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Research Article

A Core Band of a Bounded Operator in the Order-Continuous Ideal of a Banach Lattice

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Abstract

Every bounded operator $T : E \to Y$ from a Banach lattice E into a Banach space Y admits a canonical localisation on the order-continuous ideal E^a. A band $\mathfrak{B}_T \subseteq E^a$ is introduced as the largest band of E^a on which T is AM-compact. Equivalently, an element $x \in E^a$ belongs to $\mathfrak{B}_T$ precisely when $T([-|x|, |x|])$ is relatively compact in Y. This order-interval compactness is quantified by a lattice seminorm μ_T on E^a, defined via the Kuratowski measure of noncompactness of $T([-|x|, |x|])$. The kernel of μ_T coincides with $\mathfrak{B}_T$, thereby establishing a link with the restriction characterisation for LW-compact operators.

Keywords: order-continuous ideal; band; AM-compact operator; LW-compact operator; L-weakly compact set; Kuratowski measure.

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1. Introduction

Compactness phenomena in Banach lattices are strongly influenced by the order structure [6,9]. Two ways of incorporating this structure are to test compactness on order intervals (AM-compactness) and to test compactness on sets governed by disjointness (L-weakly compact sets and LW-compact operators). While LW-compactness is an operator class in its own right, the present paper adopts a complementary *domain-side* viewpoint: given an arbitrary bounded linear operator $T : E \to Y$ from a Banach lattice into a Banach space, one can canonically isolate the largest portion of the order-continuous ideal E^a on which T acts compactly on *principal* order intervals. In recent years, several operator classes refining or combining such compactness conditions have been studied, including b-AM-compact operators, b-AM-Dunford–Pettis operators, and DW-compact operators [2,4,8]. The present work complements these developments by providing a canonical localisation of AM-compactness inside E^a for an *arbitrary* bounded operator.

The rest of the paper is organised as follows. Section 2 is concerned with preliminaries. Section 3 associates to T an operator-dependent order ideal in E, determined by the relative compactness of the image of the principal interval generated by a single vector. Intersecting with E^a produces a canonically determined band in E^a. This band is order closed (hence a band) and enjoys a maximality property: it is the largest band of E^a on which the restriction of T is AM-compact. Section 4 assigns to T a lattice seminorm on E^a using the Kuratowski measure of noncompactness of the image of a principal order interval. The kernel of this seminorm coincides with the core band, so the same localisation is encoded numerically. Section 5 records the interface with LW-compactness. Using the restriction characterisation recalled in Section 2, LW-compactness of T is equivalent to AM-compactness of $T|_{E^a}$, and this is reflected by the fact that the core band coincides with E^a precisely in the LW-compact case. Conclusion and some related questions are given in Section 6.

2. Preliminaries

Throughout this paper, E denotes a real Banach lattice and Y a Banach space. The positive cone of E is denoted by E_+. For $x, y \in E$, disjointness is written $x \perp y$ and means $|x| \wedge |y| = 0$. A sequence (x_n) in E is called *disjoint* if $x_n \perp x_m$ whenever $n \neq m$. For $x \in E_+$, the order interval generated by x is $[-x, x] = \{z \in E : \ -x \leq z \leq x\}$. For a subset $A \subseteq E$, its *solid hull* is

$$\mathrm{sol}(A) = \{x \in E : \ \exists\, a \in A \text{ such that } |x| \leq |a|\},$$

the smallest solid set containing A. We write B_Y for the closed unit ball of Y.

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A norm bounded set $A \subseteq E$ is said to be *L-weakly compact* if every disjoint sequence in $\mathrm{sol}(A)$ converges to 0 in norm; see [5,9]. A bounded linear operator $T : E \to Y$ is *LW-compact* if $T(A)$ is relatively compact in Y for every L-weakly compact set $A \subseteq E$; see [1,7]. The operator T is *AM-compact* if $T([-x, x])$ is relatively compact in Y for every $x \in E_+$; see [6,9].

