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Research Article

Exact sequences of direct systems of hypergroups

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Abstract

This paper investigates the categorical properties of direct limits in the context of canonical hypergroups. Fundamental results are established concerning the construction and the preservation of exactness for direct limits of directed systems of hypergroups. The main contributions are as follows: (1) a rigorous proof of the universal property of direct limits in the category of canonical hypergroups, demonstrating their existence via an explicit quotient construction; (2) a theorem showing that exact sequences of directed systems induce exact sequences at the limit level; and (3) the development of essential commutative diagram techniques for hypergroup homomorphisms in directed systems. These results extend classical algebraic constructions to the hypergroup setting, where the multivalued nature of operations requires a careful treatment of equivalence classes and compatibility conditions.

Keywords: direct systems; direct limits; exact sequence; morphism.

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1. Introduction

Classical algebraic structures, such as groups and rings, are characterized by binary operations in which the composition of any two elements yields a single, uniquely determined element of the set. Algebraic hyperstructures represent a significant generalization of this foundational concept. In a hyperstructure, the composition of two elements, instead of producing a single element, results in a nonempty set of elements. This fundamental departure from classical algebra provides a powerful framework for modeling more complex and nondeterministic relationships.

The formal theory of hyperstructures was inaugurated by the French mathematician F. Marty in 1934, during the Eighth Congress of Scandinavian Mathematicians, where he provided the first rigorous definition of a hypergroup [7]. Marty not only established the theoretical foundations of the hyperstructures theory but also demonstrated its applicability in several mathematical contexts, including:

1. the analysis of certain quotient structures in group theory,
2. the study of algebraic functions and their decomposition,
3. properties of rational fractions, where the behavior of functions under composition may lead to set-valued outcomes.

The concept of direct limits, also referred to as inductive limits, occupies a fundamental position in various branches of mathematics, including algebra, category theory, and topology. Direct limits provide a systematic framework for constructing a unified object from a collection of mathematical structures interconnected through a directed system. This construction is particularly valuable when addressing infinite processes or when unifying structures defined in a localized manner [2]. The utility of the direct limit is rooted in its universal property and its behavior as an exact functor in relevant categories. These characteristics render it an indispensable component of local-to-global arguments throughout modern mathematics [1–3, 6, 8].

Rakhsh Khorshid and Ostadhadi-Dehkordi [9] introduced the concept of the direct limit on G-sets of n-ary semihypergroups, which represents a nonadditive modification of the classical construction in module theory. This notion holds significant importance in homological algebra, and the authors of [9] established several key properties along with illustrative examples. In addition, Davvaz and Ghadiri [5] investigated the category of H_v-modules, demonstrating that direct limits invariably exist within this framework.

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This paper investigates the categorical properties of direct limits within the framework of canonical hypergroups. We establish foundational results concerning the construction and the preservation of exactness for direct limits of directed systems of hypergroups. The primary contributions of this work are as follows:

1. **Existence and Universal Property:**
   A rigorous proof is provided for the existence of direct limits in the category of canonical hypergroups by means of an explicit quotient construction, verifying their universal property.

2. **Exactness Preservation:**
   It is shown that exact sequences of directed systems induce exact sequences at the limit level, establishing that $\operatorname{Im}\widehat{f} = \ker\widehat{g}$ for the induced morphisms.

2. Preliminaries

We begin by reviewing some fundamental concepts and classical results pertaining to hypergroups.

A hypergroupoid $(H, \circ)$ is defined as a nonempty set H equipped with a hyperoperation $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of all nonempty subsets of H. For $x \in H$ and nonempty subsets $A, B \subseteq H$, the following properties hold:

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, \quad A \circ x = A \circ \{x\}, \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

We call $(H, \circ)$ a semihypergroup if for all $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$, which means that

$$\bigcup_{u\in y\circ z} x \circ u = \bigcup_{v\in x\circ y} v \circ z.$$

Also, $(H, \circ)$ is said to be a quasihypergroup if for $x \in H$, it holds that $x \circ H = H \circ x = H$. A hypergroupoid $(H, \circ)$ that is both a semihypergroup and a quasihypergroup is called a hypergroup [10].

Example 2.1. *Let $\mathbb{R}$ denote the set of real numbers. Then, $\mathbb{R}$ forms a semihypergroup with respect to the following hyperoperation:*

$$x \circ y = \{z \in \mathbb{R} : n \leq z < n + 1\},$$

where $x, y \in \mathbb{R}$ and $n = max\{[x], [y]\}$.

Example 2.2. *Let $(\mathbb{Z}, +)$ be the ring of integers. Then,$(\mathbb{Z}, \oplus)$ is a hypergroup with following hyperoperation:*

$$n \oplus n = \{t \in \mathbb{Z} : t \leq n\} \ \text{ and } \ n \oplus m = max\{n, m\},$$

where $n, m \in \mathbb{Z}$ and $n \neq m$.

