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Research Article

Certain incomplete sums of reciprocals of the binomial coefficients

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Abstract

Incomplete binomial sums are sums taken over a range less than their complete coefficient range. In this paper, two incomplete analogues of the following well-known identity for the sum of the reciprocals of the binomial coefficients are deduced:

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}.$$

Combining the obtained identities and some previously published results, an elegant combinatorial identity is also derived.

Keywords: binomial coefficients; binomial sums; combinatorial identities; finite sum; generating function.

2020 Mathematics Subject Classification: 11F67, 11F99, 11B83, 33B15.

1. Introduction

In 1981, Rockett [11] proved the following identity: For any nonnegative integer n,

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}. \tag{1}$$

This identity was the subject of Problem 1 in the afternoon session of the 1958 Putnam Examination and then recorded first by Comtet [3, Exercise 15, p. 294]; also, see [6, Exercise 5.100, pp. 542–543]. In fact, as stated in [12], the history of this identity goes back to 1947. This identity is entry (2.25) in Gould's collection [5], where a paper by Staver [13] is quoted. Formula (1) attracted the attention of several mathematicians, who published various new proofs and generalizations; for instance, Pla [10] used generating functions, Trif [15] and Sury [14] used gamma-beta integrals, Batır and Sofo [2], and Batır and Chen [1] employed linear recurrence relations. In 2002, Mansour [9] made a systematic study on the sums of type (1), and derived many related identities using the generating function method.

In this paper, two new identities for the following incomplete variants of identity (1) are established by using the generating function method:

$$\sum_{k=0}^{n} \frac{1}{\binom{2n}{k}} \quad \text{and} \quad \sum_{k=0}^{n} \frac{1}{\binom{3n}{k}}.$$

Applying these identities and some previously published results, a new finite binomial sum identity is also offered. It is worth noting that incomplete binomial sums were studied by Kollár [8], Worsch [16], Dence [4] and Katsuure [7]. For example, Kollár proved that

$$\sum_{k=0}^{n} \binom{2n}{k} H_k = 2^{2n-1}\left(H_{2n} - 2\sum_{k=1}^{2n} \frac{1}{k2^k}\right) + \sum_{k=1}^{n} \frac{1}{(2k)2^{2k}} \binom{2k}{k},$$

where H_n is the nth harmonic number $\sum_{k=1}^{n} \frac{1}{k}$ and $H_0 = 0$. For convenience, throughout this paper the empty sum $\sum_{k=1}^{0}$ is taken to be zero.

This introductory section concludes with a recall of two special functions, namely the gamma function Γ and the beta function $\mathrm{B}(s,t)$, which are defined, respectively, by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t \quad (\Re(z) > 0)$$

and

$$\mathrm{B}(s,t) = \int_0^1 u^{t-1}(1-u)^{s-1} \mathrm{d}u = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \quad (\Re(s), \Re(t) > 0). \tag{2}$$

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2. Main results

In this section, the main findings of the paper are presented.

Theorem 2.1. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{n} \frac{1}{\binom{2n}{k}} = \frac{2n+1}{2^{2n+2}} \sum_{k=1}^{2n+1} \frac{2^k}{k} + \frac{1}{2\binom{2n}{n}}. \tag{3}$$

Proof. We define

$$b_n := \sum_{k=0}^{n} \frac{1}{\binom{2n}{k}}.$$

We first shall show that the following integral representation is valid: For $x \in \mathbb{R}$ with $|x| < 1$,

$$\sum_{n=0}^{\infty} \frac{b_n}{2n+1} x^{2n} = \frac{1}{x^4} \int_0^1 \frac{\mathrm{d}u}{\left(\frac{1}{x^2} - u + u^2\right)\left(\frac{1}{x^2} - u^2\right)}.$$

Using formula (2), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{b_n}{2n+1} x^{2n} &= \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1} \sum_{k=0}^{n} \frac{k!(2n-k)!}{(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} \sum_{k=0}^{n} \Gamma(k+1)\Gamma(2n-k+1) \\
&= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} \sum_{k=0}^{n} \Gamma(2n+2) B(k+1, 2n-k+1) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} B(k+1, 2n-k+1) x^{2n} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \int_0^1 u^k (1-u)^{2n-k} \mathrm{d}u \right) x^{2n}.
\end{aligned} \tag{4}$$

