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Research Article

Canonical form for the zero-one matrix induced by the Markov map

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Abstract

A condition ensuring the invariance of the "digital" entropy of a cyclic permutation under refinement of the interval partition is established. Also, a "canonical form" for the 0-1 matrix induced by a Markov map is introduced, which provides a structured representation that captures and elucidates the dynamics of the map on the interval.

Keywords: "digital" entropy; canonical form; 0-1 matrix; Markov maps; spectral radius; cyclic permutations; characteristic polynomial.

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1. Introduction

For any given permutation θ of $m+1$ objects, we define the canonical θ-linear map L_θ, also known as the "connect-the-dots" map of θ. This map $L_\theta : I \to I$ is defined such that $L_\theta = \theta$ on the discrete set $T = \{0, 1, \ldots, m\}$ and is linear on each subinterval $I_i = (i, i+1)$ for $i = 0, 1, \ldots, m-1$, where $I = [0, m]$. Together, L_θ and T induce a 0-1 matrix A of size m. In [3], Fisher demonstrated that if θ is a cyclic permutation of length $m+1$, then the coefficients of the characteristic polynomial $\det(\lambda I_m - A)$ of A must all be odd. Also, in [4], Swanson and Volkmer demonstrated that if θ is a unimodal cyclic permutation, then every coefficient of the characteristic polynomial $\det(\lambda I_m - A)$ of A must be either -1 or 1. Then, in [5], the present author introduced the concept of "digital" entropy, $T(\theta)$, for a unimodal cyclic permutation θ, defined as

$$T(\theta) = .\tau(\varepsilon_1)\,\tau(\varepsilon_2)\,\tau(\varepsilon_3)\cdots$$

where the function $\tau(\varepsilon_i)$ is given as follows:

$$\tau(\varepsilon_i) = \begin{cases} 2 & \text{if} \quad i < m+1 \text{ and } \varepsilon_i = -1, \\ 0 & \text{if} \quad i < m+1 \text{ and } \varepsilon_i = +1, \\ 1 & \text{if} \quad i \geq m+1. \end{cases}$$

Here, the coefficients ε_i are derived from the characteristic polynomial of A, expressed as:

$$\det(\lambda I_m - A) = \lambda^m + \varepsilon_1 \lambda^{m-1} + \varepsilon_2 \lambda^{m-2} + \cdots + \varepsilon_{m-1}\lambda + \varepsilon_m.$$

Our primary concern is whether the "digital" entropy $T(\theta)$ of θ remains invariant under any refinement S of T. This question is addressed through an examination of the behavior of elements in $S \setminus T$ under L_θ. Propositions 2.1 and 2.2 of this article suggest that $T(\theta)$ is preserved if all elements in $S \setminus T$ converge to T under L_θ; otherwise, it is not necessarily preserved.

On the other hand, in [2], Byers and Boyarsky demonstrated that the spectral radii of two matrices, A and Z, induced by f with T and by f with S, respectively, are the same. Here, f is any given piecewise-continuous Markov map with respect to the partition point T, and S is any refinement of T. The propositions presented in this article build upon and provide further details of the work done in [2]. In the proofs of these propositions, we use V_ℓ, a matrix of change of bases, which is given in [3]. Compared to Z, both $V_\ell Z V_\ell^{-1}$ and $PV_\ell Z V_\ell^{-1} P^{-1}$ offer much more insight into the dynamics of f on S, where P is a proper permutation matrix. In this context, the matrix $PV_\ell Z V_\ell^{-1} P^{-1}$, introduced in this article, can be seen as a "canonical form" of the 0-1 matrix Z induced by the given Markov map f with respect to the set S.

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2. Statements of the main results

We begin by presenting a few definitions that are used in the rest of this article. Most of these concepts are well-established in the literature, and we refer the reader primarily to [1,2,5] for their origins and context.

