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*Research Article*

# Exponential elliptic Sombor index of molecular trees

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**Abstract**

Let $G$ be a simple connected graph of order $n$ with the vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, denote by $d_G(v)$ the degree of $v$ in $G$. The exponential elliptic Sombor index of $G$ is denoted by $e^{ESO}(G)$ and is defined as $e^{ESO}(G) = \sum_{v_iv_j \in E(G)} e^{\left(d_G(v_i)+d_G(v_j)\right)\sqrt{d_G^2(v_i)+d_G^2(v_j)}}$. In this paper, the minimum and maximum values of the exponential elliptic Sombor index of molecular trees of order $n$ are obtained. Those molecular trees that achieve the obtained minimum and maximum values are also characterized.

**Keywords:** molecular tree; exponential elliptic Sombor index; extremal value.

**2020 Mathematics Subject Classification:** 05C09, 05C50, 05C92.

## 1. Introduction

All the graphs considered in this paper are simple and connected. Let $G$ be such a graph with the vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, denote by $d_G(v)$ the degree of $v$ in $G$. A tree is a connected acyclic graph. A molecular tree (or chemical tree) is a tree of maximum degree at most four.

From a geometric perspective, Gutman [2] introduced the Sombor index, which is defined as

$$SO(G) = \sum_{v_iv_j \in E(G)} \sqrt{d_G^2(v_i) + d_G^2(v_j)}.$$

The extremal properties of the Sombor index have been the subject of many publications, as demonstrated by the numerous results reported in [4].

In 2024, Gutman, Furtula, and Oz [3] introduced a new vertex-degree-based topological index, called the elliptic Sombor index, using a novel geometric method. The elliptic Sombor index of a graph $G$ is defined as

$$ESO(G) = \sum_{v_iv_j \in E(G)} (d_G(v_i) + d_G(v_j))\sqrt{d_G^2(v_i) + d_G^2(v_j)}. \tag{1}$$

In [3], several basic mathematical properties of this new index were established, and an extremal problem about this index for trees was studied. In [7], the maximum value of the elliptic Sombor index of trees with a given diameter or matching number or number of pendent vertices was determined. The extremal values of the elliptic Sombor index of unicyclic graphs were also found in [7]. Furthermore, those trees and unicyclic graphs that achieve the obtained extremal values were characterized in [7]. In [1], the extremal values of the elliptic Sombor index of chemical graphs and chemical trees with an equal number of vertices were determined. In [5], the bicyclic graph of a given order with the maximum elliptic Sombor index was determined. Furthermore, the ordering relations in benzenoid systems with respect to the elliptic Sombor index were given in [6].

The exponential elliptic Sombor index of a graph $G$ is defined as follows:

$$e^{ESO}(G) = \sum_{v_iv_j \in E(G)} e^{(d_G(v_i)+d_G(v_j))\sqrt{d_G^2(v_i)+d_G^2(v_j)}}. \tag{2}$$

In this paper, extremal values of the exponential elliptic Sombor index of molecular trees with a given order are determined. Those molecular trees that achieve the obtained extremal values are also characterized.

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## 2. Preliminaries

Let $\mathcal{CT}_n$ be the set of all molecular trees of order $n$, and $T \in \mathcal{CT}_n$. A vertex $v$ with $d_T(v) = i$ is called an $i$-degree vertex of $T$. An edge $e = uv$ with $d_T(u) = i$ and $d_T(v) = j$ is called an $(i, j)$-edge of $T$. The numbers of $i$-degree vertices and $(i, j)$-edges in $T$ are denoted by $n_i = n_i(T)$ and $m_{ij} = m_{ij}(T)$, respectively. Then we have

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = n, \\ n_1 + 2n_2 + 3n_3 + 4n_4 = 2(n-1), \\ m_{12} + m_{13} + m_{14} = n_1, \\ m_{12} + 2m_{22} + m_{23} + m_{24} = 2n_2, \\ m_{13} + m_{23} + 2m_{33} + m_{34} = 3n_3, \\ m_{14} + m_{24} + m_{34} + 2m_{44} = 4n_4. \end{cases} \tag{3}$$

