**Contributions to Mathematics**
www.shahindp.com/locate/cm

*Contrib. Math.* **10** (2024) 58–62
DOI: 10.47443/cm.2024.079

*Research Article*

# Hull operators in bounded posets

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(Received: 4 December 2024. Received in revised form: 24 December 2024. Accepted: 25 December 2024. Published online: 26 December 2024.)



**Abstract**

Hull operators have always been studied for power sets. In this article, hull operators are generalized to the scope of posets. A construction theorem and a classification theorem are stated and proved. Particular cases of hull operators, such as closure operators, are also considered. Nontrivial examples are provided and analyzed. Moreover, an approach to hull operators in category theory is discussed.

**Keywords:** bounded poset; hull operator; limit operator; universal algebras.

**2020 Mathematics Subject Classification:** 06A06, 06A15.

## 1. Introduction

The linear span, the convex hull, and the topological closure are operators of sets, which share similar properties such as extensionality (or extensivity), monotonicity (or increasingness), and idempotency. These are precisely the three main characteristics of hull operators [7] that obviously find their origins in the three previously mentioned operators. Hull operators, also known in the classic literature as closure operators, have always been commonly studied in the context of power sets [4]; however, they can also be considered in more general settings such as Boolean algebras and bounded lattices [5, 6]. Some authors consider closure operators as a particular class of hull operators (see Example 2.1). Closure operators [1, 2] are well known in topology since they serve to characterize topologies.

The notion of hull operator [7] originated from linear algebra, topology, and operator theory; certainly, it can also be generalized to category theory. Let $Z$ be a set and let $\mathcal{P}(Z)$ denote the power set of $Z$. A function

$$H : \mathcal{P}(Z) \to \mathcal{P}(Z)$$

is a hull operator if it satisfies the following three conditions for all $A, B \in \mathcal{P}(Z)$:

- Extensivity or extensionality: $A \subseteq H(A)$.
- Increasingness or monotonicity: $A \subseteq B \Rightarrow H(A) \subseteq H(B)$.
- Idempotency: $H(H(A)) = H(A)$.

Note that hull operators do not necessarily map $\varnothing$ to $\varnothing$. Indeed, the linear span in a vector space is a hull operator, mapping the empty set to the null subspace. However, in category theory, the empty set is the null object of the category of sets, and the null vector space is the null object of the category of vector spaces, meaning that the linear span maps the null object of the category of sets to the null object of the category of vector spaces.

The rest of the paper needs basic notions and concepts of order theory, so they are recalled now. A poset is a partially ordered set. A bounded poset is a poset with a maximum and a minimum element, which are commonly denoted by $1$ and $0$, respectively. A poset is said to verify the sup property if every subset with an upper bound has a supremum. Similarly, the inf property is defined. It is not hard to show that a poset satisfies the sup property if and only if it satisfies the inf property. In a poset $X$ satisfying the inf property, a subset $Y \subseteq X$ is called inf-closed provided that every bounded below subset $Z \subseteq Y$ has its inf in $Y$. In a similar way, sup-closed subsets can be defined. The notions of finitely inf-closed and finitely sup-closed can be easily inferred from the previous ones by considering a finite $Z$.

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Lattices are special cases of posets where every pair of arbitrary elements has an infimum and a supremum. Due to the associativity of the binary internal operations $\wedge$ and $\vee$, in a lattice $X$ with the inf property, it occurs that a subset $Y \subseteq X$ is finitely inf-closed if and only if $y_1 \wedge y_2 \in Y$ for every $y_1, y_2 \in Y$. A dual characterization applies to finitely sup-closed subsets. Bear in mind that every bounded poset satisfying the inf property is a lattice. Indeed, if $x$ and $y$ are elements of the poset, then $\{x, y\}$ is bounded below by $0$ and bounded above by $1$; therefore, there exists $x \wedge y$ because the poset satisfies the inf property, and there exists $x \vee y$ because the poset also satisfies the sup property.

In a bounded lattice $X$, an element $x \in X$ is said to have a complement $y \in X$ provided that $x \wedge y = 0$ and $x \vee y = 1$. In a distributive bounded lattice, complements are unique if they exist. A distributive complemented lattice is a bounded lattice, which is distributive and every element has a complement. It is not hard to realize that distributive complemented lattices can be characterized as Boolean algebras.