Definition 2.1. *Let $I = [0, m]$ be a closed interval, and let $T = \{0, 1, \ldots, m\}$ be a set of partition points of I. Consider $f : I \to I$, a piecewise-continuous Markov map with respect to the partition points T. Specifically, f satisfies the following conditions:*

(i) f is strictly monotonic and continuous on each subinterval $I_i = (i, i+1)$ for $i = 0, 1, \ldots, m-1$;

(ii) The following limits exist and are elements of T:

$$f\left(0^+\right) = \lim_{x \to 0^+} f(x),$$

$$f\left(i^+\right) = \lim_{x \to i^+} f(x) \text{ and } f\left(i^-\right) = \lim_{x \to i^-} f(x) \text{ for } i = 1, \ldots, m-1, \text{ and}$$

$$f\left(m^-\right) = \lim_{x \to m^-} f(x).$$

Definition 2.2. *Let $A = \left[a_{ij}\right]_{m \times m}$ be the 0-1 matrix induced by f with T. Let $S = T \cup U$, where $U = \{u_1, u_2, \ldots, u_n\}$ satisfies $0 < u_1 < u_2 < \cdots < u_n < m$, $T \cap U = \phi$, and S is a partition of the interval $[0, m]$. Additionally, S is a refinement of T (that is, $T \subset S$), and f with S induces a 0-1 matrix $Z = \left[z_{ij}\right]_{(m+n)\times(m+n)}$. We define the length from $u \in U$ to T under f to be k, where $k \in \mathbb{N}$ is the smallest positive integer such that $f^0(u), f^1(u), \ldots, f^{k-1}(u) \in U$ and $f^k(u) \in T$.*

Definition 2.3. *Let J be a subinterval on which f is strictly monotonic. We denote J by J^+ (respectively, J^-) if f is monotone increasing (respectively, monotone decreasing). The sign of the slope of the tangent line to $y = f(x)$ at $(u, f(u))$ is defined as positive (respectively, negative) if u is contained in a subinterval J^+ (respectively, J^-). In this case, we write sgn$(f'(u)) = +1$ (respectively, sgn$(f'(u)) = -1$), noting that f is not necessarily differentiable at u. We say that $U = \{u_1, u_2, \ldots, u_n\}$ forms the orbit of a $(+)$-signed n-cycle (respectively, a $(-)$-signed n-cycle) if $U = \{u_1, u_2, \ldots, u_n\} = \{f^s(u_i) \mid s \geq 0\}$ for any $i = 1, 2, \ldots, n$, and $\prod_{i=1}^{n}$sgn$(f'(u_i)) = +1$ (respectively, $\prod_{i=1}^{n}$sgn$(f'(u_i)) = -1$).*

Now, we are ready to state the main results of this article.

Proposition 2.1. *Suppose that U consists of n points $u_1, u_2, \ldots, u_n$ such that the length from every u_i to T under f is a finite value $k_i \in \mathbb{N}$, for $i = 1, 2, \ldots, n$. Then, the characteristic polynomial $Ch_Z(\lambda)$ of Z is expressed as the product of the characteristic polynomial $Ch_A(\lambda)$ of A and λ^n, that is,*

$$Ch_Z(\lambda) = Ch_A(\lambda)\, \lambda^n.$$

Proposition 2.2. *Suppose that U consists of the orbit of a $(\pm)$-signed n-cycle. Then,*

$$Ch_Z(\lambda) = Ch_A(\lambda)\left(\lambda^n \mp 1\right).$$

As a corollary of Proposition 2.1, we obtain the following result:

Corollary 2.1. *Let θ be any given unimodal cyclic permutation with length $m+1$, and let $T(\theta)$ denote the digital entropy of θ. Then $T(\theta)$ remains invariant under a refinement of the partition of the interval if all n newly added points are eventually periodic points that converge to T, the orbit of the $(m+1)$-cycle of type θ.*