Let $f(x, y) = (x + y)\sqrt{x^2 + y^2}$. Then by (2), we have

$$\begin{aligned} e^{ESO}(T) &= \sum_{1 \le i \le j \le 4} m_{ij} e^{f(i,j)} \\ &= e^{3\sqrt{5}} m_{12} + e^{4\sqrt{10}} m_{13} + e^{5\sqrt{17}} m_{14} + e^{8\sqrt{2}} m_{22} + e^{5\sqrt{13}} m_{23} + e^{12\sqrt{5}} m_{24} + e^{18\sqrt{2}} m_{33} + e^{35} m_{34} + e^{32\sqrt{2}} m_{44}. \end{aligned} \tag{4}$$

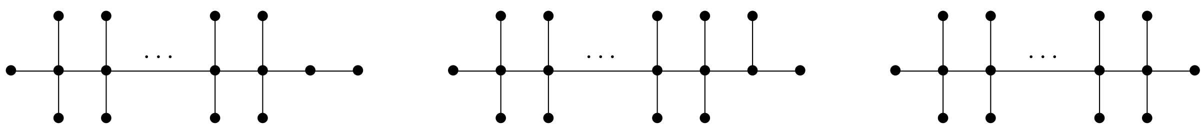
We now define three subsets $\mathcal{T}_0(n)$, $\mathcal{T}_1(n)$ and $\mathcal{T}_2(n)$ of $\mathcal{CT}_n$ ($n \ge 5$) that will be used in Theorem 3.2.

(1) $\mathcal{T}_0(n)$ consists of molecular trees on $n \equiv 0 \pmod 3$ vertices with $m_{13} = m_{22} = m_{23} = m_{33} = m_{34} = 0$, $m_{12} = m_{24} = 1$, $m_{14} = \frac{2n-3}{3}$, and $m_{44} = \frac{n-6}{3}$.

(2) $\mathcal{T}_1(n)$ consists of molecular trees on $n \equiv 1 \pmod 3$ vertices with $m_{12} = m_{22} = m_{23} = m_{24} = m_{33} = 0$, $m_{13} = 2$, $m_{34} = 1$, $m_{14} = \frac{2n-5}{3}$, and $m_{44} = \frac{n-7}{3}$.

(3) $\mathcal{T}_2(n)$ consists of molecular trees on $n \equiv 2 \pmod 3$ vertices with $m_{12} = m_{13} = m_{22} = m_{23} = m_{24} = m_{33} = m_{34} = 0$, $m_{14} = \frac{2n+2}{3}$, and $m_{44} = \frac{n-5}{3}$.

For each of the three subsets of $\mathcal{CT}_n$ defined above, a molecular tree is shown in Figure 2.1. This means that all of these subsets are nonempty.



(1) $T \in \mathcal{T}_0(n)$ (2) $T \in \mathcal{T}_1(n)$ (3) $T \in \mathcal{T}_2(n)$

**Figure 2.1:** Some examples of molecular trees.

## 3. Molecular trees with the extremal exponential elliptic Sombor indices

In this section, we give the minimum and maximum values of the exponential elliptic Sombor indices over $\mathcal{CT}_n$, and characterize those molecular trees that achieve the minimum and maximum values, respectively.

**Theorem 3.1.** *Let $T \in \mathcal{CT}_n$ with $n \ge 3$. Then*

$$e^{ESO}(T) \ge e^{8\sqrt{2}} n - 3e^{8\sqrt{2}} + 2e^{3\sqrt{5}}, \tag{5}$$

*where the equality holds if and only if $T$ is a path of order $n$.*

**Proof.** Let $T \in \mathcal{CT}_n$. Solving the system (3) with unknowns $m_{12}, m_{22}, n_1, n_2, n_3, n_4$, we obtain

$$\begin{cases} m_{12} = -\frac{2m_{13}}{3} - \frac{m_{14}}{2} + \frac{m_{23}}{3} + \frac{m_{24}}{2} + \frac{2m_{33}}{3} + \frac{5m_{34}}{6} + m_{44} + 2, \\ m_{22} = -\frac{m_{13}}{3} - \frac{m_{14}}{2} - \frac{4m_{23}}{3} - \frac{3m_{24}}{2} - \frac{5m_{33}}{3} - \frac{11m_{34}}{6} - 2m_{44} + n - 3, \\ n_1 = \frac{m_{13}}{3} + \frac{m_{14}}{2} + \frac{m_{23}}{3} + \frac{m_{24}}{2} + \frac{2m_{33}}{3} + \frac{5m_{34}}{6} + m_{44} + 2, \\ n_2 = -\frac{2m_{13}}{3} - \frac{3m_{14}}{4} - \frac{2m_{23}}{3} - \frac{3m_{24}}{4} - \frac{4m_{33}}{3} - \frac{17m_{34}}{12} - \frac{3m_{44}}{2} + n - 2, \\ n_3 = \frac{1}{3}(m_{13} + m_{23} + 2m_{33} + m_{34}), \\ n_4 = \frac{1}{4}(m_{14} + m_{24} + m_{34} + 2m_{44}). \end{cases} \tag{6}$$