The *order-continuous ideal* E^a is the (norm closed) ideal consisting of all $x \in E$ such that for every net $(u_\alpha) \subseteq E_+$ with $u_\alpha \downarrow 0$, one has

$$\big\| \, |x| \wedge u_\alpha \big\| \longrightarrow 0.$$

Equivalently, $x \in E^a$ if and only if the order interval $[-|x|, |x|]$ is L-weakly compact in E; see [9,10]. In particular, the norm on E^a is order continuous: whenever $0 \le x_\alpha \uparrow x$ in E^a, one has $\|x - x_\alpha\| \to 0$; see [9,10].

Remark 2.1. *Norm compactness does not imply L-weak compactness in general. For $E = \ell^\infty$ and $\mathbf{1} = (1, 1, 1, \dots)$, the singleton $\{\mathbf{1}\}$ is norm compact, but its solid hull $[-\mathbf{1}, \mathbf{1}]$ contains the disjoint sequence (e_n) with $\|e_n\|_\infty = 1$ for all n, so $\{\mathbf{1}\}$ is not L-weakly compact.*

A key interface between LW-compactness and the order-continuous ideal is the following restriction characterisation; see [1, Proposition 2.1].

Proposition 2.1. *A bounded operator $T : E \to Y$ is LW-compact if and only if the restriction $T|_{E^a} : E^a \to Y$ is AM-compact.*

We shall repeatedly use the following simple “interval addition” observation.

Lemma 2.1. *Let $a, b \in E_+$. Then*

$$[-(a + b),\ a + b] = [-a, a] + [-b, b].$$

Consequently, for all $x, y \in E$,

$$[-|x + y|,\ |x + y|] \subseteq [-|x|, |x|] + [-|y|, |y|].$$

Proof. First note that if $u \in [-a, a]$ and $v \in [-b, b]$, then $-(a + b) \le u + v \le a + b$, and hence,

$$[-a, a] + [-b, b] \subseteq [-(a + b), a + b].$$

Conversely, let $w \in [-(a + b), a + b]$, i.e. $|w| \le a + b$. Define

$$u := (w \vee (-a)) \wedge a \in [-a, a], \qquad v := w - u.$$

Using the lattice truncation identity, one has

$$|v| \le (|w| - a)^+.$$

Since $|w| \le a + b$, it follows that

$$(|w| - a)^+ \le b,$$

and hence $|v| \le b$, i.e. $v \in [-b, b]$. Therefore $w = u + v \in [-a, a] + [-b, b]$, proving

$$[-(a + b), a + b] \subseteq [-a, a] + [-b, b].$$

The consequence follows from the inequality $|x + y| \le |x| + |y|$. The equality just proved with $a = |x|$ and $b = |y|$. □

3. The Core Ideal and the Core Band Inside E^a

Let $T : E \to Y$ be a bounded linear operator. The purpose of this section is to isolate, on the domain side, those vectors whose *principal* order intervals are mapped by T into relatively compact subsets of Y. This leads first to an order ideal $\mathcal{C}(T) \subseteq E$, and then, by intersection with the order-continuous ideal, to a canonical band $\mathfrak{B}_T \subseteq E^a$.

The Core Ideal

Definition 3.1. *Let $T : E \to Y$ be a bounded operator. The* order-interval core ideal *associated with T is defined as the set*

$$\mathcal{C}(T) = \Big\{ x \in E : \ T([-|x|, |x|]) \text{ is relatively compact in } Y \Big\}.$$

The next lemma is a convenient approximation device: elements of a larger order interval can be decomposed into a part lying in a smaller order interval plus a norm-controlled remainder.