Proof. Let $Ch_A(\lambda) = \lambda^m + \varepsilon_1\lambda^{m-1} + \varepsilon_2\lambda^{m-2} + \cdots + \varepsilon_{m-1}\lambda + \varepsilon_m$. Then $T(\theta) = .\tau(\varepsilon_1)\tau(\varepsilon_2)\tau(\varepsilon_3)\cdots$. On the other hand, by Proposition 2.1,

$$Ch_Z(\lambda) = Ch_A(\lambda)\,\lambda^n = \lambda^{m+n} + \varepsilon_1\lambda^{m-1+n} + \varepsilon_2\lambda^{m-2+n} + \cdots + \varepsilon_{m-1}\lambda^{1+n} + \varepsilon_m\lambda^n.$$

Thus, $T(\theta)$, as computed from $Ch_Z(\lambda)$, is also $.\tau(\varepsilon_1)\tau(\varepsilon_2)\tau(\varepsilon_3)\cdots$. □

3. A lemma

In [3], Fisher introduces a matrix of change of bases, defined as follows:

Definition 3.1. *For any $\ell \geq 2$, let V_ℓ be the $\ell \times \ell$ matrix with 1's on and above the diagonal and 0's below the diagonal. Then V_ℓ^{-1} is the inverse of V_ℓ, and it has 1's on the diagonal, -1's on the super-diagonal, and 0's elsewhere.*

It is evident that V_ℓ satisfies the properties given in the next lemma.

Lemma 3.1. *Let M be any given $\ell \times \ell$ square matrix. Then the following statements hold.*

(i) The i-th row of $V_\ell M$ is the sum of the i-th row, the $(i+1)$-th row, ..., and the ℓ-th row of M, for any $1 \leq i \leq \ell$. In particular:

(ii) If the j-th column and the $(j+1)$-th column of M are identical, then the j-th column and the $(j+1)$-th column of $V_\ell M$ are also identical, for any $1 \leq j \leq \ell - 1$.

(iii) The 1st column of MV_ℓ^{-1} is the 1st column of M, and the j-th column of MV_ℓ^{-1} is the negative of the $(j-1)$-th column of M plus the j-th column of M, for any $2 \leq j \leq \ell$. In particular:

(iv) If the $(j-1)$-th column and the j-th column of M are identical, then the j-th column of MV_ℓ^{-1} is the zero column vector $\mathbf{0}_{\ell \times 1}$, for any $2 \leq j \leq \ell$.

Using Lemma 3.1, we proceed to prove Proposition 2.1 and Proposition 2.2 in the subsequent sections.

4. Proof of Proposition 2.1

Proof. Let $I = [0, m]$, $T = \{0, 1, \ldots, m\}$, and $f : I \to I$ be a piecewise-continuous Markov map with respect to T. Let $A = [a_{ij}]_{m \times m}$ be the 0-1 matrix induced by f with T. Then, $Ch_A(\lambda) = Ch_A(\lambda)\, \lambda^0$ is obviously true. We reorder the n points $u_1, u_2, \ldots, u_n$ in U by their distance to T, and relabel them as follows:

$$\{u_1, u_2, \ldots, u_n\} = \left\{u_{(1,1)}, u_{(1,2)}, \ldots, u_{(1,k_1)}, u_{(2,1)}, u_{(2,2)}, \ldots, u_{(2,k_2)}, \ldots\right\}$$

where $u_{(l,1)}, u_{(l,2)}, \ldots, u_{(l,k_l)}$ are the k_l points at distance l from T, for $l = 1, 2, \ldots$, and $k_1 + k_2 + \cdots = n$. We then relabel the points as follows:

$$u_{(1,1)}, u_{(1,2)}, \ldots, u_{(1,k_1)}, u_{(2,1)}, u_{(2,2)}, \ldots, u_{(2,k_2)}, \ldots$$

becoming

$$v_1, v_2, \ldots, v_{k_1}, v_{k_1+1}, v_{k_1+2}, \ldots, v_{k_1+k_2}, \ldots.$$

Let $P = [p_{ij}]_{(m+k)\times(m+k)}$ be the 0-1 matrix induced by f with $T \cup \{v_1, v_2, \ldots, v_k\}$. Suppose that

$$Ch_P(\lambda) = \det(\lambda I_{m+k} - P) = Ch_A(\lambda)\, \lambda^k \text{ (inductive hypothesis).}$$