Replacing the values of $m_{12}$ and $m_{22}$ in (4), and then after simplification, we have

$$e^{ESO}(T) = A_0 + A_{13}m_{13} + A_{14}m_{14} + A_{23}m_{23} + A_{24}m_{24} + A_{33}m_{33} + A_{34}m_{34} + A_{44}m_{44}, \tag{7}$$

where

$$\begin{cases} A_0 = e^{8\sqrt{2}}n - 3e^{8\sqrt{2}} + 2e^{3\sqrt{5}}, \\ A_{13} = -\frac{1}{3}e^{8\sqrt{2}} - \frac{2e^{3\sqrt{5}}}{3} + e^{4\sqrt{10}} \approx 283628.0, \\ A_{14} = -\frac{1}{2}e^{8\sqrt{2}} - \frac{e^{3\sqrt{5}}}{2} + e^{5\sqrt{17}} \approx 8.97822 \times 10^8, \\ A_{23} = -\frac{4}{3}e^{8\sqrt{2}} + \frac{e^{3\sqrt{5}}}{3} + e^{5\sqrt{13}} \approx 6.7399 \times 10^7, \\ A_{24} = -\frac{3}{2}e^{8\sqrt{2}} + \frac{e^{3\sqrt{5}}}{2} + e^{12\sqrt{5}} \approx 4.50136 \times 10^{11}, \\ A_{33} = -\frac{5}{3}e^{8\sqrt{2}} + e^{18\sqrt{2}} + \frac{2e^{3\sqrt{5}}}{3} \approx 1.13588 \times 10^{11}, \\ A_{34} = e^{35} - \frac{11e^{8\sqrt{2}}}{6} + \frac{5e^{3\sqrt{5}}}{6} \approx 1.58601 \times 10^{15}, \\ A_{44} = -2e^{8\sqrt{2}} + e^{32\sqrt{2}} + e^{3\sqrt{5}} \approx 4.50739 \times 10^{19}. \end{cases} \tag{8}$$

Since all $A_{13}, A_{14}, A_{23}, A_{24}, A_{33}, A_{34}, A_{44}$ are positive, Equation (7) gives

$$e^{ESO}(T) \geq A_0,$$

where the equality holds if and only if

$$m_{13} = m_{14} = m_{23} = m_{24} = m_{33} = m_{34} = m_{44} = 0,$$

that is, $T$ is a path of order $n$. □

**Theorem 3.2.** *Let $T \in \mathcal{CT}_n$ with $n \geq 5$. Then*

$$e^{ESO}(T) \leq \begin{cases} \frac{1}{3}\left(e^{32\sqrt{2}} + 2e^{5\sqrt{17}}\right)n - 2e^{32\sqrt{2}} + e^{3\sqrt{5}} + e^{12\sqrt{5}} - e^{5\sqrt{17}}, & \textit{if } n \equiv 0 \pmod 3, \\ \frac{1}{3}\left(e^{32\sqrt{2}} + 2e^{5\sqrt{17}}\right)n + 2e^{4\sqrt{10}} + e^{35} - \frac{7e^{32\sqrt{2}}}{3} - \frac{5e^{5\sqrt{17}}}{3}, & \textit{if } n \equiv 1 \pmod 3, \\ \frac{1}{3}\left(e^{32\sqrt{2}} + 2e^{5\sqrt{17}}\right)n - \frac{5e^{32\sqrt{2}}}{3} + \frac{2e^{5\sqrt{17}}}{3}, & \textit{if } n \equiv 2 \pmod 3, \end{cases} \tag{9}$$

*where the equality holds if and only if $T \in \mathcal{T}_i(n)$ for $n \equiv i \pmod 3$ $(i = 0, 1, 2)$.*