Let $|x| \le r < 1$. Since

$$\left| x^{2n} \sum_{k=0}^{n} u^k (1-u)^{2n-k} \right| \le (n+1) r^{2n}$$

and

$$\sum_{n=0}^{\infty} (n+1) r^{2n} < \infty,$$

the series

$$\sum_{n=0}^{\infty} \left(x^{2n} \sum_{k=0}^{n} u^k (1-u)^{2n-k} \right)$$

is uniformly convergent on $u \in (0,1)$ for $x \in [-r, r]$ by Weierstrass M-test. Thus, we can interchange the order of summation and integration in (4). This leads to

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{b_n}{2n+1} x^{2n} &= \int_0^1 \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} u^k (1-u)^{2n-k} \right) x^{2n} \right) \mathrm{d}u \\
&= \int_0^1 \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} u^k (1-u)^{n-k} \right) (x^2 - x^2 u)^n \right) \mathrm{d}u.
\end{aligned}$$

Using the Cauchy product formula for the product of two convergent series, we have

$$\sum_{n=0}^{\infty} \frac{b_n}{2n+1} x^{2n} = \int_0^1 \left(\sum_{n=0}^{\infty} \left(ux^2(1-u)\right)^n \right) \left(\sum_{n=0}^{\infty} \left(x^2(1-u)^2\right)^n \right) du$$

$$= \int_0^1 \frac{\mathrm{d}u}{(1-ux^2(1-u))(1-x^2(1-u)^2)}$$

$$= \int_0^1 \frac{\mathrm{d}t}{(1-x^2t(1-t))(1-x^2t^2)} \quad (t=1-u)$$

$$= \frac{1}{x^4}\int_0^1 \frac{\mathrm{d}t}{\left(\frac{1}{x^2}-t+t^2\right)\left(\frac{1}{x^2}-t^2\right)}. \tag{5}$$

By partial fraction decomposition, we obtain

$$\frac{1}{x^4\left(\frac{1}{x^2}+t^2-t\right)\left(\frac{1}{x^2}-t^2\right)} = -\frac{1}{2}\frac{2tx^2-x^2-x^2+4}{(x-2)(x+2)\left(t^2x^2-tx^2+1\right)} + \frac{1}{2x(x-2)\left(t-\frac{1}{x}\right)} + \frac{1}{2x(x+2)\left(t+\frac{1}{x}\right)}$$

$$= -\frac{1}{2}\frac{2tx^2-x^2}{\left(x^2-4\right)\left(t^2x^2-tx^2+1\right)} + \frac{1}{2}\frac{1}{\left(t^2x^2-tx^2+1\right)}$$

$$+ \frac{1}{2x(x-2)\left(t-\frac{1}{x}\right)} + \frac{1}{2x(x+2)\left(t+\frac{1}{x}\right)}.$$

Integrating both sides from $t=0$ to $t=1$ yields

$$\frac{1}{x^4}\int_0^1 \frac{\mathrm{d}t}{\left(\frac{1}{x^2}+t^2-t\right)\left(\frac{1}{x^2}-t^2\right)} = -\frac{1}{2(x^2-4)}\int_0^1 \frac{2tx^2-x^2}{t^2x^2-tx^2+1}\mathrm{d}t + \frac{1}{2}\int_0^1 \frac{\mathrm{d}t}{t^2x^2-tx^2+1}$$

$$+ \frac{1}{2x(x-2)}\int_0^1 \frac{\mathrm{d}t}{t-\frac{1}{x}} + \frac{1}{2x(x+2)}\int_0^1 \frac{\mathrm{d}t}{t+\frac{1}{x}}. \tag{6}$$

