{\varsigma_2}^{L(\varsigma)}r^{-1/\lambda}(s)ds}{\int_{\varsigma_2}^{\varsigma}r^{-1/\lambda}(s)ds}.$$

Substituting this into (23), we obtain

$$w'(\varsigma)+\frac{p_0^{\lambda}}{\eta_0}v'(\varsigma)\leq -K_0\theta(\varsigma)Q(\varsigma)\left(\frac{\int_{\varsigma_2}^{L(\varsigma)}r^{-1/\lambda}(s)ds}{\int_{\varsigma_2}^{\varsigma}r^{-1/\lambda}(s)ds}\right)^{\lambda}+F(\varsigma).$$

Integrating from $\varsigma_3(>\varsigma_2)$ to ς, we obtain

$$\int_{\varsigma_3}^{\varsigma}\left[K_0\theta(u)Q(u)\left(\frac{\int_{\varsigma_2}^{L(u)}r^{-1/\lambda}(s)ds}{\int_{\varsigma_2}^{u}r^{-1/\lambda}(s)ds}\right)^{\lambda}-F(u)\right]du\leq w(\varsigma_3)+\frac{p_0^{\lambda}}{\eta_0}v(\varsigma_3),$$

which contradicts (8). □

In the following, we give an oscillation criterion for (1) with the non-canonical form.

Theorem 2.2. *Let $\lambda\geq 1$. Assume that conditions* (H1)–(H4) *and* (4) *are satisfied. Suppose further that there exists a function $L(\varsigma)\in C([\varsigma_0,\infty))$ satisfies* (7)*. If there exists a positive function $\theta\in C^1([\varsigma_0,\infty))$ such that*

$$\limsup_{\varsigma\to\infty}\int_{\varsigma_*}^{\varsigma}\left(2^{1-\lambda}k\theta(s)Q(s)\left(\frac{R(\varrho(s))R(s)}{R(L(s))R(\eta(s))}\right)^{\lambda}-C_1(s)\right)ds=\infty \tag{26}$$

and (8) *hold for sufficiently large $\varsigma_1\geq\varsigma_0$ and for some $\varsigma_*>\varsigma_1$, then every solution $\varkappa(\varsigma)$ of Equation* (1) *is oscillatory.*

Proof. Assume that Equation (1) has a nonoscillatory solution $\varkappa(\varsigma)$ on $[\varsigma_0,\infty)$. Without loss of generality, suppose that it is an eventually positive solution. By (1) and (4), there exists a $\varsigma_1\in[\varsigma_0,\infty)$ such that $z'(\varsigma)>0$ or $z'(\varsigma)<0$ for $\varsigma\in[\varsigma_1,\infty)$. In the case of $z'(\varsigma)>0$, the proof is the same as that of Theorem 2.1. Now, assume that $z'(\varsigma)<0$. Since $\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)'\leq 0$, we have $r(s)(z'(s))^{\lambda}\leq r(\varsigma)(z'(\varsigma))^{\lambda}$ and $s\geq\varsigma\geq\varsigma_1$. It follows that

$$z'(s)\leq\frac{r^{1/\lambda}(\varsigma)z'(\varsigma)}{r^{1/\lambda}(s)}.$$

Integrating from ς to l and letting $l\to\infty$, we obtain

$$0\leq z(\varsigma)+r^{1/\lambda}(\varsigma)z'(\varsigma)R(\varsigma),\quad \varsigma\geq\varsigma_1.$$

Hence,

$$\frac{z'(\varsigma)}{z(\varsigma)}\geq-\frac{1}{r^{1/\lambda}(\varsigma)R(\varsigma)},\quad \varsigma\geq\varsigma_1. \tag{27}$$

Consequently, we have

$$\left(\frac{z(\varsigma)}{R(\varsigma)}\right)'\geq 0. \tag{28}$$

Now, define a generalized Riccati substitution

$$w_1(\varsigma):=\theta(\varsigma)\left[\frac{r(\varsigma)(z'(\varsigma))^{\lambda}}{z^{\lambda}(\varsigma)}+\frac{1}{R^{\lambda}(\varsigma)}\right],\quad \varsigma\geq\varsigma_1. \tag{29}$$

Then, $w_1(\varsigma) > 0$ for all $\varsigma \geq \varsigma_1$, and by the virtue of (27), we have

$$\begin{aligned} w_1'(\varsigma) &= \frac{\theta'(\varsigma)}{\theta(\varsigma)} w_1(\varsigma) + \theta(\varsigma) \frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^\lambda\right)'}{z^\lambda(\varsigma)} - \lambda\theta(\varsigma) r(\varsigma) \frac{(z'(\varsigma))^{\lambda+1}}{z^{\lambda+1}(\varsigma)} + \frac{\lambda\theta(\varsigma)}{r^{1/\lambda}(\varsigma) R^{\lambda+1}(\varsigma)} \\ &= \frac{\theta'(\varsigma)}{\theta(\varsigma)} w_1(\varsigma) + \theta(\varsigma) \frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^\lambda\right)'}{z^\lambda(\varsigma)} - \lambda\theta(\varsigma) r(\varsigma) \left[\frac{w_1(\varsigma)}{\theta(\varsigma) r(\varsigma)} - \frac{1}{r(\varsigma) R^\lambda(\varsigma)}\right]^{(\lambda+1)/\lambda} + \frac{\lambda\theta(\varsigma)}{r^{1/\lambda}(\varsigma) R^{\lambda+1}(\varsigma)}. \end{aligned} \tag{30}$$