Example 2.3. *Let $(G, \cdot)$ be a group and $\{A_g\}_{g\in G}$ be a collection of nonempty pairwise disjoint sets. Then, $H = \bigcup_{g\in G} A_g$ forms a hypergroup under the hyperoperation defined by $x \otimes y = A_g, \quad g = g_1 \cdot g_2, \quad x \in A_{g_1}$, and $y \in A_{g_2}$. We denote this hypergroup by $H[G]$.*

Definition 2.1. *Let$(G_1, \circ)$ and $(G_2, *)$ be two hypergroups. Then, a map $\varphi : G_1 \longrightarrow G_2$ is called a hypergroup homomorphism if for every $x, y \in G_1$, the following equality $\varphi(x \circ y) = \varphi(x) * \varphi(y)$ holds, where the image of the set $x \circ y$ under φ is given as follows:*

$$\varphi(x \circ y) = \bigcup_{z\in x\circ y} \varphi(z).$$

Definition 2.2 (see [4])**.** *A hypergroup $(H, +)$ is called canonical if*

(1) *it is commutative*;

(2) *it has a scalar identity (also called scalar unit), which means that $\exists \ 0 \in H : \forall x \in H, \ \ x \circ 0 = 0 \circ x = x$;*

(3) *every element has a unique inverse, which means that for all $x \in H$, there exists a unique $-x \in H$, such that*

$$0 \in x + (-x) \cap (-x) + x;$$

(4) *it is reversible, which means that if $x \in y + z$, then there exist the inverses $-y$ of y and $-z$ of z , such that $z \in -y \circ x$ and $y \in x \circ -z$.*

Clearly, the identity of a canonical hypergroup is unique. The following elementary facts follow from the axioms:

(1) For every $x \in H$, $-(-x) = x$.

(2) For every $x, y \in H$, $-(x+y) = -x - y$, where $-A = \{-a : a \in A\}$.

Definition 2.3 (see [10]). *A map $\varphi : (G_1, +) \longrightarrow (G_2, +)$ between two canonical hypergroups is said to be a homomorphism if it satisfies the following conditions:*

(1) *for every $x, y \in G_1$, $\varphi(x+y) = \varphi(x) + \varphi(y)$;*

(2) $\varphi(0) = 0$.

3. Direct limit of hypergroups

A partially ordered set I is said to be a directed set if for each pair $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Definition 3.1. *Let $\{G_i\}_{i \in I}$ be a family of hypergroups indexed by a directed set I. If, for every pair $i, j \in I$ with $i \leq j$, there exists a homomorphism $\varphi_{ij} : G_i \longrightarrow G_j$ such that the following axioms are satisfied, then the pair (G_i, φ_{ij}) is called a direct system over the directed set I:*

(1) *For every $i \in I$, the homomorphism φ_{ii} is the identity map on G_i.*

(2) *For every $i \leq j \leq k$, the homomorphisms satisfy*

$$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}.$$

Example 3.1. *Let $I = (\mathbb{N}, \leq)$ be the set of natural numbers equipped with the standard ordering ($n \leq m$, if n divides m). Then, for each $n \in \mathbb{N}$, $G_n = \mathbb{Z}_n$ is a hypergroup with the following hyperoperation:*

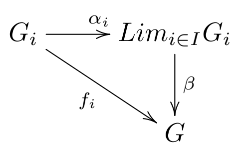
$$[a] \oplus [b] = \{[a+b], [a+b+1]\}.$$

For $n \mid m$, we define $\varphi_{nm} : G_n \longrightarrow G_m$ by $\varphi_{nm}([a]_n) = [a]_m$. Hence, (G_n, φ_{nm}) is a direct system.

Definition 3.2. *Let (G_i, φ_{ij}) be a direct system over a directed set I. Then the direct limit of this system, denoted by $Lim_{i \in I} G_i$, is a hypergroup equipped with a family of homomorphisms $\alpha_i : G_i \longrightarrow Lim_{i \in I} G_i$ satisfying the following conditions:*

(1) *$\alpha_j \circ \varphi_{ij} = \alpha_i$ for all $i \leq j$;*

(2) *for every hypergroup G and every family of homomorphisms $f_i : G_i \longrightarrow G$ such that $f_i = f_j \circ \varphi_{ij}$ whenever $i \leq j$, there exists a unique homomorphism $\beta : Lim_{i \in I} G_i \longrightarrow G$ making the following diagram commute for all $i \leq j$:*



Let (G_i, φ_{ij}) be a direct system of hypergroups and G_∞ be the disjoint union of G_i. Then, we define a binary relation "$\equiv$" on G_∞ as follows:

$$g_i \equiv g_j \quad \Longleftrightarrow \quad \exists \ k \geq i, j : \quad \varphi_{ik}(g_i) = \varphi_{jk}(g_j),$$

where $g_i \in G_i$ and $g_j \in G_j$. It is straightforward to verify that "$\equiv$" is an equivalence relation. We denote by $\widehat{x}$ the equivalence class of an element $x \in G_\infty$. Let $\widehat{G}$ denote the set of all equivalence classes. For every $g_i \in G_i$ and $g_j \in G_j$, we define

$$\widehat{g_i} \odot \widehat{g_j} = \{\widehat{x} : x \in g_k \circ g_k' : g_k = \varphi_{ik}(g_i), \ \ g_k' = \varphi_{jk}(g_j), \ \ k \geq i, j\}.$$

Proposition 3.1. *Let (G_i, φ_{ij}) be a direct system of hypergroups and $a_j = \varphi_{ij}(a_i)$. Then, $\widehat{a}_i = \widehat{a}_j$.*

Proof. For $j \in I$, we have $a_j = \varphi_{jj}(a_j) = \varphi_{ij}(a_i)$, and hence, $\widehat{a}_i = \widehat{a}_j$. □

Proposition 3.2. *Let (G_i, φ) be a direct system of hypergroups and $b \in G_i$. Then, for every $i \leq j$, $\widehat{\varphi_{ij}(b)} = \widehat{b}$, where $\widehat{b}$ denotes the equivalence class of b in the direct limit of $\widehat{G}$.*