Let $Q = [q_{ij}]_{(m+k+1)\times(m+k+1)}$ be the 0-1 matrix induced by f with $T \cup \{v_1, v_2, \ldots, v_k, v_{k+1}\}$. Denote $T \cup \{v_1, v_2, \ldots, v_k\}$ as

$$\{w_0, w_1, \ldots, w_{m+k+1}\}, \text{ where } 0 = w_0 < w_1 < \cdots < w_{m+k+1} = m.$$

Since $v_{k+1} \in \left(w_{i_{k+1}}, w_{i_{k+1}+1}\right)$ for some $0 \leq i_{k+1} \leq m+k$, v_{k+1} divides the subinterval $\left(w_{i_{k+1}}, w_{i_{k+1}+1}\right)$ into two new subintervals $\left(w_{i_{k+1}}, v_{k+1}\right)$ and $\left(v_{k+1}, w_{i_{k+1}+1}\right)$. Therefore, the addition of the $(i_{k+1}+1)$-th row and the $(i_{k+1}+2)$-th row of Q corresponds to the $(i_{k+1}+1)$-th row of P.

Also, since the x-coordinates of the intersection points of the horizontal line $y = v_{k+1}$ and the graph $y = f(x)$ are not in $T \cup \{v_1, v_2, \ldots, v_k, v_{k+1}\}$, the $(i_{k+1}+1)$-th column and the $(i_{k+1}+2)$-th column of Q are identical, and both correspond to the $(i_{k+1}+1)$-th column of P.

Thus, by Lemma 3.1.(i) and (ii), striking out the $(i_{k+1}+2)$-th row and the $(i_{k+1}+2)$-th column of $V_{m+k+1}Q$ results in a matrix identical to $V_{m+k}P$. Additionally, the $(i_{k+1}+1)$-th column and the $(i_{k+1}+2)$-th column of $V_{m+k+1}Q$ are identical.

Thus, by Lemma 3.1.(iii) and (iv), striking out the $(i_{k+1}+2)$-th row and $(i_{k+1}+2)$-th column of $(V_{m+k+1}Q)\, V_{m+k+1}^{-1}$ results in a matrix identical to $(V_{m+k}P)\, V_{m+k}^{-1}$. Moreover, the $(i_{k+1}+2)$-th column of $(V_{m+k}P)\, V_{m+k}^{-1}$ is the zero column vector.

Let $P_{\sigma_{k+1}}$ be the permutation matrix of the cyclic permutation σ_{k+1}:

$$\sigma_{k+1} = (i_{k+1}+2, i_{k+1}+3, \ldots, m+k+1)$$

represented as

$$\sigma_{k+1} = \begin{pmatrix} 1 & 2 & \cdots & i_{k+1}+1 & i_{k+1}+2 & i_{k+1}+3 & \cdots & m+k & m+k+1 \\ 1 & 2 & \cdots & i_{k+1}+1 & i_{k+1}+3 & i_{k+1}+4 & \cdots & m+k+1 & i_{k+1}+2 \end{pmatrix}.$$

Using the cofactor expansion along the $(m+k+1)$-th column and the inductive hypothesis, we obtain

$$\begin{aligned} Ch_Q(\lambda) &= \det\left(\lambda I_{m+k+1} - P_{\sigma_{k+1}}\left(V_{m+k+1} Q V_{m+k+1}^{-1}\right) P_{\sigma_{k+1}}^{-1}\right) \\ &= \begin{vmatrix} \lambda I_{m+k} - V_{m+k} P V_{m+k}^{-1} & \mathbf{0}_{(m+k)\times 1} \\ [* * \cdots *]_{1\times(m+k)} & \lambda \end{vmatrix}_{(m+k+1)\times(m+k+1)} \\ &= \det\left(\lambda I_{m+k} - V_{m+k} P V_{m+k}^{-1}\right)\lambda \\ &= Ch_P(\lambda)\,\lambda \\ &= \left(Ch_A(\lambda)\,\lambda^k\right)\lambda \\ &= Ch_A(\lambda)\,\lambda^{k+1}. \end{aligned}$$