Apply the inequality (see [18], Lemma 1)

$$A_2^{(1+\lambda)/\lambda} - (A_2 - B_2)^{(1+\lambda)/\lambda} \leq \frac{B_2^{1/\lambda}}{\lambda} \left[(1+\lambda) A_2 - B_2\right], \quad A_2 B_2 \geq 0, \tag{31}$$

for $A_2 := w_1(\varsigma)/(\theta(\varsigma) r(\varsigma))$ and $B_2 := 1/\left(r(\varsigma) R^\lambda(\varsigma)\right)$, we have

$$\left[\frac{w_1(\varsigma)}{\theta(\varsigma) r(\varsigma)} - \frac{1}{r(\varsigma) R^\lambda(\varsigma)}\right]^{(\lambda+1)/\lambda} \geq \left(\frac{w_1(\varsigma)}{\theta(\varsigma) r(\varsigma)}\right)^{(\lambda+1)/\lambda} - \frac{1}{\lambda r^{1/\lambda}(\varsigma) R(\varsigma)} \left[(1+\lambda) \frac{w_1(\varsigma)}{\theta(\varsigma) r(\varsigma)} - \frac{1}{r(\varsigma) R^\lambda(\varsigma)}\right]. \tag{32}$$

This with (30) leads to

$$w_1'(\varsigma) \leq \frac{\theta'(\varsigma)}{\theta(\varsigma)} w_1(\varsigma) + \theta(\varsigma) \frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^\lambda\right)'}{z^\lambda(\varsigma)} - \frac{\lambda}{\theta^{1/\lambda}(\varsigma) r^{1/\lambda}(\varsigma)} w_1^{(\lambda+1)/\lambda}(\varsigma) + \frac{(1+\lambda)}{r^{1/\lambda}(\varsigma) R(\varsigma)} w_1(\varsigma) + \frac{(\lambda-1)\theta(\varsigma)}{r^{1/\lambda}(\varsigma) R^{\lambda+1}(\varsigma)}. \tag{33}$$

Apply the inequality (6), we obtain

$$\left(\frac{\theta'(\varsigma)}{\theta(\varsigma)} + \frac{(1+\lambda)}{r^{1/\lambda}(\varsigma) R(\varsigma)}\right) w_1(\varsigma) - \frac{\lambda}{\theta^{1/\lambda}(\varsigma) r^{1/\lambda}(\varsigma)} w_1^{(\lambda+1)/\lambda}(\varsigma) \leq \frac{\theta(\varsigma) r(\varsigma) \left(A_1(\varsigma)\right)^{\lambda+1}}{(\lambda+1)^{\lambda+1}}. \tag{34}$$

It follows that

$$\begin{aligned} w_1'(\varsigma) &\leq \theta(\varsigma) \frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^\lambda\right)'}{z^\lambda(\varsigma)} + \frac{\theta(\varsigma) r(\varsigma) \left(A_1(\varsigma)\right)^{\lambda+1}}{(\lambda+1)^{\lambda+1}} + \frac{(\lambda-1)\theta(\varsigma)}{r^{1/\lambda}(\varsigma) R^{\lambda+1}(\varsigma)} \\ &\leq \theta(\varsigma) \frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^\lambda\right)'}{z^\lambda(\eta(\varsigma))} + \frac{\theta(\varsigma) r(\varsigma) \left(A_1(\varsigma)\right)^{\lambda+1}}{(\lambda+1)^{\lambda+1}} + \frac{(\lambda-1)\theta(\varsigma)}{r^{1/\lambda}(\varsigma) R^{\lambda+1}(\varsigma)}. \end{aligned} \tag{35}$$

Now, define the function

$$v_1(\varsigma) := \theta(\varsigma) \left[\frac{r(\eta(\varsigma))(z'(\eta(\varsigma)))^\lambda}{z^\lambda(\eta(\varsigma))} + \frac{1}{R^\lambda(\eta(\varsigma))}\right], \quad \varsigma \geq \varsigma_1. \tag{36}$$