Proof. Suppose $x \in \widehat{\varphi_{ij}(b)}$, where $x \in G_t$. Then, $x \equiv \varphi_{ij}(b)$. Now, consider an index $k \geq j, t$. Applying the connecting homomorphism φ_{jk}, we have $\varphi_{tk}(x) = \varphi_{jk}(\varphi_{ij}(b)) = \varphi_{ik}(b)$. Hence, we conclude that $\widehat{x} = \widehat{b}$. Therefore, $\widehat{\varphi_{ij}(b)} = \widehat{b}$. □

Proposition 3.3. *The relation $\odot$ is well-defined on $\widehat{G}$.*

Proof. Suppose $\widehat{a_i} = \widehat{b_s}$, where $x \in G_t$ and $\widehat{a_j'} = \widehat{b_t'}$. Then, by Definition 3.2, there exist indices, $k_1 \geq i, s$ and $k_2 \geq j, t$ such that $\varphi_{ik_1}(a_i) = \varphi_{sk_1}(b_s)$ and $\varphi_{jk_2}(a_j') = \varphi_{tk_2}(b_t')$. Let $\widehat{x} \in \widehat{a_i} \odot \widehat{a_j'}$. Then, for some $k_3 \geq i, j$, we have $x \in g_{k_3} \circ g_{k_3}'$, $g_{k_3} = \varphi_{ik_3}(a_i)$, and $g_{k_3}' = \varphi_{jk_3}(a_j')$. For every $k \geq k_1, k_2, k_3$, the following equivalences hold:

$$\begin{aligned}
x \in g_{k_3} \circ g_{k_3}' \quad &\Longleftrightarrow \quad x \in \varphi_{ik}(a_i) \circ \varphi_{jk}(a_j') \\
&\Longleftrightarrow x \in (\varphi_{k_1k} \circ \varphi_{ik_1}(a_i)) \circ (\varphi_{k_1k} \circ \varphi_{jk_1}(a_j')) \\
&\Longleftrightarrow x \in (\varphi_{k_1k} \circ \varphi_{sk_1}(b_s)) \circ (\varphi_{k_2k} \circ \varphi_{jk_2}(b_t')) \\
&\Longleftrightarrow x \in \varphi_{sk}(b_s) \circ \varphi_{jk}(b_t') \\
&\Longleftrightarrow x \in \widehat{b_s} \circ \widehat{b_t'}.
\end{aligned}$$

Therefore, $\widehat{a_i} \odot \widehat{a_j'} = \widehat{b_s} \odot \widehat{b_t'}$, and hence, $\odot$ is a well-defined hyperoperation. □

Proposition 3.4 (Direct Limit Semihypergroup)**.** *The hyperoperation $\odot$ is associative and $(\widehat{G}, \odot)$ is a semihypergroup.*

Proof. Suppose that $\widehat{x_i}, \widehat{y_j}, \widehat{z_t}$ are arbitrary elements of $\widehat{G}$. We will show that the hyperoperation $\odot$ is associative, i.e.,

$$(\widehat{x} \odot \widehat{y}) \odot \widehat{z} = \widehat{x} \odot (\widehat{y} \odot \widehat{z}).$$

Let $\widehat{x} \in (\widehat{x_i} \odot \widehat{y_j}) \odot \widehat{z_t}$. Then, by the definition of the hyperoperation $\odot$,

$$\exists \ x_{k_1} \in g_{k_1} \circ g_{k_1}' : \widehat{x} \in \widehat{x}_{k_1} \odot \widehat{z}_t,$$

for every $k_1 \geq i, j$ and $g_{k_1} = \varphi_{ik_1}(x_i)$, $g_{k_1}' = \varphi_{jk_1}(y_j)$. Hence, there exists $k_2 \geq k, t$ such that $x \in h_{k_2} \circ h_{k_2}'$, where

$$h_{k_2} = \varphi_{k_1k_2}(x_{k_1}), \quad h_{k_2}' = \varphi_{tk_2}(z_t).$$

Now, let $k \geq k_1, k_2, t$. Then we can apply the connecting morphisms to work at the common index k. The following equivalence holds, relying on the compatibility properties of the direct system:

$$x \in h_{k_2} \circ h_{k_2}' \Longleftrightarrow x \in h_k \circ h_k'.$$

We now show the equivalence with

$$\begin{aligned}
\widehat{x} \in (\widehat{x_i} \odot \widehat{y_j}) \odot \widehat{z_t} \quad &\Longleftrightarrow x \in \varphi_{k_1k}(x_{k_1}) \circ \varphi_{tk}(z_t) \\
&\Longleftrightarrow x \in \varphi_{k_1k} \circ (\varphi_{ik_1}(x_i) \circ \varphi_{jk}(y_j)) \circ \varphi_{tk}(z_t) \\
&\Longleftrightarrow x \in (\varphi_{k_1k} \circ \varphi_{ik_1}(x_i) \circ \varphi_{jk}(y_j)) \circ \varphi_{tk}(z_t) \\
&\Longleftrightarrow x \in (\varphi_{ik}(x_i) \circ \varphi_{jk}(y_j)) \circ \varphi_{tk}(z_t) \\
&\Longleftrightarrow x \in (\varphi_{ik_1}(x_i) \circ \varphi_{jk_1}(y_j)) \circ \varphi_{tk}(z_t) \\
&\Longleftrightarrow x \in \varphi_{ik_1}(x_i) \circ (\varphi_{jk_1}(y_j) \circ \varphi_{tk}(z_t)) \\
&\Longleftrightarrow x \in \widehat{x_i} \odot (\widehat{y_j} \odot \widehat{z_t}).
\end{aligned}$$