Since both the base step and the inductive step have been proven, we conclude that, for any $k \geq 0$, $Ch_P(\lambda) = Ch_A(\lambda)\,\lambda^k$, where $P = [p_{ij}]_{(m+k)\times(m+k)}$ is the 0-1 matrix induced by f with $T \cup \{v_1, v_2, \ldots, v_k\}$. In particular,

$$Ch_Z(\lambda) = Ch_A(\lambda)\,\lambda^n,$$

where $Z = [z_{ij}]_{(m+n)\times(m+n)}$ is the 0-1 matrix induced by f with $S = T \cup \{u_1, u_2, \ldots, u_n\} = T \cup \{v_1, v_2, \ldots, v_n\}$. □

5. Proof of Proposition 2.2

Proof. First, we relabel $U = \{u_1, u_2, \ldots, u_n\}$ as follows:

$$\{u_1, u_2, \ldots, u_n\} = \{v_1, v_2, \ldots, v_n\},$$

where $f(v_1) = v_2$, $f(v_2) = v_3$, …, $f(v_{n-1}) = v_n$, and $f(v_n) = v_1$. (For convenience, we may denote v_1 as v_{n+1} if necessary). Then, the mapping $f : [0, m] \to [0, m]$ with $S = T \cup U$ induces the matrix

$$Z = [z_{ij}]_{(m+n)\times(m+n)}.$$

Then, $f(v_1) = v_2$ in the graph $y = f(x)$ is expressed in Z as either

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

at the i_1-th and (i_1+1)-th rows, and the i_2-th and (i_2+1)-th columns, respectively, if the slope of the graph $y = f(x)$ at $x = v_1$ is positive or negative, for some $1 \leq i_1, i_2 \leq m+n-1$, and i_1 may also be denoted as i_{n+1} if necessary.

Inductively, $f(v_k) = v_{k+1}$ $(k = 2, 3, \ldots, n-1)$ in the graph $y = f(x)$ is expressed in Z as either

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

at the i_k-th and (i_k+1)-th rows, and the i_{k+1}-th and $(i_{k+1}+1)$-th columns, respectively, if the slope of the graph $y = f(x)$ at $x = v_k$ is positive or negative, for some $1 \leq i_{k+1} \leq m+n-1$.

Finally, $f(v_n) = v_1$ in the graph $y = f(x)$ is expressed in Z as either

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

at the i_n-th and (i_n+1)-th rows, and the i_1-th and (i_1+1)-th columns, respectively, if the slope of the graph $y = f(x)$ at $x = v_n$ is positive or negative.

We recall that the sign of the n-cycle is defined as the product of the signs of the slopes of the graph $y = f(x)$ at $x = v_k$ for $k = 1, 2, \ldots, n$. Notice that the x-coordinates of the intersection points of the horizontal line $y = v_{k+1}$ and the graph $y = f(x)$ are not in $S = T \cup \{v_1, v_2, \ldots, v_n\}$, except for the x-coordinate of the intersection point $(v_k, v_{k+1}) = (v_k, f(v_k))$ for $1 \le k \le n$. Consequently, the i_{k+1}-th column and the $(i_{k+1}+1)$-th column of Z are identical, except at the i_k-th row and the (i_k+1)-th row. Thus, by Lemma 3.1.(i) and (ii), the i_{k+1}-th column and the $(i_{k+1}+1)$-th column of $V_{m+n}Z$ are identical, except at the (i_k+1)-th rows. Notice that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

at the i_k-th and (i_k+1)-th rows, and the i_{k+1}-th and $(i_{k+1}+1)$-th columns in Z has been transformed into

$$\begin{bmatrix} p_k & p_k \\ p_k - 1 & p_k \end{bmatrix}, \text{ or } \begin{bmatrix} q_k & q_k \\ q_k & q_k - 1 \end{bmatrix},$$

at the same position in $V_{m+n}Z$, for some $p_k \in \mathbb{Z}^+$ or for some $q_k \in \mathbb{Z}^+$, respectively.