Hence, $v_1(\varsigma) > 0$ and

$$\begin{aligned} v_1'(\varsigma) &= \frac{\theta'(\varsigma)}{\theta(\varsigma)} v_1(\varsigma) + \theta(\varsigma) \frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^\lambda\right)'}{z^\lambda(\eta(\varsigma))} - \lambda\eta'(\varsigma)\theta(\varsigma) r(\eta(\varsigma)) \frac{(z'(\eta(\varsigma)))^{\lambda+1}}{z^{\lambda+1}(\eta(\varsigma))} + \frac{\lambda\eta'(\varsigma)\theta(\varsigma)}{r^{1/\lambda}(\eta(\varsigma)) R^{\lambda+1}(\eta(\varsigma))} \\ &= \frac{\theta'(\varsigma)}{\theta(\varsigma)} v_1(\varsigma) + \theta(\varsigma) \frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^\lambda\right)'}{z^\lambda(\eta(\varsigma))} - \lambda\eta'(\varsigma)\theta(\varsigma) r(\eta(\varsigma)) \left[\frac{v_1(\varsigma)}{\theta(\varsigma) r(\eta(\varsigma))} - \frac{1}{r(\eta(\varsigma)) R^\lambda(\eta(\varsigma))}\right]^{(\lambda+1)/\lambda} \\ &\quad + \frac{\lambda\eta'(\varsigma)\theta(\varsigma)}{r^{1/\lambda}(\eta(\varsigma)) R^{\lambda+1}(\eta(\varsigma))}. \end{aligned} \tag{37}$$

On the other hand, by (31), we have

$$\left[\frac{v_1(\varsigma)}{\theta(\varsigma) r(\eta(\varsigma))} - \frac{1}{r(\eta(\varsigma)) R^\lambda(\eta(\varsigma))}\right]^{(\lambda+1)/\lambda} \geq \left(\frac{v_1(\varsigma)}{\theta(\varsigma) r(\eta(\varsigma))}\right)^{(\lambda+1)/\lambda} - \frac{1}{\lambda r^{1/\lambda}(\eta(\varsigma)) R(\eta(\varsigma))} \left[(1+\lambda) \frac{v_1(\varsigma)}{\theta(\varsigma) r(\eta(\varsigma))} - \frac{1}{r(\eta(\varsigma)) R^\lambda(\eta(\varsigma))}\right].$$

Hence, from (37), it follows that

$$\begin{aligned} v_1'(\varsigma) \leq{}& \frac{\theta'(\varsigma)}{\theta(\varsigma)} v_1(\varsigma) + \theta(\varsigma) \frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^\lambda\right)'}{z^\lambda(\eta(\varsigma))} \\ & - \frac{\lambda\eta'(\varsigma)}{\theta^{1/\lambda}(\varsigma) r^{1/\lambda}(\eta(\varsigma))} v_1^{(\lambda+1)/\lambda}(\varsigma) + \frac{(1+\lambda)\eta'(\varsigma)}{r^{1/\lambda}(\eta(\varsigma)) R(\eta(\varsigma))} v_1(\varsigma) + \frac{(\lambda-1)\eta'(\varsigma)\theta(\varsigma)}{r^{1/\lambda}(\eta(\varsigma)) R^{\lambda+1}(\eta(\varsigma))}. \end{aligned} \tag{38}$$

By virtue of (6), we have

$$\left(\frac{\theta'(\varsigma)}{\theta(\varsigma)}+\frac{(1+\lambda)\eta'(\varsigma)}{r^{1/\lambda}(\eta(\varsigma))R(\eta(\varsigma))}\right)v_1(\varsigma)-\frac{\lambda\eta'(\varsigma)}{\theta^{1/\lambda}(\varsigma)r^{1/\lambda}(\eta(\varsigma))}v_1^{(\lambda+1)/\lambda}(\varsigma)\leq\frac{\theta(\varsigma)r(\eta(\varsigma))\left(B_1(\varsigma)\right)^{\lambda+1}}{(\lambda+1)^{\lambda+1}(\eta'(\varsigma))^{\lambda}}$$
$$\leq\frac{\theta(\varsigma)r(\eta(\varsigma))\left(B_1(\varsigma)\right)^{\lambda+1}}{\eta_0^{\lambda}(\lambda+1)^{\lambda+1}}. \tag{39}$$

This with (38) leads to

$$v_1'(\varsigma)\leq\theta(\varsigma)\frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)'}{z^{\lambda}(\eta(\varsigma))}+\frac{\theta(\varsigma)r(\eta(\varsigma))\left(B_1(\varsigma)\right)^{\lambda+1}}{\eta_0^{\lambda}(\lambda+1)^{\lambda+1}}+\frac{(\lambda-1)\eta'(\varsigma)\theta(\varsigma)}{r^{1/\lambda}(\eta(\varsigma))R^{\lambda+1}(\eta(\varsigma))}. \tag{40}$$