Thus, $\odot$ is associative, and therefore,$(\widehat{G}, \odot)$ is a semihypergroup. □

Proposition 3.5 (Direct Limit Quasihypergroup)**.** *Let (G_i, φ_{ij}) be a direct system of hypergroups. Then, $(\widehat{G}, \odot)$ is a quasihypergroup.*

Proof. Suppose that $\widehat{x_j}, \widehat{x_i} \in \widehat{G}$ are arbitrary elements. We will prove that the reproduction axiom holds in $(\widehat{G}, \odot)$, i.e., $\widehat{x_i} \odot \widehat{G} = \widehat{G}$. Let $k \geq i, j$ be a common index. Then, we define $a_k = \varphi_{ik}(x_i)$ and $b_k = \varphi_{jk}(x_j)$. Since $a_k, b_k \in G_k$ and G_k is a hypergroup, by the reproduction axiom, $b_k \in G_k = a_k \circ G_k$. Thus, $b_k \in a_k \circ x$ for some $x \in G_k$. Hence,

$$\widehat{x_j} = \widehat{b_k} \in \widehat{a_k} \odot \widehat{x} \in \widehat{x_i} \odot \widehat{G}.$$

Therefore, $\widehat{x}_i \odot \widehat{G} = \widehat{G}$. By a symmetric argument, using the other part of the reproduction axiom in G_k, we have $\widehat{G} \odot \widehat{x}_i = \widehat{G}$. Hence, the reproduction axiom is satisfied, and therefore, $(\widehat{G}, \odot)$ is a quasihypergroup. □

Theorem 3.1. *Let (G_i, φ_{ij}) be a direct system of hypergroups, and let $\widehat{G}$ be its direct limit equipped with the hyperoperation $\odot$. Then, $(\widehat{G}, \odot)$ is a hypergroup.*

Proof. The conclusion follows from Propositions 3.4 and 3.5. □

Theorem 3.2. *Let (G_i, φ_{ij}) be a direct system of hypergroups index by I. Then, $(\widehat{G}, \odot)$ satisfies the universal property of the direct limit.*

Proof. For each $i \in I$, we define the map $\alpha_i : G_i \longrightarrow \widehat{G}$, given by $\alpha_i(a_i) = \widehat{a}_i$. In what follows, we verify the homomorphism property for every $a_i, b_i \in G_i$:

$$\begin{aligned}
\alpha_i(a_i) \odot \alpha_i(b_i) &= \widehat{a}_i \odot \widehat{b}_i \\
&= \{\, \widehat{b}_k : b_k \in a_k \circ b_k,\ a_k = \varphi_{ik}(a_i),\ b_k = \varphi_{ik}(b_i),\ \ k \geq i, j\} \\
&= \{\alpha_k(b_k) : b_k \in a_k \circ b_k,\ a_k = \varphi_{ik}(a_i),\ b_k = \varphi_{ik}(b_i),\ \ k \geq i, j\} \\
&= \{\alpha_k(a_k \circ b_k) :\ a_k = \varphi_{ik}(a_i),\ b_k = \varphi_{ik}(b_i),\ \ k \geq i, j\} \\
&= \alpha_k(\varphi_{ik}(a_i) \circ \varphi_{ik}(b_i)),\ \ k \geq i, j \\
&= \alpha_k(\varphi_{ik}(a_i \circ b_i)) \\
&= \alpha_i(a_i \circ b_i).
\end{aligned}$$

Hence, α_i is a homomorphism. Thus, for every $a_i \in G_i$, we have

$$\alpha_j \circ \varphi_{ij}(a_i) = \alpha_j(\varphi_{ij}(a_i)) = \widehat{\varphi_{ij}(a_i)}.$$

By Proposition 3.1, $\widehat{a_i} = \widehat{\varphi_{ij}(a_i)}$ for $i \leq j$. This implies that $\alpha_j \circ \varphi_{ij} = \alpha_i$. Let H be a hypergroup and $\{f_i : G_i \longrightarrow H,\ i \in I\}$ be a family of homomorphism with $f_i = f_j \circ \varphi_{ij}$. Now, we define $\beta : \widehat{G} \longrightarrow H$ by $\beta(\widehat{a}_i) = f_i(a_i)$. First, we show that β is well-defined. Suppose $\widehat{a}_i = \widehat{a}_j$. Then, there exists $k \geq i, j$, such that $\varphi_{ik}(a_i) = \varphi_{jk}(b_j)$. Then, $f_k(\varphi_{ik}(a_i) = f_k(\varphi_{jk}(b_j))$. Hence, $f_i(a_i) = f_j(a_j)$, and therefore, β is well-defined. Now, let $\widehat{a}_i, \widehat{b}_j \in \widehat{G}$. Then, we have

$$\begin{aligned}
\beta(\widehat{a}_i \odot \widehat{b}_j) &= \{\beta(\widehat{x}) : x \in a_k \circ a_k^{'},\ \ a_k = \varphi_{ik}(a_i),\ \ a_k^{'} = \varphi_{jk}(b_j),\ \ k \geq i, j\} \\
&= \{f_k(x) : x \in a_k \circ a_k^{'},\ \ a_k = \varphi_{ik}(a_i),\ \ a_k^{'} = \varphi_{jk}(b_j),\ \ k \geq i, j\} \\
&= f_k(a_k \circ a_k^{'}) \\
&= f_k(a_k) \circ f_k(a_k^{'}) \\
&= f_k(\varphi_{ik}(a_i)) \circ f_k(\varphi_{jk}(b_j)) \\
&= f_i(a_i) \circ f_j(b_j) \\
&= \beta(\widehat{a}_i) \circ \beta(\widehat{b}_j).
\end{aligned}$$