**Proof.** Let $T \in \mathcal{CT}_n$. Solving the system (3) with unknowns $m_{12}, m_{24}, n_1, n_2, n_3, n_4$, we obtain

$$\begin{cases} m_{12} = \frac{n+3}{3} - \frac{7m_{13}}{9} - \frac{2m_{14}}{3} - \frac{m_{22}}{3} - \frac{m_{23}}{9} + \frac{m_{33}}{9} + \frac{2m_{34}}{9} + \frac{m_{44}}{3}, \\ m_{24} = \frac{2n-6}{3} - \frac{2m_{13}}{9} - \frac{m_{14}}{3} - \frac{2m_{22}}{3} - \frac{8m_{23}}{9} - \frac{10m_{33}}{9} - \frac{11m_{34}}{9} - \frac{4m_{44}}{3}, \\ n_1 = \frac{n+3}{3} + \frac{2m_{13}}{9} + \frac{m_{14}}{3} - \frac{m_{22}}{3} - \frac{m_{23}}{9} + \frac{m_{33}}{9} + \frac{2m_{34}}{9} + \frac{m_{44}}{3}, \\ n_2 = \frac{n-2}{2} - \frac{m_{13}}{2} - \frac{m_{14}}{2} + \frac{m_{22}}{2} - \frac{m_{33}}{2} - \frac{m_{34}}{2} - \frac{m_{44}}{2}, \\ n_3 = \frac{1}{3}(m_{13} + m_{23} + 2m_{33} + m_{34}), \\ n_4 = \frac{n-3}{6} - \frac{m_{13}}{18} + \frac{m_{14}}{6} - \frac{m_{22}}{6} - \frac{2m_{23}}{9} - \frac{5m_{33}}{18} - \frac{m_{34}}{18} + \frac{m_{44}}{6}. \end{cases} \tag{10}$$

Replacing the values of $m_{12}$ and $m_{24}$ in (4), then after simplification, we have

$$e^{ESO}(T) = B_0 + B_{13}m_{13} + B_{14}m_{14} + B_{22}m_{22} + B_{23}m_{23} + B_{33}m_{33} + B_{34}m_{34} + B_{44}m_{44}, \tag{11}$$

where

$$\begin{cases} B_0 = \frac{1}{3}\left(e^{3\sqrt{5}} + 2e^{12\sqrt{5}}\right)n + e^{3\sqrt{5}} - 2e^{12\sqrt{5}}, \\ B_{13} = -\frac{7}{9}e^{3\sqrt{5}} - \frac{2e^{12\sqrt{5}}}{9} + e^{4\sqrt{10}} \approx -1.0003 \times 10^{11}, \\ B_{14} = -\frac{2}{3}e^{3\sqrt{5}} - \frac{e^{12\sqrt{5}}}{3} + e^{5\sqrt{17}} \approx -1.49147 \times 10^{11}, \\ B_{22} = e^{8\sqrt{2}} - \frac{e^{3\sqrt{5}}}{3} - \frac{2e^{12\sqrt{5}}}{3} \approx -3.00091 \times 10^{11}, \\ B_{23} = -\frac{1}{9}e^{3\sqrt{5}} - \frac{8e^{12\sqrt{5}}}{9} + e^{5\sqrt{13}} \approx -4.00053 \times 10^{11}, \\ B_{33} = e^{18\sqrt{2}} + \frac{e^{3\sqrt{5}}}{9} - \frac{10e^{12\sqrt{5}}}{9} \approx -3.86563 \times 10^{11}, \\ B_{34} = e^{35} + \frac{2e^{3\sqrt{5}}}{9} - \frac{11e^{12\sqrt{5}}}{9} \approx 1.58546 \times 10^{15}, \\ B_{44} = e^{32\sqrt{2}} + \frac{e^{3\sqrt{5}}}{3} - \frac{4e^{12\sqrt{5}}}{3} \approx 4.50739 \times 10^{19}. \end{cases} \tag{12}$$

In order to find the maximum value of $e^{ESO}(T)$, we consider the following four cases.