It follows from (35), (40), and (14) that

$$w_1'(\varsigma)+\frac{p_0^{\lambda}}{\eta_0}v_1'(\varsigma)\leq\theta(\varsigma)\frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)'+\frac{p_0^{\lambda}}{\eta_0}\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)'}{z^{\lambda}(\eta(\varsigma))}+C_1(\varsigma) \tag{41}$$
$$\leq-2^{1-\lambda}k\theta(\varsigma)Q(\varsigma)\frac{z^{\lambda}(\varrho(\varsigma))}{z^{\lambda}(L(\varsigma))}\frac{z^{\lambda}(L(\varsigma))}{z^{\lambda}(\varsigma)}\frac{z^{\lambda}(\varsigma)}{z^{\lambda}(\eta(\varsigma))}+C_1(\varsigma).$$

However, from the fact that $\varrho(\varsigma)\geq L(\varsigma)$, $\eta(\varsigma)\leq\varsigma$, and (28), it follows that

$$w_1'(\varsigma)+\frac{p_0^{\lambda}}{\eta_0}v_1'(\varsigma)\leq-2^{1-\lambda}k\theta(\varsigma)Q(\varsigma)\left(\frac{R(\varrho(\varsigma))R(\varsigma)}{R(L(\varsigma))R(\eta(\varsigma))}\right)^{\lambda}\frac{z^{\lambda}(L(\varsigma))}{z^{\lambda}(\varsigma)}+C_1(\varsigma).$$

This with the fact that $L(\varsigma)\leq\varsigma$ and $z'(\varsigma)<0$ yields

$$w_1'(\varsigma)+\frac{p_0^{\lambda}}{\eta_0}v_1'(\varsigma)\leq-2^{1-\lambda}k\theta(\varsigma)Q(\varsigma)\left(\frac{R(\varrho(\varsigma))R(\varsigma)}{R(L(\varsigma))R(\eta(\varsigma))}\right)^{\lambda}+C_1(\varsigma).$$

Integrating from ς_1 to ς, we obtain

$$\int_{t_1}^{t}\left(2^{1-\lambda}k\theta(s)Q(s)\left(\frac{R(\varrho(s))R(s)}{R(L(s))R(\eta(s))}\right)^{\lambda}-C_1(s)\right)ds\leq w_1(\varsigma_1)+\frac{p_0^{\lambda}}{\eta_0}v_1(\varsigma_1),$$

which contradicts (26). □

Theorem 2.3. *Assume that conditions* (H1)−(H4) *and* (4) *are satisfied. Suppose further that there exists a function* $L(\varsigma)\in C\left([\varsigma_0,\infty)\right)$ *satisfying* (7)*. If there exists a positive function* $\theta\in C^1\left([\varsigma_0,\infty)\right)$ *such that*

$$\limsup_{\varsigma\to\infty}\int_{\varsigma_*}^{\varsigma}\left(K_0\theta(s)Q(s)\left(\frac{R(\varrho(s))R(s)}{R(L(s))R(\eta(s))}\right)^{\lambda}-C(s)\right)ds=\infty \tag{42}$$

and (8) *hold for sufficiently large* $\varsigma_1\geq\varsigma_0$ *and for some* $\varsigma_*>\varsigma_1$*, then every solution* $\varkappa(\varsigma)$ *of Equation* (1) *is oscillatory.*

Proof. Following the same lines of the proof of Theorem 2.2, we arrive at (30), and consequently, we have

$$w'(\varsigma)=\frac{\theta'(\varsigma)}{\theta(\varsigma)}w_1(\varsigma)+\theta(\varsigma)\frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)'}{z^{\lambda}(\varsigma)}-\lambda\theta(\varsigma)r(\varsigma)\left[\frac{w_1(\varsigma)}{\theta(\varsigma)r(\varsigma)}-\frac{1}{r(\varsigma)R^{\lambda}(\varsigma)}\right]^{(\lambda+1)/\lambda}+\frac{\lambda\theta(\varsigma)}{r^{1/\lambda}(\varsigma)R^{\lambda+1}(\varsigma)}$$
$$=\theta'(\varsigma)r(\varsigma)\left[\frac{w_1(\varsigma)}{\theta(\varsigma)r(\varsigma)}\right]+\theta(\varsigma)\frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)'}{z^{\lambda}(\varsigma)}-\lambda\theta(\varsigma)r(\varsigma)\left[\frac{w_1(\varsigma)}{\theta(\varsigma)r(\varsigma)}-\frac{1}{r(\varsigma)R^{\lambda}(\varsigma)}\right]^{(\lambda+1)/\lambda}+\frac{\lambda\theta(\varsigma)}{r^{1/\lambda}(\varsigma)R^{\lambda+1}(\varsigma)}. \tag{43}$$