Hence, β is a homomorphism and $\beta \circ \alpha_i = f_i$. Let $\beta^{'} : \widehat{G} \longrightarrow H$ be a another homomorphism such that $\beta^{'} \circ \alpha_i = f_i$ for every $i \in I$. Then, for $\widehat{a}_i \in \widehat{G}$, we have

$$\beta^{'}(\widehat{a}_i) = \beta^{'}(\alpha_i(a_i)) = (\beta^{'} \circ \alpha_i)(a_i) = f_i(a_i) = (\beta \circ \alpha_i)(a_i) = \beta(\widehat{a}_i).$$

Therefore, $\beta = \beta^{'}$. Consequently, we conclude that $(\widehat{G}, \alpha_i)_{i \in I}$ is the direct limit of the direct system (G_i, φ_{ij}) in the category of hypergroups. □

Let (G_i, φ_{ij}) be a direct system of hypergroups. Then, $(\widehat{G}, \odot)$ is called a direct system hypergroup (DSH, for short)

Definition 3.3. *Let (G_i, φ_{ij}) be a direct system of hypergroups over a directed set I and $(\widehat{G}, \odot)$ be a DSH such that $A \subseteq G_\infty$ and $B \subseteq \widehat{G}$. Then, $\widehat{A} = \{\widehat{x} : x \in A\}$ and $B^{'} = \{x \in G_\infty : \widehat{x} \in B\}$.*

Proposition 3.6. *Let (G_i, φ_{ij}) be a direct system of hypergroups and $B \subseteq \widehat{G}$ be a subsemihypergroup. Then, $B^{'} \cap G_i = \emptyset$ or $B^{'} \cap G_i$ is a subsemihypergroup of G_i for every $i \in I$.*

Proof. Suppose for $i \in I$, $B^{'} \cap G_i \neq \emptyset$ and $x, y \in B^{'} \cap G_i$. Hence, $\widehat{x}, \widehat{y} \in \widehat{G}$. Since B is a subsemihypergroup of $\widehat{G}$, it is closed under the hyperoperation $\odot$. Hence, $\widehat{x} \odot \widehat{y} \subseteq B$. Now, consider any $a \in x \circ y$. Because $\varphi_{ii} = id_{G_i}$, we have

$$a \in \varphi_{ii}(x) \circ \varphi_{ii}(y).$$

This implies that $\widehat{a} \in \widehat{x} \odot \widehat{y} \subseteq B$. Hence, $\widehat{a} \in B$ and $a \in B^{'}$. Then, $B^{'} \cap G_i$ is closed under the hyperoperation $\circ$. Obviously, the hyperoperation $\circ$ is associative on $B^{'} \cap G_i$. Therefore, $(B^{'} \cap G_i, \circ)$ is closed under the hyperoperation $\circ$. Therefore, $B^{'} \cap G_i$ is a subsemihypergroup of G_i. □

Proposition 3.7. *Let (G_i, φ_{ij}) be a direct system of hypergroups and let $(\widehat{G}, \odot)$ be a DSH, and $A \subseteq G_t$ be a subhypergroup. Then, its image $\widehat{A}$ in $\widehat{G}$ is also a subhypergroup.*

Proof. Suppose $\widehat{x}_i, \widehat{x}_j \in \widehat{A}$. Then, $\widehat{x}_i = \widehat{a}_t$ and $\widehat{x}_j = \widehat{b}_t$ for some $a_t, b_t \in A$. Let $\widehat{x} \in \widehat{x}_i \odot \widehat{x}_j$. Then, by the definition of the direct limit hyperoperation, there exists $k_1 \geq i, j$, such that $x \in \varphi_{ik_1}(x_i) \circ \varphi_{jk_1}(x_j)$. By the direct limit construction, there exist indices $k_2 \geq i, t$ and $k_3 \geq j, t$, such that $\varphi_{ik_2}(x_i) = \varphi_{tk_2}(a_t)$ and $\varphi_{jk_3}(x_j) = \varphi_{jk_3}(x_j) = \varphi_{tk_3}(b_t)$. Let $k_4 = \max\{k_2, k_3\}$. By compatibility of the direct system, we have $\varphi_{ik_4}(x_i) = \varphi_{k_2k_4} \circ \varphi_{ik_2}(x_i) = \varphi_{k_2k_4} \circ \varphi_{tk_2}(a_2) = \varphi_{tk_4}(b_t)$ and

$$\varphi_{jk_4}(x_j) = \varphi_{k_3k_4} \circ \varphi_{jk_3}(x_j) = \varphi_{k_3k_4} \circ \varphi_{tk_3}(b_t) = \varphi_{tk_4}(b_t).$$

Also, for $k = \max\{k_1, k_4\}$, the hyperoperation in G_k satisfies the following:

$$\varphi_{k_1k}(x) \in \varphi_{k_1k}(\varphi_{ik_1}(x_i) \circ \varphi_{jk_1}(x_j)) = (\varphi_{k_1k} \circ \varphi_{jk_1}(x_j)) \circ (\varphi_{k_1k} \circ \varphi_{jk_1}(x_j)) = \varphi_{ik}(x_i) \circ \varphi_{jk}(x_j).$$