Lemma 3.1. *Let $x, y \in E_+$ with $y \le x$. Then every $z \in [-x, x]$ admits a decomposition $z = z_1 + z_2$ such that*

$$z_1 \in [-y, y] \quad \text{and} \quad \|z_2\| \le \|x - y\|.$$

Proof. Let us fix $z \in [-x, x]$ and define

$$z_1 := (z \vee (-y)) \wedge y, \qquad z_2 := z - z_1.$$

Then $z_1 \in [-y, y]$. Moreover,

$$|z_2| \le (|z| - y)^+,$$

where $u^+ := u \vee 0$. Since $|z| \le x$ and $y \le x$, one has

$$(|z| - y)^+ \le (x - y)^+ = x - y,$$

and hence, $|z_2| \le x - y$. Therefore, $\|z_2\| \le \|x - y\|$. □

Proposition 3.1. *For every bounded operator $T : E \to Y$, the set $\mathcal{C}(T)$ is a norm closed order ideal of E.*

Proof. If $T = 0$, then $T([-|x|, |x|]) = \{0\}$ is relatively compact for every $x \in E$, and hence $\mathcal{C}(T) = E$, which implies the required result. Let us assume henceforth that $\|T\| > 0$.

Let $T : E \to Y$ be bounded and recall that

$$\mathcal{C}(T) = \{x \in E : \ T([-|x|, |x|]) \text{ is relatively compact in } Y\}.$$

If $|x| \le |y|$ and $y \in \mathcal{C}(T)$, then

$$[-|x|, |x|] \subseteq [-|y|, |y|],$$

and hence $T([-|x|, |x|]) \subseteq T([-|y|, |y|])$ is relatively compact, and therefore, $x \in \mathcal{C}(T)$. Thus, $\mathcal{C}(T)$ is an order ideal. To see that it is a linear subspace, take $x, y \in \mathcal{C}(T)$. By Lemma 2.1, it holds that

$$[-|x + y|, |x + y|] \subseteq [-|x|, |x|] + [-|y|, |y|].$$

Applying T yields

$$T([-|x + y|, |x + y|]) \subseteq T([-|x|, |x|]) + T([-|y|, |y|]).$$

Since the sum of two relatively compact sets is relatively compact, it follows that $x + y \in \mathcal{C}(T)$. Scalar homogeneity follows since

$$[-|\lambda x|, |\lambda x|] = |\lambda| \, [-|x|, |x|] \quad \text{and hence} \quad T([-|\lambda x|, |\lambda x|]) = |\lambda| \, T([-|x|, |x|]).$$

Consequently, the relative compactness is preserved under the scalar multiplication. Hence, $\mathcal{C}(T)$ is an order ideal in the algebraic sense.

Finally, $\mathcal{C}(T)$ is norm closed. Let $x_n \to x$ in norm with $x_n \in \mathcal{C}(T)$. Fix $\varepsilon > 0$ and choose N such that

$$\|x - x_N\| < \frac{\varepsilon}{2\|T\|}.$$

Set $y := |x| \wedge |x_N|$. Then, $0 \le y \le |x|$ and

$$[-y, y] \subseteq [-|x_N|, |x_N|],$$

and hence, $T([-y, y]) \subseteq T([-|x_N|, |x_N|])$ is relatively compact, and therefore, it is totally bounded. Now, take any $z \in [-|x|, |x|]$. By Lemma 3.1, we can write $z = z_1 + z_2$ with $z_1 \in [-y, y]$ and $\|z_2\| \le \||x| - y\|$. Moreover,

$$\||x| - y\| = \big\|(|x| - |x_N|)^+\big\| \le \big\||x| - |x_N|\big\| \le \|x - x_N\|,$$

and hence, $\|z_2\| < \varepsilon/(2\|T\|)$. Consequently,

$$Tz = Tz_1 + Tz_2 \in T([-|x_N|, |x_N|]) + \varepsilon B_Y.$$

Since $T([-|x_N|, |x_N|])$ is totally bounded, its ε-neighbourhood is also totally bounded, and therefore $T([-|x|, |x|])$ is totally bounded. Hence, $T([-|x|, |x|])$ is relatively compact, which shows that $x \in \mathcal{C}(T)$.