Applying the inequality (6) with $A=\theta'(\varsigma)r(\varsigma)$, $B=\lambda\theta(\varsigma)r(\varsigma)$, $U=\frac{w_1(\varsigma)}{\theta(\varsigma)r(\varsigma)}$ and $R=\frac{1}{r(\varsigma)R^{\lambda}(\varsigma)}$, we obtain

$$\theta'(\varsigma)r(\varsigma)\left[\frac{w_1(\varsigma)}{\theta(\varsigma)r(\varsigma)}\right]-\lambda\theta(\varsigma)r(\varsigma)\left[\frac{w_1(\varsigma)}{\theta(\varsigma)r(\varsigma)}-\frac{1}{r(\varsigma)R^{\lambda}(\varsigma)}\right]^{(\lambda+1)/\lambda}\leq\frac{\theta'(\varsigma)}{R^{\lambda}(\varsigma)}+\frac{r(\varsigma)(\theta'(\varsigma)^{\lambda+1})}{(\lambda+1)^{\lambda+1}(\theta(\varsigma))^{\lambda}}.$$

Substituting into (43), we have

$$w_1^{'}(\varsigma)\leq\theta(\varsigma)\frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)'}{z^{\lambda}(\varsigma)}+\frac{\lambda\theta(\varsigma)}{r^{\frac{1}{\lambda}}(\varsigma)R^{\lambda+1}(\varsigma)}+\frac{\theta'(\varsigma)}{R^{\lambda}(\varsigma)}+\frac{r(\varsigma)\left(\theta'(\varsigma)\right)^{\lambda+1}}{(\lambda+1)^{\lambda+1}\left(\theta(\varsigma)\right)^{\lambda}}.$$

Since $z'(\varsigma) < 0$ and $\eta(\varsigma) \leq \varsigma$, it follows that

$$w_1'(\varsigma) \leq \theta(\varsigma)\frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)'}{z^{\lambda}(\eta(\varsigma))} + \frac{\lambda\theta(\varsigma)}{r^{\frac{1}{\lambda}}(\varsigma)R^{\lambda+1}(\varsigma)} + \frac{\theta'(\varsigma)}{R^{\lambda}(\varsigma)} + \frac{r(\varsigma)\left(\theta'(\varsigma)\right)^{\lambda+1}}{(\lambda+1)^{\lambda+1}\left(\theta(\varsigma)\right)^{\lambda}}. \tag{44}$$

Now, define the function $v_1(\varsigma)$ as in (36). Then, we have

$$\begin{aligned} v_1'(\varsigma) &= \theta'(\varsigma)r(\eta(\varsigma))\left[\frac{v_1(\varsigma)}{\theta(\varsigma)r(\eta(\varsigma))}\right] + \theta(\varsigma)\frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)'}{z^{\lambda}(\eta(\varsigma))} + \frac{\lambda\eta'(\varsigma)\theta(\varsigma)}{r^{\frac{1}{\lambda}}(\eta(\varsigma))R^{\lambda+1}(\eta(\varsigma))} \\ &\quad - \lambda\eta'(\varsigma)\theta(\varsigma)r(\eta(\varsigma))\left[\frac{v_1(\varsigma)}{\theta(\varsigma)r(\eta(\varsigma))} - \frac{1}{r(\eta(\varsigma))R^{\lambda}(\eta(\varsigma))}\right]^{(\lambda+1)/\lambda}. \end{aligned} \tag{45}$$

Applying the inequality (6) with $M = \theta'(\varsigma)r(\eta(\varsigma))$, $U = \frac{v_1(\varsigma)}{\theta(\varsigma)r(\eta(\varsigma))}$, $N = \lambda\eta'(\varsigma)\theta(\varsigma)r(\eta(\varsigma))$ and $R = \frac{1}{r(\eta(\varsigma))R^{\lambda}(\eta(\varsigma))}$, we have

$$\theta'(\varsigma)r(\eta(\varsigma))\left[\frac{v_1(\varsigma)}{\theta(\varsigma)r(\eta(\varsigma))}\right] - \lambda\eta'(\varsigma)\theta(\varsigma)r(\eta(\varsigma))\left[\frac{v_1(\varsigma)}{\theta(\varsigma)r(\eta(\varsigma))} - \frac{1}{r(\eta(\varsigma))R^{\lambda}(\eta(\varsigma))}\right]^{(\lambda+1)/\lambda} \leq \frac{\theta'(\varsigma)}{R^{\lambda}(\eta(\varsigma))} + \frac{r(\eta(\varsigma))\left(\theta'(\varsigma)\right)^{\lambda+1}}{(\lambda+1)^{\lambda+1}\left(\eta'(\varsigma)\theta(\varsigma)\right)^{\lambda}}.$$

Substituting into (45), we obtain

$$v_1'(\varsigma) \leq \theta(\varsigma)\frac{\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)'}{z^{\lambda}(\eta(\varsigma))} + \frac{\lambda\eta'(\varsigma)\theta(\varsigma)}{r^{\frac{1}{\lambda}}(\eta(\varsigma))R^{\lambda+1}(\eta(\varsigma))} + \frac{\theta'(\varsigma)}{R^{\lambda}(\eta(\varsigma))} + \frac{r(\eta(\varsigma))\left(\theta'(\varsigma)\right)^{\lambda+1}}{(\lambda+1)^{\lambda+1}\left(\eta'(\varsigma)\theta(\varsigma)\right)^{\lambda}}. \tag{46}$$