This implies that $\widehat{\varphi_{k_1k}(x)} \in \widehat{x}_i \odot \widehat{x}_j$. Also, $\widehat{\varphi_{k_1k}(x)} = \widehat{x}$ implies that $\widehat{x} \in \widehat{x}_i \odot \widehat{x}_j$. Hence, $\widehat{x}_i \odot \widehat{x}_j \subseteq \widehat{A}$ and $\widehat{A}$ is closed under hyperoperation $\odot$. Obviously, the hyperoperation $\odot$ is associative. Let $\widehat{x} \in \widehat{A}$. Then, $\widehat{x} \odot \widehat{A} \subseteq \widehat{A}$. Also, $\widehat{x} = \widehat{a}$ for some $a \in A$. Hence, for some $k_1 \geq t, j$, $\varphi_{tk_1}(a) = \varphi_{jk_1}(x)$, where $x \in G_j$. Let $\widehat{b} \in \widehat{A}$. Then, $\widehat{b} = \widehat{a}_1$ for some $a_1 \in A$ and for $k_2 \geq i, t$, $\varphi_{ik_2}(b) = \varphi_{tk_2}(a_1)$. Since, $(A, \circ)$ is a hypergroup, $a_1 \in a \circ a_2$ for some $a_2 \in A$. Let $k = \max\{k_1, k_2\}$. Then,

$$\varphi_{ik}(b) = \varphi_{k_2k} \circ \varphi_{ik_2}(b) = \varphi_{k_2k} \circ \varphi_{tk_2}(a_1) = \varphi_{tk}(a_1)$$

and

$$\varphi_{tk}(a) = \varphi_{k_1k} \circ \varphi_{tk_1}(a) = \varphi_{k_1k} \circ \varphi_{jk_1}(x) = \varphi_{jk}(x).$$

Also, $\varphi_{tk}(a_1) \in \varphi_{tk_2}(a \circ a_2) = \varphi_{tk}(a) \circ \varphi_{tk}(a_2)$. This implies that $\widehat{\varphi_{tk}}(a_1) \in \widehat{a} \odot \widehat{a}_2 = \widehat{x} \odot \widehat{A}$. Now, $\widehat{\varphi_{ik}}(b) = \widehat{b}$ implies that $\widehat{b} \in \widehat{x} \odot \widehat{A}$ and $\widehat{x} \odot \widehat{A} = \widehat{A}$. Similarly, $\widehat{A} \odot \widehat{x} = \widehat{A}$. Therefore, $(A, \odot)$ is closed under $\odot$, associative, and satisfies the reproduction axiom. Thus, $(A, \odot)$ is a hypergroup. □

Theorem 3.3. *Let (G_i, φ_{ij}) be a direct system of canonical hypergroups over a directed set I. Then, the pair $(\widehat{G}, \odot)$ is a canonical hypergroup, where $\widehat{G}$ is the direct limit of the system and $\odot$ is the induced hyperoperation.*

Proof. For every $\widehat{a}_i \in G_i$, the hyperoperation $\odot$ satisfies

$$\widehat{a_i} \odot \widehat{0}_j = \{\widehat{b} : b \in a_k + 0_k, \;\; a_k = \varphi_{ik}(a_i), \;\; 0_k = \varphi_{jk}(0_j), \;\; k \geq i, j\}.$$

Since, 0_k is the scaler identity in G_k, we have $a_k + 0_k = a_k$.Consequently, $\widehat{a}_i \odot \widehat{0}_j = \widehat{a}_k$. By Proposition 3.1, $\widehat{a}_i = \widehat{a}_k$, which implies that $\widehat{a}_i \odot \widehat{0}_j = \widehat{a}_i$. This shows that $\widehat{0}_j$ acts as a scalar identity in $\widehat{G}$.

For every $i \in I$, G_i is a commutative hypergroup. Then, $(\widehat{G}, \odot)$ is a commutative hypergroup. Now, let $\widehat{a}_k \in \widehat{b}_i \odot \widehat{c}_j$. Then, for $k \geq i, j$, we have $a_k \in b_k + c_k$. By the reproduction axiom, we have

$$b_k \in a_k + (-c_k) \;\text{ and }\; c_k \in a_k + (-b_k).$$

This implies that $\widehat{b}_k \in \widehat{a}_k \odot (-\widehat{c}_k)$ and $\widehat{c}_k \in \widehat{a}_k \odot (-\widehat{b}_k)$. Therefore, $(\widehat{G}, \odot)$ is a canonical hypergroup. □

4. Exact sequences of direct limits

In this section, we introduces morphisms between direct systems of hypergroups, generalizing the categorical notion of compatible homomorphisms to the hyperalgebraic setting.

Definition 4.1. *Let (G_i, φ_{ij}) and (H_i, ψ_{ij}) be direct systems of hypergroups and let $f = \{f_i\}$ be a family of maps such that $f_i : G_i \longrightarrow H_i$ makes the following commutes:*

$$\begin{array}{ccc} G_i & \xrightarrow{\varphi_{ij}} & G_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ H_i & \xrightarrow[\psi_{ij}]{} & H_j \end{array}$$

for every $i \leq j$. Then, f is called morphism of direct systems.