Clearly, we have

$$\int_0^1 \frac{2tx^2-x^2}{t^2x^2-tx^2+1}\mathrm{d}t = 0, \tag{7}$$

$$\int_0^1 \frac{\mathrm{d}t}{t+\frac{1}{x}} = \log(x+1), \tag{8}$$

$$\int_0^1 \frac{\mathrm{d}t}{t-\frac{1}{x}} = \log(1-x). \tag{9}$$

Using $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ $(|x|<1)$, we obtain

$$\int_0^1 \frac{\mathrm{d}t}{t^2x^2-tx^2+1} = \int_0^1 \sum_{k=0}^{\infty} t^n(1-t)^n x^{2n} dt.$$

Since $|x|<1$, we have $\left|x^{2n}t^n(1-t)^n\right| < \frac{1}{4^n}$ and $\sum_{n=0}^{\infty}\frac{1}{4^n}$ is convergent, the series $\sum_{n=0}^{\infty} x^{2n}t^n(1-t)^n$ is uniformly convergent on $[0,1]$ by Weierstrass M-test. So, we can reverse the order of integration and summation. This gives

$$\int_0^1 \frac{\mathrm{d}t}{t^2x^2-tx^2+1} = \sum_{n=0}^{\infty}\left(\int_0^1 t^n(1-t)^n dt\right)x^{2n}. \tag{10}$$

Employing (2), it follows that

$$\int_0^1 t^n(1-t)^n\mathrm{d}t = \mathrm{B}(n+1,n+1) = \frac{\Gamma^2(n+1)}{\Gamma(2n+2)} = \frac{1}{(2n+1)\binom{2n}{n}}.$$

Thus, from (10), we obtain

$$\int_0^1 \frac{\mathrm{d}t}{t^2x^2-tx^2+1} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n+1)\binom{2n}{n}}. \tag{11}$$

From (8) and (9), it follows that

$$\frac{1}{2x(x-2)}\int_0^1 \frac{dt}{t-\frac{1}{x}}+\frac{1}{2x(x+2)}\int_0^1 \frac{\mathrm{d}t}{t+\frac{1}{x}}=\frac{\log(1-x)}{2x(x-2)}+\frac{\log(1+x)}{2x(x+2)}. \tag{12}$$

Applying the power series expansions, we obtain

$$\frac{\log(1-x)}{2x(x-2)}+\frac{\log(1+x)}{2x(x+2)}=\frac{1}{4}\left(\sum_{n=0}^{\infty}\frac{x^n}{2^n}\right)\left(\sum_{n=0}^{\infty}\frac{x^n}{n+1}\right)+\frac{1}{4}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^n}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^n\right). \tag{13}$$

Making use of the Cauchy product formula for series, from (12) and (13), we deduce

$$\begin{aligned}
&\frac{1}{2x(x-2)}\int_0^1 \frac{\mathrm{d}t}{t-\frac{1}{x}}+\frac{1}{2x(x+2)}\int_0^1 \frac{\mathrm{d}t}{t+\frac{1}{x}}\\
&=\frac{1}{4}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{2^{n-k}}\frac{1}{k+1}\right)x^n+\frac{1}{4}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{2^{n-k}}\frac{1}{k+1}\right)(-x)^n\\
&=\frac{1}{4}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{2^{n-k}}\frac{1}{k+1}\right)(1+(-1)^n)x^n\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{2n}\frac{2^k}{k+1}\right)\frac{x^{2n}}{2^{2n+1}}.
\end{aligned} \tag{14}$$

From (5), (6), (7), (11) and (14), we conclude that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{b_n}{2n+1}x^{2n}&=\sum_{n=0}^{\infty}\frac{x^{2n}}{2(2n+1)\binom{2n}{n}}+\sum_{n=0}^{\infty}\left(\sum_{k=0}^{2n}\frac{2^k}{k+1}\right)\frac{x^{2n}}{2^{2n+1}}\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{2(2n+1)\binom{2n}{n}}+\frac{1}{2^{2n+1}}\sum_{k=0}^{2n}\frac{2^k}{k+1}\right)x^{2n}.
\end{aligned}$$