Combining (44) with (46) and using (14), we obtain

$$\begin{aligned} w_1'(\varsigma) + \frac{p_0^{\lambda}}{\eta_0}v_1'(\varsigma) &\leq \theta(\varsigma)\frac{\left(r(\varsigma)\left(z'(\varsigma)\right)^{\lambda}\right)' + \frac{p_0^{\lambda}}{\eta_0}\left(r(\eta(\varsigma))\left(z'(\eta(\varsigma))\right)^{\lambda}\right)'}{z^{\lambda}(\eta(\varsigma))} + \lambda\theta(\varsigma)\left[\frac{1}{r^{\frac{1}{\lambda}}(\varsigma)R^{\lambda+1}(\varsigma)} + \frac{p_0^{\lambda}}{\eta_0}\frac{\eta'(\varsigma)}{r^{\frac{1}{\lambda}}(\eta(\varsigma))R^{\lambda+1}(\eta(\varsigma))}\right] \\ &\quad + \theta'(\varsigma)\left[\frac{1}{R^{\lambda}(\varsigma)} + \frac{p_0^{\lambda}}{\eta_0}\frac{1}{R^{\lambda}(\eta(\varsigma))}\right] + \frac{\left(\theta'(\varsigma)\right)^{\lambda+1}}{(\lambda+1)^{\lambda+1}\left(\theta(\varsigma)\right)^{\lambda}}\left[r(\varsigma) + \frac{p_0^{\lambda}}{\eta_0}\frac{r(\eta(\varsigma))}{\left(\eta'(\varsigma)\right)^{\lambda}}\right] \\ &\leq -K_0\theta(\varsigma)Q(\varsigma)\frac{z^{\lambda}(\varrho(\varsigma))}{z^{\lambda}(\eta(\varsigma))} + C(\varsigma). \end{aligned} \tag{47}$$

Following the same argument as in the proof of Theorem 2.2, with (47) in place of (41), we obtain

$$w_1'(\varsigma) + \frac{p_0^{\lambda}}{\eta_0}v_1'(\varsigma) \leq -K_0\theta(\varsigma)Q(\varsigma)\left(\frac{R(\varrho(\varsigma))R(\varsigma)}{R(L(\varsigma))R(\eta(\varsigma))}\right)^{\lambda} + C(\varsigma).$$

Integrating from ς_1 to ς, we have

$$\int_{t_1}^{t}\left(K_0\theta(s)Q(s)\left(\frac{R(\varrho(s))R(s)}{R(L(s))R(\eta(s))}\right)^{\lambda} - C(s)\right)ds \leq w_1(\varsigma_1) + \frac{p_0^{\lambda}}{\eta_0}v_1(\varsigma_1),$$

which contradicts (42). □

Example 2.1. Consider the following differential equation

$$\left(\left(\left(\varkappa(\varsigma) + p_0 x\left(\lambda\varsigma\right)\right)'\right)^{\lambda}\right)' + \frac{q_0}{\varsigma^{\lambda+1}}x^{\lambda}\left(\mu\varsigma\right) = 0, \quad \varsigma \geq 1, \tag{48}$$

where $\lambda, \mu \in (0,1]$. Here, $r(\varsigma) = 1,\ p(\varsigma) = p_0,\ \eta_0 = \lambda,\ q(\varsigma) = \frac{q_0}{\varsigma^{\lambda+1}}$, λ is a positive integer and $\varrho(\varsigma) = \mu\varsigma$. It is clear that

$$\int_{\varsigma_0}^{\infty} r^{-1/\lambda}(s)ds = \infty.$$

Choosing $L(\varsigma) = \varrho(\varsigma) = \mu\varsigma$ and $\theta(\varsigma) = \varsigma^{\lambda}$, we have

$$F(\varsigma) = \frac{1}{(\lambda+1)^{\lambda+1}}\frac{\left(\theta'(\varsigma)\right)^{\lambda+1}}{\theta^{\lambda}(\varsigma)}\left(r(\varsigma) + \frac{p_0^{\lambda}}{\eta_0^{\lambda+1}}r(\eta(\varsigma))\right) = \frac{1}{\varsigma}\left(\frac{\lambda}{\lambda+1}\right)^{\lambda+1}\left(1 + \frac{p_0^{\lambda}}{\lambda^{\lambda+1}}\right).$$