Therefore, $\mathcal{C}(T)$ is a norm closed order ideal of E. □

Remark 3.1. *The operator T is AM-compact on E if and only if $\mathcal{C}(T) = E$. Moreover, if T is compact then $\mathcal{C}(T) = E$. However, in general, $\mathcal{C}(T)$ may be trivial (e.g. for $T = \mathrm{Id}_{L^1[0,1]}$, see Example 4.2).*

The Core Band in E^a

Definition 3.2. *Let $T : E \to Y$ be a bounded operator. The* core band *of T in E^a is defined as the set*

$$\mathfrak{B}_T := \mathcal{C}(T) \cap E^a = \Big\{ x \in E^a : \ T([-|x|, |x|]) \text{ is relatively compact in } Y \Big\}.$$

Theorem 3.1. *Let $T : E \to Y$ be bounded.*

(a). *$\mathfrak{B}_T$ is an order ideal in E^a.*

(b). *$\mathfrak{B}_T$ is order closed in E^a; hence $\mathfrak{B}_T$ is a band in E^a.*

(c). *The restriction $T|_{\mathfrak{B}_T} : \mathfrak{B}_T \to Y$ is AM-compact.*

(d). *If $\mathcal{B} \subseteq E^a$ is a band and $T|_{\mathcal{B}}$ is AM-compact, then $\mathcal{B} \subseteq \mathfrak{B}_T$. In particular, $\mathfrak{B}_T$ is the largest band of E^a on which T is AM-compact.*

Proof. If $T = 0$, then $\mathfrak{B}_T = E^a$ and all assertions are clear. Let us assume henceforth that $\|T\| > 0$.

(a). Since $\mathfrak{B}_T = \mathcal{C}(T) \cap E^a$ and $\mathcal{C}(T)$ is an order ideal of E (see Proposition 3.1), $\mathfrak{B}_T$ is an order ideal of E^a.

(b). Let $0 \le x_\alpha \uparrow x$ in E^a with $x_\alpha \in \mathfrak{B}_T$. Order continuity of the norm on E^a yields $\|x - x_\alpha\| \to 0$. Let us fix $\varepsilon > 0$ and choose α such that

$$\|x - x_\alpha\| < \frac{\varepsilon}{2\|T\|}.$$

Since $T([-x_\alpha, x_\alpha])$ is relatively compact, it is totally bounded. Let us choose $y_1, \ldots, y_n \in Y$ such that

$$T([-x_\alpha, x_\alpha]) \subseteq \bigcup_{i=1}^{n} B(y_i, \frac{\varepsilon}{2}).$$

For $z \in [-x, x]$, Lemma 3.1 gives $z = z_1 + z_2$ with $z_1 \in [-x_\alpha, x_\alpha]$ and $\|z_2\| \le \|x - x_\alpha\|$. Let us pick i with

$$\|Tz_1 - y_i\| < \frac{\varepsilon}{2}.$$

Then,

$$\|Tz - y_i\| \le \|Tz_1 - y_i\| + \|T\| \, \|z_2\| < \frac{\varepsilon}{2} + \|T\| \, \|x - x_\alpha\| < \varepsilon.$$

Thus, $T([-x, x])$ is totally bounded and hence relatively compact. Therefore, $x \in \mathfrak{B}_T$.

(c). This part is immediate from the definition of $\mathfrak{B}_T$.

(d). If $x \in \mathcal{B}$, then $[-|x|, |x|] \subseteq \mathcal{B}$ because $\mathcal{B}$ is an ideal. The AM-compactness of $T|_{\mathcal{B}}$ yields relative compactness of $T([-|x|, |x|])$. Hence, $x \in \mathfrak{B}_T$. □

4. A Measure of Noncompactness Encoding

This section provides a quantitative refinement of the preceding localisation. Instead of asking whether $T([-|x|, |x|])$ is relatively compact, one measures its deviation from relative compactness via a standard measure of noncompactness.

Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness on bounded subsets of Y; see [3]. Recall that $\alpha(A) = 0$ if and only if A is relatively compact. Moreover, for bounded subsets $A, B \subseteq Y$ and $\lambda \in \mathbb{R}$,

$$A \subseteq B \Rightarrow \alpha(A) \le \alpha(B), \qquad \alpha(\lambda A) = |\lambda| \, \alpha(A), \qquad \alpha(A + B) \le \alpha(A) + \alpha(B).$$

Definition 4.1. *Let $T : E \to Y$ be a bounded operator. The mapping $\mu_T : E^a \to [0, \infty)$ is defined by*

$$\mu_T(x) := \alpha\big(T([-|x|, |x|])\big), \qquad x \in E^a.$$

Proposition 4.1. *The mapping μ_T is a lattice seminorm on E^a, it is monotone in modulus (that is, $|x| \le |y| \Rightarrow \mu_T(x) \le \mu_T(y)$), and satisfies $\ker(\mu_T) = \mathfrak{B}_T$.*

Proof. Let us fix $x \in E^a$. Since the principal interval $[-|x|, |x|]$ depends only on $|x|$, we have $\mu_T(x) = \mu_T(|x|)$; in particular, μ_T is a lattice functional.

Let $\lambda \in \mathbb{R}$. Using $[-|\lambda x|, |\lambda x|] = |\lambda|\,[-|x|, |x|]$ and the homogeneity $\alpha(\lambda A) = |\lambda|\,\alpha(A)$ of the Kuratowski measure of noncompactness, we obtain

$$\mu_T(\lambda x) = \alpha\big(T([-|\lambda x|, |\lambda x|])\big) = \alpha\big(|\lambda|\,T([-|x|, |x|])\big) = |\lambda|\,\mu_T(x).$$

For $x, y \in E^a$, Lemma 2.1 yields

$$[-|x+y|, |x+y|] \subseteq [-|x|, |x|] + [-|y|, |y|],$$

and hence, by linearity of T,

$$T([-|x+y|, |x+y|]) \subseteq T([-|x|, |x|]) + T([-|y|, |y|]).$$

Monotonicity of α with respect to inclusion and the estimate $\alpha(A+B) \le \alpha(A) + \alpha(B)$ give

$$\mu_T(x+y) = \alpha\big(T([-|x+y|, |x+y|])\big) \le \alpha\big(T([-|x|, |x|]) + T([-|y|, |y|])\big) \le \mu_T(x) + \mu_T(y).$$

Thus, μ_T is a seminorm.

Moreover, if $|x| \le |y|$, then $[-|x|, |x|] \subseteq [-|y|, |y|]$, and hence,

$$\mu_T(x) = \alpha\big(T([-|x|, |x|])\big) \le \alpha\big(T([-|y|, |y|])\big) = \mu_T(y),$$

showing monotonicity in modulus.

Finally, $\mu_T(x) = 0$ if and only if $T([-|x|, |x|])$ is relatively compact (equivalently, $\alpha(T([-|x|, |x|])) = 0$; see [3]). By Definition 3.2, this is equivalent to $x \in \mathfrak{B}_T$. Hence, $\ker(\mu_T) = \mathfrak{B}_T$. □

Proposition 4.2. *For every $x \in E^a$ one has*

$$0 \le \mu_T(x) \le 2\|T\|\,\|x\|.$$

In particular, μ_T is norm continuous on E^a.

Proof. For any bounded $A \subseteq Y$, it holds that $\alpha(A) \le \operatorname{diam}(A)$ (see [3]). Hence,

$$\mu_T(x) = \alpha\big(T([-|x|, |x|])\big) \le \operatorname{diam}\big(T([-|x|, |x|])\big).$$

If $u, v \in [-|x|, |x|]$, then $\|u - v\| \le 2\|x\|$, and hence,

$$\|Tu - Tv\| \le \|T\|\,\|u - v\| \le 2\|T\|\,\|x\|.$$

Taking the supremum over u, v yields the claimed bound. □

Remark 4.1. *If $T : E \to Y$ has finite rank, then $\mu_T(x) = 0$ for every $x \in E^a$, and hence, $\mathfrak{B}_T = E^a$. Indeed, $T([-|x|, |x|])$ is bounded in a finite-dimensional subspace of Y, and therefore relatively compact.*