Comparing the coefficients of x^{2n}, we obtain the desired identity. □

Theorem 2.2. *Let n be a non-negative integer. Then the following identity holds:*

$$\begin{aligned}
\sum_{k=0}^{n}\frac{1}{\binom{3n}{k}}=\frac{1}{\binom{3n}{n}}+\frac{3n+1}{2^{3n}}\Bigg[&\frac{1}{12}\sum_{k=1}^{n}\frac{2^{3k}}{k}+\sum_{k=0}^{n-1}\frac{2^{3k}}{3k+2}+\frac{1}{2}\sum_{k=0}^{n}\frac{2^{3k}}{3k+1}\\
&-\frac{1}{2}\sum_{k=0}^{n}\frac{2^{3k}}{(3k+1)\binom{3k}{k}}+\frac{1}{3}\sum_{k=0}^{n-1}\frac{2^{3k}}{(3k+2)\binom{3k+1}{k}}\Bigg].
\end{aligned} \tag{15}$$

Proof. We define

$$c_n:=\sum_{k=0}^{n}\frac{1}{\binom{3n}{k}}.$$

First, we shall show that for all $x\in\mathbb{R}$ with $|x|<1$, it holds that

$$\sum_{n=0}^{\infty}\frac{c_n}{3n+1}x^{3n}=\int_0^1\frac{\mathrm{d}u}{(1-x^3u^3)(1-x^3u^2(1-u))}.$$

Making use of formula (2), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{c_n}{3n+1}x^{3n}&=\sum_{n=0}^{\infty}\frac{x^{3n}}{3n+1}\sum_{k=0}^{n}\frac{k!(3n-k)!}{(3n)!}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k+1,3n-k+1)x^{3n}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\int_0^1 u^k(1-u)^{3n-k}\mathrm{d}u\right)x^{3n}.
\end{aligned} \tag{16}$$

Let $|x| \leq r < 1$. Since

$$\left| x^{3n} \sum_{k=0}^{n} u^k (1-u)^{3n-k} \right| \leq (n+1) r^{3n}$$

and the series $\sum_{n=0}^{\infty} (n+1) r^{3n}$ is convergent,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} u^k (1-u)^{3n-k} x^{3n}$$

is uniformly convergent by Weierstrass M-test. Thus, we can interchange the order of summation and integration in (16). This yields

$$\sum_{n=0}^{\infty} \frac{c_n}{3n+1} x^{3n} = \int_0^1 \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} u^k (1-u)^{3n-k} \right) x^{3n} \right) \mathrm{d}u = \int_0^1 \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} u^k (1-u)^{n-k} \right) \left(x^3 (1-u)^2 \right)^n \right) \mathrm{d}u.$$

Applying once more the Cauchy product formula for two convergent series, we obtain

$$\sum_{n=0}^{\infty} \frac{c_n}{3n+1} x^{3n} = \int_0^1 \left(\sum_{n=0}^{\infty} \left(x^3 u (1-u)^2 \right)^n \right) \left(\sum_{n=0}^{\infty} \left(x^3 (1-u)^3 \right)^n \right) \mathrm{d}u = \int_0^1 \frac{\mathrm{d}u}{\left(1 - x^3 u^3\right)\left(1 - x^3 u^2 (1-u)\right)}.$$

By partial fraction decomposition, we have

$$\frac{1}{(1-u^3x^3)(1-u^2(1-u)x^3)} = \frac{-2u^2x^3 + ux^3 + x^3 - 4}{(x^3-8)(u^3x^3 - u^2x^3 + 1)} + \frac{2u^2x^3 + ux^3 + 4}{(x^3-8)(u^3x^3 - 1)}$$

or

$$\begin{aligned}
\frac{1}{(1-u^3x^3)(1-u^2(1-u)x^3)} &= \frac{2}{3} \frac{3x^3u^2}{\left(8-x^3\right)\left(1-x^3u^3\right)} - \frac{u}{1-x^3u^3} + \frac{8u}{\left(8-x^3\right)\left(1-x^3u^3\right)} \\
&+ \frac{4}{\left(8-x^3\right)\left(1-x^3u^3\right)} + \frac{1}{1-x^3u^2(1-u)} - \frac{4}{\left(8-x^3\right)\left(1-x^3u^2(1-u)\right)} \\
&+ \frac{2}{3} \frac{3x^3u^2 - 2x^3u}{\left(8-x^3\right)\left(1-x^3u^2(1-u)\right)} - \frac{u}{3\left(1-x^3u^2(1-u)\right)} + \frac{8u}{3\left(8-x^3\right)\left(1-x^3u^2(1-u)\right)}.
\end{aligned}$$