**Case 1.** $n_3 = 0$.
In this case, we have $m_{13} = m_{23} = m_{33} = m_{34} = 0$. By (11), we have

$$e^{ESO}(T) = B_0 + B_{14}m_{14} + B_{22}m_{22} + B_{44}m_{44}. \tag{13}$$

**Subcase 1.1.** $n_2 = 0$.
Note that $m_{12} = m_{22} = m_{24} = 0$. By (3), we have $n_4 = \frac{n-2}{3}$, $n_1 = \frac{2n+2}{3}$, $m_{14} = \frac{2n+2}{3}$, and $m_{44} = \frac{n-5}{3}$. So $n \equiv 2 \pmod 3$ and $T \in \mathcal{T}_2(n)$. Thus, by (13), we have

$$e^{ESO}(T) = B_0 + \frac{2n+2}{3}B_{14} + \frac{n-5}{3}B_{44}. \tag{14}$$

**Subcase 1.2.** $n_2 = 1$.
Note that $m_{22} = 0$, $m_{12} \le 1$, and $m_{12} + m_{24} = 2$. By the first two equations in (3), we have $n_4 = \frac{n-3}{3}$ and $n_1 = \frac{2n}{3}$. So, $n \equiv 0 \pmod 3$.

If $m_{12} = 1$, then $m_{24} = 1$. By the third and sixth equations in (3), we have $m_{14} = \frac{2n-3}{3}$ and $m_{44} = \frac{n-6}{3}$. By (13), we have

$$e^{ESO}(T) = B_0 + \frac{2n-3}{3}B_{14} + \frac{n-6}{3}B_{44}.$$

If $m_{12} = 0$, then $m_{24} = 2$. By the third and sixth equations in (3), we have $m_{14} = \frac{2n}{3}$ and $m_{44} = \frac{n-9}{3}$. By (13), we have

$$e^{ESO}(T) = B_0 + \frac{2n}{3}B_{14} + \frac{n-9}{3}B_{44}.$$

Note that

$$\frac{2n-3}{3}B_{14} + \frac{n-6}{3}B_{44} > \frac{2n}{3}B_{14} + \frac{n-9}{3}B_{44}.$$

Thus, for the considered subcase, we have

$$e^{ESO}(T) \le B_0 + \frac{2n-3}{3}B_{14} + \frac{n-6}{3}B_{44}, \tag{15}$$

where the equality is true if and only if $m_{12} = m_{24} = 1$, $m_{14} = \frac{2n-3}{3}$, and $m_{44} = \frac{n-6}{3}$, that is, $T \in \mathcal{T}_0(n)$.

**Subcase 1.3.** $n_2 \ge 2$.
By the first two equations in (3), we have that $3n_1 + 2n_2 = 2n + 2$ and $n_2 + 3n_4 = n - 2$. Then,

$$n_1 = \frac{2n - 2n_2 + 2}{3} \quad \text{and} \quad n_4 = \frac{n - n_2 - 2}{3}.$$

Note that $m_{12} \le n_2$, $m_{44} \le n_4 - 1$, and $m_{12} + m_{14} = n_1$. Hence,

$$m_{14} = n_1 - m_{12} \ge n_1 - n_2 = \frac{2n - 5n_2 + 2}{3}$$

and

$$m_{44} \le n_4 - 1 = \frac{n - n_2 - 5}{3}.$$

By (12) and (13), we have

$$\begin{aligned} e^{ESO}(T) &\le B_0 + \frac{2n - 5n_2 + 2}{3}B_{14} + \frac{n - n_2 - 5}{3}B_{44} \\ &= B_0 + \frac{2n+2}{3}B_{14} + \frac{n-5}{3}B_{44} - \left(\frac{5}{3}B_{14} + \frac{1}{3}B_{44}\right)n_2. \end{aligned}$$

Note that

$$\frac{5}{3}B_{14} + \frac{1}{3}B_{44} \approx 1.50246 \times 10^{19} > 0.$$

So,

$$\begin{aligned} e^{ESO}(T) &\le B_0 + \frac{2n+2}{3}B_{14} + \frac{n-5}{3}B_{44} - 2\left(\frac{5}{3}B_{14} + \frac{1}{3}B_{44}\right) \\ &= B_0 + \frac{2n-8}{3}B_{14} + \frac{n-7}{3}B_{44}. \end{aligned} \tag{16}$$

**Case 2.** $n_3 = 1$ and $n_2 = 0$.
Observe that $m_{12} = m_{22} = m_{23} = m_{24} = m_{33} = 0$, and $m_{13} + m_{34} = 3$. By (11), we have

$$e^{ESO}(T) = B_0 + B_{13}m_{13} + B_{14}m_{14} + B_{34}m_{34} + B_{44}m_{44}. \tag{17}$$

By the first two equations in (3), we have $n_4 = \frac{n-4}{3}$ and $n_1 = \frac{2n+1}{3}$. So, $n \equiv 1 \pmod 3$. Note that $m_{13} \le 2$. Table 3.1 gives the values of $m_{14}$ and $m_{44}$ for all possible choices of $(m_{13}, m_{34})$ by the third and sixth equations in (3).