Proposition 4.1. *Let (G_i, φ_{ij}) and (H_i, ψ_{ij}) be direct systems of hypergroups over a directed set I, and let $f = \{f_i\}$ be a morphism of direct systems. Then, for every $x \in G_i$, the following compatibility condition holds in the direct limit:*

$$\widehat{f_j \circ \varphi_{ij}} = \widehat{f_i(x)}.$$

Proof. Suppose $b \in \widehat{f_j \circ \varphi_{ij}}$, where $b \in G_s$. In the direct limit $\widehat{H}$, we have $b \equiv f_j(\varphi_{ij}(x))$ for $k \geq s, j$. Hence,

$$\psi_{sk}(b) = \psi_{jk}(f_j(\varphi_{ij}(x)).$$

Since the following diagram

$$\begin{array}{ccc} G_j & \xrightarrow{\varphi_{jk}} & G_k \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f_k} \\ H_j & \xrightarrow[\psi_{ik}]{} & H_k \end{array}$$

is commutative, we have $\psi_{sk}(b) = \psi_{jk}(f_j(\varphi_{ij}(x)) = f_k \circ \varphi_{jk} \circ \varphi_{ij}(x) = f_k(\varphi_{iK}(x)$. Additionally, because the diagram

$$\begin{array}{ccc} G_i & \xrightarrow{\varphi_{jk}} & G_k \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_k} \\ H_i & \xrightarrow[\psi_{ik}]{} & H_k \end{array}$$

is commutative, it follows that $\psi_{sk}(b) = f_k(\varphi_{iK}(x) = \psi_{ik}(f_i(x))$. Therefore, $\widehat{b} = \widehat{f_i(x)}$, which shows that $\widehat{f_j \circ \varphi_{ij}} = \widehat{f_i(x)}$. □

Proposition 4.2 (Induced Map on Direct Limits of Hypergroups)**.** *Let (G_i, φ_{ij}) and (H_i, ψ_{ij}) be direct systems of hypergroups over a directed set I and $f = \{f_i\}$ be a family of maps such that $f_i : G_i \longrightarrow H_i$. Then, there exists a map $\widehat{f} : \widehat{G} \longrightarrow \widehat{H}$ between the direct limits.*

Proof. Suppose that $\widehat{f} : \widehat{G} \longrightarrow \widehat{H}$ defined by $\widehat{f}(\widehat{x}) = \widehat{f_i(x)}$, where $x \in G_i$. Let $\widehat{x}, \widehat{y} \in \widehat{G}$ such that $\widehat{x} = \widehat{y}$, where $x \in G_i$ and $y \in G_j$. This implies that there exists $k \geq i, j$ such that $\varphi_{ik}(x) = \varphi_{jk}(y)$. Also, we note that the following diagram is commutative:

$$\begin{array}{ccc} H_i & \xrightarrow{\varphi_{ik}} & G_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_k} \\ H_i & \xrightarrow[\psi_{ik}]{} & H_k \end{array}$$

Using the commutativity of this diagrams, we have $\psi_{ik}(f_i(x)) = f_k(\varphi_{ik}(x)) = f_k(\varphi_{jk}(y)) = \psi_{jk}(f_j(y))$. This equality means that $f_i(x)$ and $f_j(y)$ are equivalent in $\widehat{H}$. Therefore,

$$\widehat{f_i(x)} = \widehat{f_j(y)},$$

which shows that $\widehat{f}$ is well-defined. □

Theorem 4.1 (Induced Homomorphism on Direct Limits)**.** *Let (G_i, φ_{ij}) and (H_i, ψ_{ij}) be direct systems of hypergroups over a directed set I and $f = \{f_i\}$ be a family of homomorphisms. Then there exists a unique induced homomorphism $\widehat{f} : \widehat{G} \longrightarrow \widehat{H}$ between the direct limits.*

Proof. Suppose $\widehat{x}, \widehat{y} \in \widehat{G}$ with representatives $x \in G_i, y \in G_j$, for some $i \in J$. By the definition of the direct limit, there exists an index $k \geq i, j$ such that

$$\widehat{f}(\widehat{x} \odot \widehat{y}) = \bigcup_{z \in \varphi_{ik}(x) \circ \varphi_{jk}(y)} \widehat{f_k(z)}.$$

for some $k \geq i, j$. Now, since f_k is a homomorphism, we have

$$f_k(z) \in f_k(\varphi_{ik}(x) \circ \varphi_{jk}(y)) = (f_k \circ \varphi_{ik})(x) \circ (f_k \circ \varphi_{jk}(y)).$$

From the compatibility condition $f_k \circ \varphi_{ik} = \psi_{ik} \circ f_k$, it follows that $f_k(z) \in \psi_{ik}(f_i(x)) \circ \psi_{jk}(f_j(y))$. Thus, we have shown that $\widehat{f}(\widehat{x} \odot \widehat{y}) = \widehat{f}(\widehat{x}) \odot \widehat{f}(\widehat{y})$, which proves that $\widehat{f}$ is a homomorphism. □

Corollary 4.1. *Let (G_i, φ_{ij}) and (H_i, ψ_{ij}) be direct systems of hypergroups over a directed set I and let $f = \{f_i\}$ be a family of good homomorphisms. Then, the induced homomorphism $\widehat{f} : \widehat{G} \longrightarrow \widehat{H}$ between the direct limits is a good homomorphism.*

Corollary 4.2. *Let (G_i, φ_{ij}) and (H_i, ψ_{ij}) be direct systems of hypergroups over a directed set I and let $f = \{f_i\}$ be a family of epimorphism. Then the induced homomorphism $\widehat{f} : \widehat{G} \longrightarrow \widehat{H}$ between the direct limits is an epimorphism.*

Definition 4.2 (see [4])**.** *Let $G_1$and G_2 be canonical hypergroups, and let $\varphi : G_1 \longrightarrow G_2$ be a homomorphism. The kernel of φ is denoted by $\ker\varphi$ and is defined as*

$$\ker\varphi = \{x \in G_1 : \varphi(x) = 0_{G_2}\}.$$

where 0_{G_2} denotes the identity element (or neutral element) in G_2.