The primary purpose of this paper is to provide a construction theorem (Theorem 2.1) and a classification theorem (Theorem 2.2) for hull operators and closure operators in the general setting of bounded posets satisfying the inf property. Two nontrivial examples of hull operators that are not closure operators are also provided (Examples 2.2 and 2.3).

## 2. Main results

We start this section by defining a hull operator in the context of posets; obviously, the original definition of hull operators for power sets [7] served as an inspiration for this definition.

**Definition 2.1** (Hull operator)**.** *Let $X$ be a poset. A unary internal operation $H : X \to X$ is said to be a hull operator if it satisfies the following three conditions for all $x, y \in X$:*

- *Extensivity or extensionality: $x \leq H(x)$.*
- *Increasingness or monotonicity: $x \leq y \Rightarrow H(x) \leq H(y)$.*
- *Idempotency: $H(H(x)) = H(x)$.*

There are trivial examples of hull operators. For instance, if $X$ is a bounded poset, then $H(x) = 1$ for all $x \in X$ is trivially a hull operator, which has no interest. A nontrivial example follows (also inspired by the classical definition of closure operator for power sets [1, 2]).

**Example 2.1** (Closure operators)**.** *Let $X$ be a bounded lattice. A closure operator is a unary internal operation $C : X \to X$ satisfying the following conditions for all $x, y \in X$:*

- *Nullity: $C(0) = 0$.*
- *Extensivity or extensionality: $x \leq C(x)$.*
- *Additivity: $C(x \vee y) = C(x) \vee C(y)$.*
- *Idempotency: $C(x) = C(C(x))$.*

*Notice that a closure operator is necessarily increasing; thus, it is a hull operator. Indeed, if $x \leq y$, then*

$$C(y) = C(x \vee y) = C(x) \vee C(y),$$

*meaning that $C(y) \geq C(x)$.*

It is not difficult to check that a unary internal operation in a lattice is increasing if and only if it is supadditive. Indeed, let $X$ be a lattice and $T : X \to X$ a unary internal operation. If $T$ is increasing and $x, y \in X$, then $x \leq x \vee y$ and $y \leq x \vee y$. Thus, $T(x) \leq T(x \vee y)$ and $T(y) \leq T(x \vee y)$, and therefore $T(x) \vee T(y) \leq T(x \vee y)$, which implies that $T$ is supadditive. Conversely, if $x \leq y$, then

$$T(y) = T(x \vee y) \geq T(x) \vee T(y) \geq T(x),$$

meaning that $T$ is increasing. As a consequence, every closure operator is a hull operator. In fact, it is not difficult to realize that, in a bounded lattice, a hull operator $H$ is a closure operator if and only if it is null (zero is a fixed point) and additive. From now on, we work with bounded posets satisfying the inf property. Bear in mind that bounded posets satisfying the inf property are lattices.

**Proposition 2.1.** *Let $X$ be a bounded poset satisfying the inf property. Let $H : X \to X$ be a hull operator. Let*

$$Y_H := \{x \in X : H(x) = x\}$$

*be the set of fixed points of $H$. Then the following hold:*

**(a).** $H(1) = 1$.

**(b).** $H(0) = \min\{H(x) : x \in X\}$.

**(c).** *If $H(0) > 0$, then $H(0) = \min\{H(x) : x \in X \setminus \{0\}\}$ and $H(x) = H(0)$ for every $x$ satisfying $0 < x < H(0)$.*

**(d).** *$Y_H$ is inf-closed and contains $1$.*

*If, in addition, $H$ is a closure operator, then $0 \in Y_H$ and $Y_H$ is finitely sup-closed.*

**Proof.** The proof is itemized according to the items of the statement of the theorem.

**(a).** It directly follows from the extensivity that $1 \leq H(1) \leq 1$. In particular, $1 \in Y_H$.

**(b).** Observe that $0 \leq x$ for all $x \in X$, meaning that $H(0) \leq H(x)$ for all $x \in X$. As a consequence,

$$H(0) = \min\{H(x) : x \in X\}.$$

**(c).** Next, suppose that $H(0) > 0$. Then, the inequality $H(0) \leq H(x)$ holds for all $x \in X \setminus \{0\}$ and $H(0) = H(H(0))$, which implies that $H(0) \in \{H(x) : x \in X \setminus \{0\}\}$. This shows that $H(0) = \min\{H(x) : x \in X \setminus \{0\}\}$. If there exists $x$ satisfying $0 < x < H(0)$, then $H(0) \leq H(x) \leq H(H(0)) = H(0)$, and hence $H(x) = H(0)$.