Example 4.1. *Let $E = Y = \ell^\infty$ and $T = \mathrm{Id}_{\ell^\infty}$. Then $E^a = c_0$ and $\mu_T(x) = 0$ for every $x \in c_0$, so $\mathfrak{B}_T = c_0$. Indeed, for $x \in c_0^+$, the interval $[0, x]$ is norm totally bounded in c_0, and hence relatively compact in ℓ^∞.*

Example 4.2. *Let $E = Y = L^1[0,1]$ and $T = \mathrm{Id}$. Then $\mathfrak{B}_T = \{0\}$ and $\mu_T(f) > 0$ for every $0 \ne f \in L^1[0,1]$. Fix $0 \ne f \in L^1_+[0,1]$. Choose $\delta > 0$ such that $C := \{t \in [0,1] : f(t) \ge \delta\}$ has a positive measure. Since Lebesgue measure is non-atomic on C, there exist measurable sets $(A_n) \subseteq C$ with $\lambda(A_n) = \lambda(C)/2$ and $\lambda(A_n \triangle A_m) = \lambda(C)/2$ for $n \ne m$. Put $u_n := \delta\,\mathbf{1}_{A_n}$. Then $0 \le u_n \le f$, so $u_n \in [0, f]$, and for $n \ne m$,*

$$\|u_n - u_m\|_1 = \delta\,\lambda(A_n \triangle A_m) = \delta\,\frac{\lambda(C)}{2}.$$

Hence, $[0, f]$ (and therefore, $[-f, f]$) is not totally bounded. Consequently, $\alpha([-f, f]) > 0$ and $\mu_T(f) > 0$.

5. Interface With LW-Compactness

The restriction principle identifies LW-compactness of T with AM-compactness of its restriction to E^a, and the core band provides an equivalent domain-side formulation.

Theorem 5.1. *Let $T : E \to Y$ be a bounded operator. The following statements are equivalent:*

(a). *T is LW-compact;*

(b). *$T|_{E^a} : E^a \to Y$ is AM-compact;*

(c). *$\mathfrak{B}_T = E^a$;*

(d). *$\mu_T(x) = 0$ for all $x \in E^a$.*

Proof. Proposition 2.1 gives **(a)**$\Leftrightarrow$**(b)**. By definition, **(b)** holds if and only if $T([-|x|, |x|])$ is relatively compact for every $x \in E^a$, i.e. if and only if $\mathfrak{B}_T = E^a$, which is **(c)**. Finally, Proposition 4.1 yields $\ker(\mu_T) = \mathfrak{B}_T$, and therefore, **(c)**$\Leftrightarrow$**(d)**. $\square$

Remark 5.1. *If $E^a = E$ (equivalently, the norm of E is order continuous), then T is LW-compact if and only if T is AM-compact, and this is also equivalent to $\mathfrak{B}_T = E$ and $\mu_T \equiv 0$ on E.*

Proposition 5.1. *Let $T : E \to Y$, $S : Y \to Z$ and $T_1, T_2 : E \to Y$ be bounded operators, where Z is a Banach space. Then, for every $x \in E^a$,*

$$\mu_{S\circ T}(x) \le \|S\|\, \mu_T(x) \quad \textit{and} \quad \mu_{T_1+T_2}(x) \le \mu_{T_1}(x) + \mu_{T_2}(x).$$

In particular,

$$\mathfrak{B}_T \subseteq \mathfrak{B}_{S\circ T} \quad \textit{and} \quad \mathfrak{B}_{T_1} \cap \mathfrak{B}_{T_2} \subseteq \mathfrak{B}_{T_1+T_2}.$$

Proof. Let us fix $x \in E^a$ and set $I_x := [-|x|, |x|] \subseteq E$.