Definition 4.3. *Let G_1, G_2, and G_3 be canonical hypergroups. Then, a sequence $0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \longrightarrow 0$, is called exact if the following conditions hold:*

(1) *φ is surjective (onto);*

(2) *ψ is injective (one-to-one); and*

(3) $\operatorname{Im}\varphi = \ker\psi$.

Proposition 4.3. *Let (G_i, φ_{ij}) and (H_i, ψ_{ij}) be direct systems over a directed index set I and $f = \{f_i\}_{i\in I}$ be a morphism between these systems. Then, the kernel of the induced morphism $\widehat{f} : \widehat{G} \longrightarrow \widehat{H}$ in the direct limit category is given as follows:*

$$\ker\widehat{f} = \{\widehat{x} \in \widehat{G} : \exists\, k \geq i,\;\; \varphi_{ik}(x) \in \ker f_k\}.$$

Proof. Let $\widehat{f} : \widehat{G} \longrightarrow \widehat{H}$ be the induced homomorphism. Then,

$$\begin{aligned}
\ker\widehat{f} = \{\widehat{x} \in \widehat{G} : \widehat{f}(\widehat{x}) = 0_{\widehat{G}}\} \;\; &= \{\widehat{x} \in \widehat{G} : \widehat{f_i(x)} = \widehat{0}_{G_i}\} \\
&= \{\widehat{x} \in \widehat{G} : \exists\, k \geq i,\; \psi_{ik}(f_i(x)) = \psi_{ik}(f_i(0))\} \\
&= \{\widehat{x} \in \widehat{G} : \exists\, k \geq i,\; f_k(\varphi_{ik}(x)) = f_k(\varphi_{ik}(0))\} \\
&= \{\widehat{x} \in \widehat{G} : \exists\, k \geq i,\; f_k(\varphi_{ik}(x)) = 0_{H_k}\} \\
&= \{\widehat{x} \in \widehat{G} : \exists\, k \geq i,\; \varphi_{ik}(x) \in \ker f_k\}.
\end{aligned}$$

This completes the proof. □

Theorem 4.2 (Exactness of Direct Limits)**.** *Let (G_i, φ_{ij}), (H_i, ψ_{ij}), and (K_i, δ_{ij}) be direct systems of hypergroups over a directed set I. Let $f = \{f_i\}_{i\in I}$ and $g = \{g_i\}_{i\in I}$ be morphisms of direct systems such that*

(1) *$f_i : G_i \longrightarrow H_i$ and $g_i : H_i \longrightarrow K_i$ are homomorphisms for all $i \in I$;*

(2) $\operatorname{Im} f_i = \ker g_i$ *(i.e., the sequences are short exact componentwise).*

Then, the induced sequence of direct limits, $\widehat{G} \longrightarrow \widehat{H} \longrightarrow \widehat{K}$, satisfies $\operatorname{Im}\widehat{f} = \operatorname{Im}\widehat{g}$, where $\widehat{f}$ and $\widehat{g}$ are the induced homomorphisms on the direct limits.

Proof. Let $\widehat{x} \in \widehat{G}$ with $x \in G_i$. Then, $(\widehat{g} \circ \widehat{f})(\widehat{x}) = \widehat{g}(\widehat{f}(\widehat{x})) = \widehat{g}(\widehat{f_i(x)}) = \widehat{g_i \circ f_i(x)} = \widehat{0_{G_i}} = 0_{\widehat{G}}$. This establishes $\operatorname{Im}\widehat{f} \subseteq \ker\widehat{g}$. Now, to prove the reverse inclusion, assume that $\widehat{g}(\widehat{x}) = 0_{\widehat{H}}$ implies $\widehat{g_r(x)} = 0_{\widehat{H}}$. Then, $\delta_{rt}(g_r(x)) = \delta_{rt}(0_{K_t}) = 0_{K_r}$. Now, consider the following commutative diagram for the morphism:

$$\begin{array}{ccc}
H_r & \xrightarrow{\psi_{rk}} & H_k \\
{\scriptstyle g_r}\downarrow & & \downarrow{\scriptstyle g_k} \\
K_r & \xrightarrow[\delta_{rt}]{} & K_t
\end{array}$$

Hence, $(g_k \circ \psi_{rk})(x) = 0$. Since the original sequences are exact componentwise, we have $\psi_{rk}(x) \in \ker g_k = \operatorname{Im} f_k$. Hence, there exists an element $a \in G_k$ such that $f_k(a) = \psi_{rk}(x)$. Therefore,

$$\widehat{f}(\widehat{a}) = \widehat{f_k(a)} = \widehat{\psi_{rk}(x)} = \widehat{x},$$

which shows that $\ker\widehat{g} \subseteq \operatorname{Im}\widehat{f}$. Combining both inclusions proves that $\operatorname{Im}\widehat{f} = \ker\widehat{g}$. □

5. Conclusion

The present study has yielded two principal theorems concerning the categorical properties of direct limits in the category of canonical hypergroups. First, we have constructed the direct limit for an arbitrary directed system and proved that it satisfies the requisite universal property, thereby confirming its existence and uniqueness up to isomorphism. Second, and crucially, we have established that the direct limit functor preserves exactness, as demonstrated by the relation $\operatorname{Im} f = \ker g$ for the induced morphisms at the limit level. These contributions reinforce the role of the direct limit as a powerful and reliable tool in hypergroup theory. The results presented here lay the groundwork for future investigations, including the study of inverse limits, localization, and the development of sheaf theory over structures equipped with a hypergroup topology.

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