Thus, by Lemma 3.1.(iii) and (iv), the $(i_{k+1}+1)$-th column of $(V_{m+n}Z)\,V_{m+n}^{-1}$ is the zero column vector except at the (i_k+1)-th row. Notice that

$$\begin{bmatrix} p_k & p_k \\ p_k - 1 & p_k \end{bmatrix}, \text{ or } \begin{bmatrix} q_k & q_k \\ q_k & q_k - 1 \end{bmatrix},$$

at the i_k-th and (i_k+1)-th rows, and the i_{k+1}-th and $(i_{k+1}+1)$-th columns in $V_{m+n}Z$ has been transformed into $+1$, or -1, at the $(i_k+1, i_{k+1}+1)$-th entry of $(V_{m+n}Z)\,V_{m+n}^{-1}$, respectively.

Let $\{r_1, r_2, \ldots, r_m\} = \{1, 2, \ldots, m, \ldots, m+n\} \setminus \{i_1+1, i_2+1, \ldots, i_n+1\}$ where $r_1 < r_2 < \cdots < r_m$. Let

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & m+n \\ r_1 & r_2 & \cdots & r_m & i_1+1 & i_2+1 & \cdots & i_n+1 \end{pmatrix},$$

and let P be the permutation matrix corresponding to σ. Then

$$PV_{m+n}ZV_{m+n}^{-1}P^{-1} = \begin{bmatrix} \left(\overline{Z}_{11}\right)_{m\times m} & \left(\overline{Z}_{12}\right)_{m\times n} \\ \left(\overline{Z}_{21}\right)_{n\times m} & \left(\overline{Z}_{22}\right)_{n\times n} \end{bmatrix}_{(m+n)\times(m+n)}$$

where

$$\left(\overline{Z}_{12}\right)_{m\times n} = O_{m\times n}$$

and

$$\left(\overline{Z}_{22}\right)_{n\times n} = \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \cdots & \cdots & \omega_{2n} \\ \vdots & \vdots & & & \vdots \\ & & & & \\ \omega_{n1} & \omega_{n2} & \cdots & \cdots & \omega_{nn} \end{bmatrix} = \begin{bmatrix} 0 & \pm 1 & 0 & \cdots & 0 \\ 0 & 0 & \pm 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \pm 1 \\ \pm 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}_{n\times n}$$

where the sign of $\omega_{(k)(k+1)}$ corresponds to the sign of the slope of the graph $y = f(x)$ at $x = v_k$ for $1 \le k \le n-1$, and the sign of ω_{n1} corresponds to the sign of the slope of the graph $y = f(x)$ at $x = v_n$.

Next, we consider $\left(\overline{Z}_{11}\right)_{m\times m}$. First, notice that the (i,j)-th entry (where $1 \le i, j \le m$) of V_mA is given by the number of intersection points of any horizontal line $y = y_0$ where $j - 1 < y_0 < j$ and the graph $y = f(x)$ on $[i-1, m]$. Similarly, the (r_i, r_j)-th entry of $V_{m+n}Z$ is also given by the number of intersection points of any horizontal line $y = y_0$ where $j-1 < y_0 < j$ and the graph $y = f(x)$ on $[i-1, m]$. Therefore, the (i,j)-th entry of V_mA is the same as the (r_i, r_j)-th entry of $V_{m+n}Z$.

Furthermore, if r_{j+1} is not directly after r_j, then the (r_i, r_j)-th entry, the (r_i, r_j+1)-th entry, the (r_i, r_j+2)-th entry, $\ldots$, the $(r_i, r_{j+1}-1)$-th entry of $V_{m+n}Z$ are all the same.