Since $S \circ T(I_x) = S\big(T(I_x)\big)$ and the Kuratowski measure of noncompactness satisfies $\alpha(S(A)) \le \|S\|\, \alpha(A)$ for every bounded $A \subseteq Y$ (see [3]), we obtain

$$\mu_{S\circ T}(x) = \alpha\big((S \circ T)(I_x)\big) = \alpha\big(S(T(I_x))\big) \le \|S\|\, \alpha\big(T(I_x)\big) = \|S\|\, \mu_T(x).$$

Using linearity, we have

$$(T_1 + T_2)(I_x) = \{T_1(u) + T_2(u) : u \in I_x\} \subseteq T_1(I_x) + T_2(I_x).$$

By the monotonicity of α under inclusion and subadditivity $\alpha(A + B) \le \alpha(A) + \alpha(B)$ for bounded sets (see [3]), we have

$$\mu_{T_1+T_2}(x) = \alpha\big((T_1 + T_2)(I_x)\big) \le \alpha\big(T_1(I_x) + T_2(I_x)\big) \le \alpha\big(T_1(I_x)\big) + \alpha\big(T_2(I_x)\big) = \mu_{T_1}(x) + \mu_{T_2}(x).$$

Finally, we discuss inclusions of core bands. If $x \in \mathfrak{B}_T$ then $\mu_T(x) = 0$, and hence,

$$\mu_{S\circ T}(x) \le \|S\|\, \mu_T(x) = 0,$$

Thus, we have $x \in \mathfrak{B}_{S\circ T}$ and $\mathfrak{B}_T \subseteq \mathfrak{B}_{S\circ T}$. Similarly, if $x \in \mathfrak{B}_{T_1} \cap \mathfrak{B}_{T_2}$ then $\mu_{T_1}(x) = \mu_{T_2}(x) = 0$, and hence, $\mu_{T_1+T_2}(x) \le 0$. Therefore, $x \in \mathfrak{B}_{T_1+T_2}$. $\square$

Remark 5.2. *LW-compactness can be strictly weaker than AM-compactness: for $E = \ell^\infty$, Example 4.1 shows that $\mathrm{Id}_{\ell^\infty}$ is LW-compact (since $E^a = c_0$ and $\mathfrak{B}_T = E^a$) but not AM-compact on ℓ^∞. Moreover, even for compact (and hence, LW-compact) operators, one need not to have $T(E^a) \subseteq E^a$; see Example 5.1.*

Example 5.1. *Let $E = \ell^\infty$, so $E^a = c_0$, and define $T : \ell^\infty \to \ell^\infty$ by*

$$T(x) = x_1 \mathbf{1} \qquad (x = (x_n),\ \mathbf{1} = (1, 1, 1, \dots)).$$

Then, T has rank one and is compact (and hence, LW-compact). However, $T(e_1) = \mathbf{1} \notin c_0$, and hence, $T(E^a) \nsubseteq E^a$.

6. Conclusion

We introduced, for a bounded operator $T : E \to Y$, the core ideal $\mathcal{C}(T)$ of all vectors $x \in E$ for which the image of the principal order interval $[-|x|, |x|]$ is relatively compact in Y. Intersecting $\mathcal{C}(T)$ with the order-continuous ideal E^a yields the core band $\mathfrak{B}_T \subseteq E^a$, which is the largest band of E^a on which the restriction $T|_{E^a}$ is AM-compact (see Section 3). To quantify this localisation, we associated with T a lattice seminorm μ_T on E^a, defined by the Kuratowski measure of noncompactness of $T([-|x|, |x|])$ (see Section 4). The kernel of μ_T coincides with $\mathfrak{B}_T$, so the core band can be recovered numerically from the behaviour of T on principal intervals. Finally, using the restriction characterisation of LW-compact operators, we showed that T is LW-compact if and only if $\mathfrak{B}_T = E^a$, equivalently $\mu_T \equiv 0$ on E^a (see Section 5).

It seems natural to pursue several related questions. For instance, one may study localisation and quantitative cores for other compactness-type operator classes (e.g. DW-compact or b-AM-type classes), or investigate how $\mathfrak{B}_T$ behaves under order domination of positive operators.

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