Integrating both sides from $u = 0$ to $u = 1$ gives

$$\begin{aligned}
\int_0^1 \frac{\mathrm{d}u}{(1-x^3u^3)(1-x^3u^2(1-u))} &= \frac{2}{3\left(8-x^3\right)} \int_0^1 \frac{3x^3u^2}{1-x^3u^3} \mathrm{d}u - \int_0^1 \frac{u\mathrm{d}u}{1-x^3u^3} + \frac{8}{8-x^3} \int_0^1 \frac{u\mathrm{d}u}{1-x^3u^3} \\
&+ \frac{4}{8-x^3} \int_0^1 \frac{\mathrm{d}u}{1-x^3u^3} + \int_0^1 \frac{\mathrm{d}u}{1-x^3u^2(1-u)} - \frac{4}{8-x^3} \int_0^1 \frac{\mathrm{d}u}{1-x^3u^2(1-u)} \\
&+ \frac{2}{3\left(8-x^3\right)} \int_0^1 \frac{3x^3u^2 - 2x^3u}{1-x^3u^2(1-u)} \mathrm{d}u - \frac{1}{3} \int_0^1 \frac{u\mathrm{d}u}{1-x^3u^2(1-u)} \\
&+ \frac{8}{3\left(8-x^3\right)} \int_0^1 \frac{u\mathrm{d}u}{1-x^3u^2(1-u)}. \qquad (17)
\end{aligned}$$

Since

$$\frac{4}{8-x^3} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^{3n},$$

$$\frac{1}{1-u^3x^3} = \sum_{n=0}^{\infty} u^{3n} x^{3n}$$

and

$$\frac{1}{1-x^3u^2(1-u)} = \sum_{n=0}^{\infty} u^{2n} (1-u)^n x^{3n},$$

we have the following evaluations:

$$\frac{2}{3(8-x^3)}\int_0^1 \frac{3u^2x^3}{1-u^3x^3}\mathrm{d}u = -\frac{2}{3(8-x^3)}\log\left(1-x^3\right)$$

$$= \frac{1}{12}\left(\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^{3n}\right)\left(\sum_{n=0}^{\infty}\frac{x^{3n+3}}{n+1}\right)$$

$$= \frac{1}{12}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{2^{3k-3n}}{k}\right)x^{3n},$$

$$\int_0^1 \frac{u\mathrm{d}u}{1-u^3x^3} = \sum_{n=0}^{\infty}\frac{x^{3n}}{3n+2},$$

$$\frac{8}{8-x^3}\int_0^1 \frac{u\mathrm{d}u}{1-u^3x^3} = \left(\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^{3n}\right)\left(\sum_{n=0}^{\infty}\frac{x^{3n}}{3n+2}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{2^{3k-3n}}{3k+2}\right)x^{3n},$$

$$\frac{4}{8-x^3}\int_0^1 \frac{\mathrm{d}u}{1-u^3x^3} = \frac{1}{2}\sum_{n=0}^{\infty}(x/2)^{3n}\int_0^1\sum_{n=0}^{\infty}(xu)^{3n}\mathrm{d}u$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(x/2)^{3n}\sum_{n=0}^{\infty}\frac{x^{3n}}{3n+1}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(x/2)^{3n-3k}\frac{x^{3k}}{3k+1} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{2^{3k-3n}}{3k+1}\right)x^{3n},$$

$$\int_0^1 \frac{\mathrm{d}u}{1-x^3u^2(1-u)} = \sum_{n=0}^{\infty}x^{3n}\int_0^1 u^{2n}(1-u)^n\mathrm{d}u = \sum_{n=0}^{\infty}x^{3n}B(2n+1,n+1)$$