**Table 3.1** The values of $m_{14}$ and $m_{44}$ for all possible choices of $(m_{13}, m_{34})$.

| $m_{13}$ | $m_{34}$ | $m_{14}$ | $m_{44}$ |
|---|---|---|---|
| 2 | 1 | $\frac{2n-5}{3}$ | $\frac{n-7}{3}$ |
| 1 | 2 | $\frac{2n-2}{3}$ | $\frac{n-10}{3}$ |
| 0 | 3 | $\frac{2n+1}{3}$ | $\frac{n-13}{3}$ |

By (17), Table 3.1, and a direct calculation, we have

$$e^{ESO}(T) \leq B_0 + 2B_{13} + \frac{2n-5}{3}B_{14} + B_{34} + \frac{n-7}{3}B_{44}, \tag{18}$$

where the equality is true if and only if $m_{13} = 2$, $m_{34} = 1$, $m_{14} = \frac{2n-5}{3}$, and $m_{44} = \frac{n-7}{3}$, that is, $T \in \mathcal{T}_1(n)$.

**Case 3.** $n_3 = 1$ and $n_2 \geq 1$.
By the first three equations in (3), we have that $3n_1 + 2n_2 = 2n + 1$, $n_2 + 3n_4 = n - 4$, and $m_{12} + m_{13} + m_{14} = n_1$. Note that $m_{12} \leq n_2$. Then,

$$n_1 = \frac{2n - 2n_2 + 1}{3}, \quad n_4 = \frac{n - n_2 - 4}{3}, \quad \text{and} \quad m_{14} = n_1 - m_{12} - m_{13} \geq n_1 - n_2 - m_{13} = \frac{2n - 5n_2 + 1}{3} - m_{13}.$$

Note also that

$$m_{33} = 0, \quad m_{13} \leq 2, \quad m_{34} \leq 3, \quad m_{44} \leq n_4 - 1 = \frac{n - n_2 - 7}{3}, \quad \text{and} \quad B_{13} > B_{14}.$$

Thus, by (11) and (12), we have

$$\begin{aligned}
e^{ESO}(T) &\leq B_0 + B_{13}m_{13} + B_{14}m_{14} + B_{34}m_{34} + B_{44}m_{44} \\
&\leq B_0 + B_{13}m_{13} + \left(\frac{2n - 5n_2 + 1}{3} - m_{13}\right) B_{14} + 3B_{34} + \frac{n - n_2 - 7}{3}B_{44} \\
&= B_0 + (B_{13} - B_{14})m_{13} + \frac{2n - 5n_2 + 1}{3}B_{14} + 3B_{34} + \frac{n - n_2 - 7}{3}B_{44} \\
&\leq B_0 + 2(B_{13} - B_{14}) + \frac{2n - 5n_2 + 1}{3}B_{14} + 3B_{34} + \frac{n - n_2 - 7}{3}B_{44} \\
&= B_0 + 2B_{13} + \frac{2n-5}{3}B_{14} + 3B_{34} + \frac{n-7}{3}B_{44} - \left(\frac{5}{3}B_{14} + \frac{1}{3}B_{44}\right) n_2.
\end{aligned}$$

Since

$$\frac{5}{3}B_{14} + \frac{1}{3}B_{44} \approx 1.50246 \times 10^{19} > 0,$$

we have

$$\begin{aligned}
e^{ESO}(T) &\leq B_0 + 2B_{13} + \frac{2n-5}{3}B_{14} + 3B_{34} + \frac{n-7}{3}B_{44} - \left(\frac{5}{3}B_{14} + \frac{1}{3}B_{44}\right) \\
&= B_0 + 2B_{13} + \frac{2n-10}{3}B_{14} + 3B_{34} + \frac{n-8}{3}B_{44}.
\end{aligned} \tag{19}$$