Thus, by Lemma 3.1.(iv), the (i,j)-th entry of $(V_mA)\,V_m^{-1}$ is the same as the (r_i, r_j)-th entry of $(V_{m+n}Z)\,V_{m+n}^{-1}$. In other words, $(V_{m+n}Z)\,V_{m+n}^{-1}$, when excluding the (i_1+1)-th, (i_2+1)-th, $\ldots$, (i_n+1)-th, rows and columns, is the same as $(V_mA)\,V_m^{-1}$. Therefore,

$$\left(\overline{Z}_{11}\right)_{m\times m} = V_mAV_m^{-1}.$$

Consequently, we obtain

$$\begin{aligned} Ch_Z(\lambda) &= \det(\lambda I_{m+n} - Z) \\ &= \det\left(\lambda I_{m+n} - PV_{m+n} Z V_{m+n}^{-1} P^{-1}\right) \\ &= \begin{vmatrix} \lambda I_m - V_m A V_m^{-1} & O_{m\times n} \\ -\left(\overline{Z}_{21}\right)_{n\times m} & \lambda I_n - \left(\overline{Z}_{22}\right)_{n\times n} \end{vmatrix}_{(m+n)\times(m+n)} \\ &= \det\left(\lambda I_m - V_m A V_m^{-1}\right) \det\left(\lambda I_n - \left(\overline{Z}_{22}\right)_{n\times n}\right) \\ &= Ch_A(\lambda)\left(\lambda^n \pm 1\right), \end{aligned}$$

where the sign of ± 1 depends on the sign of the n-cycle. Specifically, if the sign of the n-cycle is given as $(-1)^l(+1)^{n-l}$ for some $0 \leq l \leq n$, where l is the number of negative slopes, then

$$\begin{aligned} \det\left(\lambda I_n - \left(\overline{Z}_{22}\right)_{n\times n}\right) &= \mathrm{sgn}(\iota)\,\lambda^n + \mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}(+1)^l(-1)^{n-l} \\ &= \lambda^n + (-1)^{n-1}(+1)^l(-1)^{n-l} \\ &= \lambda^n + (-1)^{l+1}. \end{aligned}$$

Here, $(-1)^{l+1} = +1$ (respectively -1), if and only if l is odd (respectively, even), which corresponds to the sign of the n-cycle: $(-1)^l(+1)^{n-l}$ being negative (respectively, positive).

This completes the proof of Proposition 2.2. □

6. Canonical form of the 0-1 matrix induced by the Markov map

By combining Proposition 2.1 and Proposition 2.2, we obtain the following result:

Proposition 6.1. *Suppose that U consists of n_0 points converging to T, and the remaining $n - n_0$ in U consist of n_i of $(-)$-signed l_i-cycles ($i = 1, \ldots, j$), and n_i of $(+)$-signed l_i-cycles ($i = j+1, \ldots, k$), where*

$$n_0 + \sum_{i=1}^{k} n_i l_i = n.$$

Then,

$$Ch_Z(\lambda) = Ch_A(\lambda)\,\lambda^{n_0} \prod_{i=1}^{j}\left(\lambda^{l_i}+1\right)^{n_i} \prod_{i=j+1}^{k}\left(\lambda^{l_i}-1\right)^{n_i}.$$

Remark 6.1. *Conversely, consider a 0-1 matrix Z of size ℓ induced by a Markov map f with $S = \{0, 1, 2, \ldots, \ell\}$. Starting with Z, we decompose S into T and U. Naturally, T is chosen to include both endpoints 0 and ℓ of the interval $I = [0, \ell]$, as well as every point where the slope of f is undefined, ensuring $f(T) \subseteq T$. The remaining points form $U = S \setminus T$. If necessary, U is further decomposed into subsets $U_0, U_1, U_2, \ldots$ by analyzing the dynamics of f on U. Through this decomposition of S into $T, U_0, U_1, U_2, \ldots$, a proper permutation σ and its corresponding permutation matrix P are obtained.With these, we construct $PV_\ell Z V_\ell^{-1} P^{-1}$. While Z and $PV_\ell Z V_\ell^{-1} P^{-1}$ are similar matrices, the latter provides a far richer representation of the dynamics of f on S. In this sense, $PV_\ell Z V_\ell^{-1} P^{-1}$, as introduced above, can be regarded as a "canonical form" of the 0-1 matrix associated with the given Markov map f and the set S.*

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