$$= \sum_{n=0}^{\infty}\frac{(2n)!n!}{(3n+1)!}x^{3n} = \sum_{n=0}^{\infty}\frac{x^{3n}}{(3n+1)\binom{3n}{n}},$$

$$\frac{4}{8-x^3}\int_0^1 \frac{\mathrm{d}u}{1-x^3u^2(1-u)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^{3n}\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n+1)\binom{3n}{n}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{2^{3k-3n}}{(3k+1)\binom{3k}{k}}\right)x^{3n},$$

$$\int_0^1 \frac{3x^3u^2-2x^3u}{1-x^3u^2(1-u)}\mathrm{d}u = 0,$$

$$\frac{8}{3(8-x^3)}\int_0^1 \frac{u\mathrm{d}u}{1-x^3u^2(1-u)} = \frac{1}{3}\left(\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^{3n}\right)\left(\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n+2)\binom{3n+1}{n}}\right)$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{2^{3k-3n}}{(3k+2)\binom{3k+1}{k}}\right)x^{3n},$$

and finally

$$\frac{1}{3}\int_0^1 \frac{u\mathrm{d}u}{1-x^3u^2(1-u)} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n+2)\binom{3n+1}{n}}.$$

Using these evaluations in (17), we obtain

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}u}{(1-u^3x^3)\,(1-u^2(1-u)x^3)} =& \frac{1}{12}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{2^{3k-3n}}{k}\right)x^{3n} - \sum_{n=0}^{\infty}\frac{x^{3n}}{3n+2}\\ &+\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{2^{3k-3n}}{3k+2}\right)x^{3n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{2^{3k-3n}}{3k+1}\right)x^{3n}\\ &+\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n+1)\binom{3n}{n}} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{2^{3k-3n}}{(3k+1)\binom{3k}{k}}\right)x^{3n}\\ &-\frac{1}{3}\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n+2)\binom{3n+1}{n}} + \frac{1}{3}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{2^{3k-3n}}{(3k+2)\binom{3k+1}{k}}\right)x^{3n},\end{aligned}$$

so that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{c_n}{3n+1}x^{3n} =& \sum_{n=1}^{\infty}\left[\frac{1}{12}\sum_{k=1}^{n}\frac{2^{3k-3n}}{k} - \frac{1}{3n+2} + \sum_{k=0}^{n}\frac{2^{3k-3n}}{3k+2} + \frac{1}{2}\sum_{k=0}^{n}\frac{2^{3k-3n}}{3k+1} + \frac{1}{(3n+1)\binom{3n}{n}}\right.\\ &\left.-\frac{1}{2}\sum_{k=0}^{n}\frac{2^{3k-3n}}{(3k+1)\binom{3k}{k}} - \frac{1}{3}\frac{1}{(3n+2)\binom{3n+1}{n}} + \frac{1}{3}\sum_{k=0}^{n}\frac{2^{3k-3n}}{(3k+2)\binom{3k+1}{k}}\right]x^{3n}.\end{aligned}$$

Comparing the coefficients of x^{3n} in both sides yields

$$\frac{c_n}{3n+1} = \frac{1}{12}\sum_{k=1}^{n}\frac{2^{3k-3n}}{k} + \sum_{k=0}^{n-1}\frac{2^{3k-3n}}{3k+2} + \frac{1}{2}\sum_{k=0}^{n}\frac{2^{3k-3n}}{3k+1} + \frac{1}{(3n+1)\binom{3n}{n}} - \frac{1}{2}\sum_{k=0}^{n}\frac{2^{3k-3n}}{(3k+1)\binom{3k}{k}} + \frac{1}{3}\sum_{k=0}^{n-1}\frac{2^{3k-3n}}{(3k+2)\binom{3k+1}{k}}.$$