Applying Theorem 2.1, we have

$$\limsup_{\varsigma\to\infty}\int_{\varsigma_*}^{\varsigma}\left[K_0\theta(s)Q(s)\left(\frac{\int_{\varsigma_1}^{L(s)}r^{-1/\lambda}(u)du}{\int_{\varsigma_1}^{s}r^{-1/\lambda}(u)du}\right)^{\lambda}-F(s)\right]ds=\limsup_{\varsigma\to\infty}\int_{\varsigma_*}^{\varsigma}\left[\frac{K_0q_0}{s}\left(\frac{\mu s-\varsigma_1}{s-\varsigma_1}\right)^{\lambda}-\frac{1}{s}\left(\frac{\lambda}{\lambda+1}\right)^{\lambda+1}\left(1+\frac{p_0^{\lambda}}{\lambda^{\lambda+1}}\right)\right]ds$$

$$=\infty,\ \text{ if } q_0>\frac{1}{K_0\mu^{\lambda}}\left(\frac{\lambda}{\lambda+1}\right)^{\lambda+1}\left(1+\frac{p_0^{\lambda}}{\lambda^{\lambda+1}}\right). \tag{49}$$

Consider the following particular case of (48) for $\lambda\geq 1$:

$$\left(\left(\left(\varkappa(\varsigma)+\frac{1}{2}x\left(\frac{2\varsigma}{3}\right)\right)'\right)^{3}\right)'+\frac{q_0}{\varsigma^4}x^3\left(\frac{\varsigma}{2}\right)=0. \tag{50}$$

By (49), every solution of (50) oscillates for $q_0>16.532$. The comparison given in the following table can be observed:

Condition	Criterion
Theorem 3 [11]	$q_0>23.892$
Theorem 6 [11]	$q_0>20.25$
Corollary 2.3 [8]	$q_0>20.25$
Theorem 2 [19]	$q_0>17.086$

We note that our results provide sharper results. Next, we consider the following special case of (48):

$$\left[\varkappa(\varsigma)+p_0x\left(\frac{\varsigma}{2}\right)\right]''+\frac{q_0}{\varsigma^2}\varkappa(\varsigma)=0,\quad t\geq 1. \tag{51}$$

Note that $\varrho(\varsigma)>\eta(\varsigma)$. According to (49), we conclude that (51) is oscillatory for $q_0>\frac{1}{4}+p_0$. Using the oscillation criterion of Theorem 2.1 in [21], the solution oscillates if $q_0>\frac{1}{2}+p_0$. Therefore, our results can be considered more general. On the other hand, note that oscillation results in [2, 20] cannot be applied to (51).

Example 2.2. Consider the following differential equation:

$$\left(\left(\left(\varkappa(\varsigma)+\frac{1}{\varsigma}x\left(\varsigma/2\right)\right)'\right)^{5}\right)'+\frac{q_0}{\varsigma^6}x^5\left(\varsigma-\cos\varsigma\right)=0,\quad \varsigma\geq 2. \tag{52}$$

Here, $r(\varsigma)=1$, $p(\varsigma)=\frac{1}{\varsigma}\leq p_0=\frac{1}{2}$, $\eta(\varsigma)=\frac{\varsigma}{2}$, $\eta_0=\frac{1}{2}$, $\lambda=5$, $q(\varsigma)=\frac{q_0}{\varsigma^6}$ and $\varrho(\varsigma)=\varsigma-\cos\varsigma$. It is clear that

$$\int_{\varsigma_0}^{\infty}r^{-1/\lambda}(s)ds=\infty.$$

Choosing $L(\varsigma)=\frac{10\varsigma}{12}$ and $\theta(\varsigma)=\varsigma^5$, we have

$$F(\varsigma)=\frac{1}{(\lambda+1)^{\lambda+1}}\frac{(\theta'(\varsigma))^{\lambda+1}}{\theta^{\lambda}(\varsigma)}\left(r(\varsigma)+\frac{p_0^{\lambda}}{\eta_0^{\lambda+1}}r(\eta(\varsigma))\right)=\frac{3}{\varsigma}\left(\frac{5}{6}\right)^{6}.$$

Applying Theorem 2.1, we obtain

$$\limsup_{\varsigma\to\infty}\int_{\varsigma_*}^{\varsigma}\left[K_0\theta(s)Q(s)\left(\frac{\int_{t_1}^{L(s)}r^{-1/\lambda}(u)du}{\int_{t_1}^{s}r^{-1/\lambda}(u)du}\right)^{\lambda}-F(s)\right]ds=\limsup_{\varsigma\to\infty}\int_{\varsigma_*}^{\varsigma}\left[\frac{1}{16}\frac{q_0}{s}\left(\frac{\frac{10s}{12}-\varsigma_1}{s-\varsigma_1}\right)^{5}-\frac{3}{s}\left(\frac{5}{6}\right)^{6}\right]ds$$

$$\geq\limsup_{\varsigma\to\infty}\int_{\varsigma_*}^{\varsigma}\left[\frac{1}{16}\frac{q_0}{s^6}\left(\frac{10s}{12}-\varsigma_1\right)^{5}-\frac{3}{s}\left(\frac{5}{6}\right)^{6}\right]ds=\infty,\ \text{if } q_0>40.$$

Consequently, every solution $\varkappa(\varsigma)$ of Equation (52) is oscillatory.