**Case 4.** $n_3 \geq 2$.
By the first three equations in (3), we have that $3n_1 + 2n_2 + n_3 = 2n + 2$, $n_2 + 2n_3 + 3n_4 = n - 2$, and $m_{12} + m_{13} + m_{14} = n_1$. Note that $m_{12} \leq n_2$. Then

$$n_1 = \frac{2n + 2 - 2n_2 - n_3}{3}, \; n_4 = \frac{n - 2 - n_2 - 2n_3}{3}, \; \text{and } m_{14} = n_1 - m_{12} - m_{13} \geq n_1 - n_2 - m_{13} = \frac{2n + 2 - 5n_2 - n_3}{3} - m_{13}.$$

Observe also that

$$m_{44} \leq n_4 - 1 = \frac{n - 5 - n_2 - 2n_3}{3} \quad \text{and} \quad m_{34} \leq 3n_3 - m_{13}.$$

Thus, by (11) and (12), we have

$$\begin{aligned}
e^{ESO}(T) &\leq B_0 + B_{13}m_{13} + B_{14}m_{14} + B_{34}m_{34} + B_{44}m_{44} \\
&\leq B_0 + B_{13}m_{13} + \left(\frac{2n + 2 - 5n_2 - n_3}{3} - m_{13}\right) B_{14} + (3n_3 - m_{13})B_{34} + \frac{n - 5 - n_2 - 2n_3}{3}B_{44} \\
&= B_0 + \frac{2n+2}{3}B_{14} + \frac{n-5}{3}B_{44} + (B_{13} - B_{14} - B_{34})\, m_{13} - \left(\frac{5}{3}B_{14} + \frac{1}{3}B_{44}\right) n_2 - \left(\frac{1}{3}B_{14} - 3B_{34} + \frac{2}{3}B_{44}\right) n_3.
\end{aligned}$$

Note that

$$\begin{aligned}
&B_{13} - B_{14} - B_{34} \approx -1.58541 \times 10^{15} < 0, \\
&\frac{5}{3}B_{14} + \frac{1}{3}B_{44} \approx 1.50246 \times 10^{19} > 0, \text{ and} \\
&\frac{1}{3}B_{14} - 3B_{34} + \frac{2}{3}B_{44} \approx 3.00445 \times 10^{19} > 0.
\end{aligned}$$

Hence,

$$e^{ESO}(T) \le B_0 + \frac{2n+2}{3}B_{14} + \frac{n-5}{3}B_{44} - 2\left(\frac{1}{3}B_{14} - 3B_{34} + \frac{2}{3}B_{44}\right)$$
$$= B_0 + \frac{2n}{3}B_{14} + 6B_{34} + \frac{n-9}{3}B_{44}. \tag{20}$$

Combining the discussions of Cases 1–4, and noting that

$$\frac{2n-8}{3}B_{14} + \frac{n-7}{3}B_{44} > 2B_{13} + \frac{2n-10}{3}B_{14} + 3B_{34} + \frac{n-8}{3}B_{44} > \frac{2n}{3}B_{14} + 6B_{34} + \frac{n-9}{3}B_{44},$$
$$\frac{2n-3}{3}B_{14} + \frac{n-6}{3}B_{44} > \frac{2n-8}{3}B_{14} + \frac{n-7}{3}B_{44},$$
$$2B_{13} + \frac{2n-5}{3}B_{14} + B_{34} + \frac{n-7}{3}B_{44} > \frac{2n-8}{3}B_{14} + \frac{n-7}{3}B_{44},$$
$$\frac{2n+2}{3}B_{14} + \frac{n-5}{3}B_{44} > \frac{2n-8}{3}B_{14} + \frac{n-7}{3}B_{44},$$

we arrive at the following conclusions:

- If $n \equiv 0 \pmod 3$, then by (15), (16), (19), and (20), $T$ has the maximum exponential elliptic Sombor index over $\mathcal{CT}_n$ if and only if $T \in \mathcal{T}_0(n)$, and this maximum value is given in (15);
- If $n \equiv 1 \pmod 3$, then by (16), (18), (19), and (20), $T$ has the maximum exponential elliptic Sombor index over $\mathcal{CT}_n$ if and only if $T \in \mathcal{T}_1(n)$, and this maximum value is given in (18);
- If $n \equiv 2 \pmod 3$, then by (14), (16), (19), and (20), $T$ has the maximum exponential elliptic Sombor index over $\mathcal{CT}_n$ if and only if $T \in \mathcal{T}_2(n)$, and this maximum value is given in (14).

This completes the proof of Theorem 3.2. □

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