Now, the required identity immediately follows. □

Theorem 2.3. *Let* $n \in \mathbb{N}$. *Then*

$$\sum_{k=1}^{n}\frac{2^{n+k}}{k\binom{n+k}{k}} = \sum_{k=1}^{n}\frac{2^k}{k}.$$

Proof. By [1, Theorem 1] (with $x=1$ and $s=n$), we have

$$\sum_{k=0}^{n}\frac{1}{\binom{2n}{k}} = \frac{2n+1}{2^{n+2}}\left(\sum_{k=0}^{n}\frac{2^{k+1}}{(k+1)\binom{n+k+1}{k+1}} + \sum_{k=0}^{n}\frac{2^{k+1}}{n+k+1}\right).$$

Setting $k+1=k'$ in both sums on the right-hand side and deleting the prime, and then applying Theorem 2.1, we obtain

$$\frac{2n+1}{2^{2n+2}}\sum_{k=1}^{2n+1}\frac{2^k}{k} + \frac{1}{2\binom{2n}{n}} = \frac{2n+1}{2^{n+2}}\left(\sum_{k=1}^{n+1}\frac{2^k}{k\binom{n+k}{k}} + \sum_{k=1}^{n+1}\frac{2^k}{n+k}\right).$$

Since

$$\frac{2n+1}{2^{n+2}}\sum_{k=1}^{n+1}\frac{2^k}{n+k} = \frac{2n+1}{2^{2n+2}}\sum_{k=1}^{2n+1}\frac{2^k}{k} - \frac{2n+1}{2^{2n+2}}\sum_{k=1}^{n}\frac{2^k}{k},$$

we obtain, after simplifying, the following identity:

$$\sum_{k=1}^{n}\frac{2^k}{k\binom{n+k}{k}} = \frac{1}{2^n}\sum_{k=1}^{n}\frac{2^k}{k} + \frac{2^{n+1}}{(2n+1)\binom{2n}{n}} - \frac{2^{n+1}}{(n+1)\binom{2n+1}{n+1}}.$$

Since

$$\frac{2^{n+1}}{(2n+1)\binom{2n}{n}} - \frac{2^{n+1}}{(n+1)\binom{2n+1}{n+1}} = 0,$$

the required identity follows. □

Remark 2.1. Using the following known identity (see [2, Corollary 1]), we can deduce alternating forms of the formulas (3) and (15):

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n+s}{k}} = \frac{n+s+1}{n+s+2}\left(\frac{(-1)^n}{\binom{n+s+1}{n+1}}+1\right), \tag{18}$$

Setting $s = n$ and $s = 2n$ in (18), we immediately obtain

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{2n}{k}} = \frac{(-1)^n}{2\binom{2n}{n}} + \frac{2n+1}{2n+2}$$

and

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{3n}{k}} = \frac{3n+1}{3n+2}\left(\frac{(-1)^n}{\binom{3n+1}{n+1}}+1\right),$$

respectively.

3. Conclusion

In this work, several contributions have been made to the study of finite sums of reciprocals of binomial coefficients. Particularly, two incomplete sums for the identity (1) have been established using the generating functions method and the beta-gamma integrals. In addition, a new combinatorial identity, stated in Theorem 2.3, is derived. It is believed that this new identity will offer a fresh perspective on sums involving reciprocals of binomial coefficients. Indeed, using this formula, the identities (1) and (3) can be written as follows:

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = 1 + \frac{n+1}{2}\sum_{k=1}^{n} \frac{2^k}{k\binom{n+k}{k}} \tag{19}$$

and

$$\sum_{k=0}^{n} \frac{1}{\binom{2n}{k}} = \frac{1}{2} + \frac{2n+1}{4}\sum_{k=1}^{2n} \frac{2^k}{k\binom{2n+k}{k}} + \frac{1}{2\binom{2n}{n}}.$$

Furthermore, using the relation

$$\frac{n+1}{k+1}\binom{n}{k} = \binom{n+1}{k+1},$$

formula (19) can be written in the following elegant form:

$$\sum_{k=1}^{n+1} \frac{1}{k\binom{n+1}{k}} = \frac{1}{n+1} + \frac{1}{2}\sum_{k=1}^{n} \frac{2^k}{k\binom{n+k}{k}}.$$

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