Example 2.3. Consider the following second-order differential equation:

$$\left(\varsigma^2\left(\varkappa(\varsigma)+p_0\varkappa(\varsigma-1)\right)'\right)'+q_0\varkappa(\varsigma)=0,\quad \varsigma\geq 2. \tag{53}$$

Here, $r(\varsigma)=\varsigma^2, \eta_0=1, q(\varsigma)=Q(\varsigma)=q_0, \eta(\varsigma)=\varsigma-1$, and $\varrho(\varsigma)=\varsigma$. We observe that

$$R(\varsigma)=\int_{\varsigma}^{\infty}r^{-1/\lambda}(s)ds=\int_{\varsigma}^{\infty}s^{-2}ds=\frac{1}{\varsigma}.$$

Choosing $L(\varsigma) = \varsigma = \varrho(\varsigma)$ and $\theta(\varsigma) = \dfrac{1}{\varsigma}$, we obtain

$$\limsup_{\varsigma\to\infty} \int_{\varsigma_*}^{\varsigma} \left[K_0\theta(s)Q(s) \left(\frac{\int_{t_1}^{L(s)} r^{-1/\lambda}(u)du}{\int_{t_1}^{s} r^{-1/\lambda}(u)du} \right)^{\lambda} - F(s) \right] ds = \limsup_{\varsigma\to\infty} \int_{\varsigma_*}^{\varsigma} \left[\frac{q_0}{s} - \frac{1}{4s^3}\left(s^2 + p_0(s-1)^2\right) \right] ds = \infty$$

and

$$\limsup_{\varsigma\to\infty} \int_{\varsigma_*}^{\varsigma} \left(2^{1-\lambda}k\theta(s)Q(s) \left(\frac{R(s)R(\varrho(s))}{R(L(s))R(\eta(s))} \right)^{\lambda} - C_1(s) \right) ds = \limsup_{\varsigma\to\infty} \int_{\varsigma_*}^{\varsigma} \left(\frac{q_0(s-1)}{s^2} - \frac{1}{4s} - p_0 \left[\frac{1}{4s} + \frac{1}{2s^2} + \frac{1}{4s^3} \right] \right) ds = \infty.$$

Thus, it follows from Theorem 2.2 that every solution $\varkappa(\varsigma)$ of Equation (53) is oscillatory, provided that $q_0 > \dfrac{1}{4}(1 + p_0)$. Note that, when $p_0 = 0$, $q_0 > 1/4$ is a sharp condition for the oscillation of the second-order Euler equation $(\varsigma^2 x'(\varsigma))' + q_0\varkappa(\varsigma) = 0$. Figure 2.1 presents some numerical solutions of Equation (53).

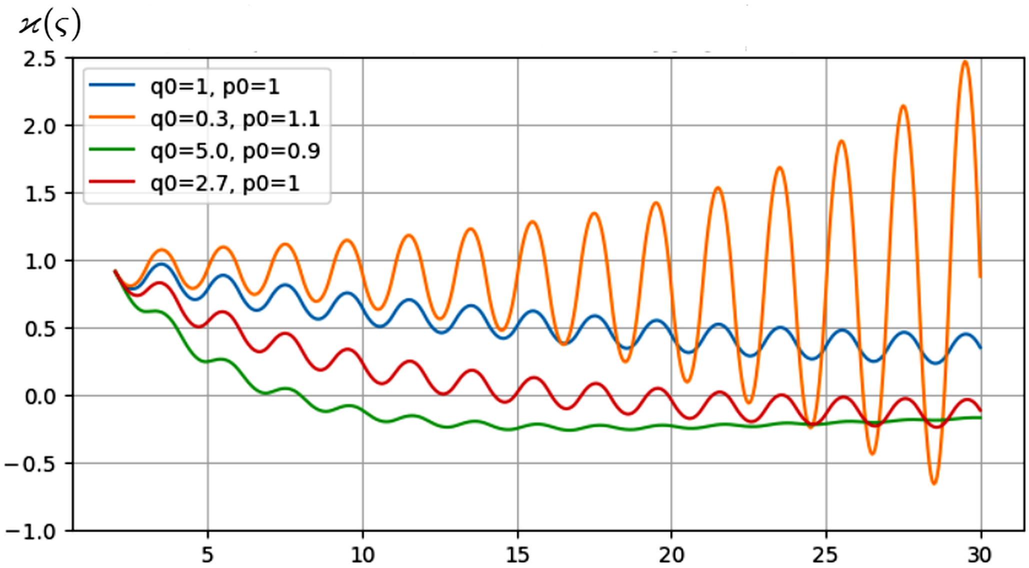

Figure 2.1: Solutions of Equation (53) for different values of q_0 and p_0.

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