**(d).** It only remains to show that $Y_H$ is inf-closed. Indeed, let $Z \subseteq Y_H$. Note that $Z$ is bounded below because $X$ has a minimum $0$. Since $X$ satisfies the inf property, there exists $z_m \in X$, the infimum of $Z$. We have to show that $z_m \in Y$. Note first that $z_m \leq H(z_m)$ by extensivity. Conversely, $z_m \leq z$ for all $z \in Z$, which means that $H(z_m) \leq H(z)$ for every $z \in Z$ by monotonocity. However, $H(z) = z$ for every $z \in Z$ because $Z \subseteq Y_H$. Then $H(z_m) \leq z$ for every $z \in Z$, which implies that $H(z_m)$ is a lower bound for $Z$. Therefore, $H(z_m) \leq z_m$. As a consequence, we obtain that $H(z_m) = z_m$, and hence $z_m \in Y_H$. This shows that $Y_H$ is inf-closed.

Finally, assume that $H$ is a closure operator. By nullity, it holds that $H(0) = 0$; so, then $0 \in Y_H$. Let us check that $Y_H$ is finitely sup-closed. Indeed, fix arbitrary elements $y_1, y_2 \in Y_H$. Since $H$ is a closure operator, it holds that

$$y_1 \vee y_2 = H(y_1) \vee H(y_2) = H(y_1 \vee y_2) \in Y_H.$$

□

Proposition 2.1 motivates the next result, which is a relevant improvement of the classical way for constructing hull operators on power sets; for instance, see [3, Proposition 1].

**Theorem 2.1** (Construction)**.** *Let $X$ be a bounded poset satisfying the inf property. Let $Y \subseteq X$ be inf-closed containing $1$. The map*

$$\begin{array}{rccl} H_Y: & X & \to & X \\ & x & \mapsto & H_Y(x) := \inf\{y \in Y : x \leq y\}. \end{array} \tag{1}$$

*is a hull operator verifying that $H_Y(X) = Y$ and $Y_{H_Y} = Y$, where $Y_{H_Y}$ is the set of fixed points of $H_Y$, as described in Proposition 2.1. If, in addition, $0 \in Y$, and $Y$ is finitely sup-closed, then $H_Y$ is a closure operator.*

**Proof.** First of all, $H_Y$ is well defined because $X$ satisfies the inf property and $\{y \in Y : x \leq y\}$ is never empty (it contains $1$), no matter which $x \in X$ is chosen. It is obvious that $H_Y(y) = y$ for every $y \in Y$, which shows that $Y \subseteq Y_{H_Y}$. Also, note that $H_Y(x) \in Y$ for every $x \in X$ because $Y$ is inf-closed. Therefore, if $x \in X$ is a fixed point of $H$, then $x = H_Y(x) \in Y$. As a consequence, $Y_{H_Y} \subseteq Y$. Next, we check that all three properties that characterize hull operators are verified for all $x, x_1, x_2 \in X$:

- Extensivity or extensionality: $x \leq H_Y(x)$. This holds by the definition.

- Increasingness or monotonicity: $x_1 \leq x_2 \Rightarrow H_Y(x_1) \leq H_Y(x_2)$. Indeed, $x_1 \leq x_2 \leq H_Y(x_2)$ and $H_Y(x_2) \in Y$. Therefore, $H_Y(x_1) \leq H_Y(x_2)$.

- Idempotency: $H_Y(H_Y(x)) = H_Y(x)$. By extensivity, $H_Y(x) \leq H_Y(H_Y(x))$. The reverse inequality will be proved by showing that $H_Y(H_Y(x))$ is a lower bound for the set $\{y \in Y : x \leq y\}$. Indeed, fix an arbitrary $y \in Y$ with $x \leq y$. By monotonicity, $H_Y(x) \leq H_Y(y) = y$. So, again by monotonicity, $H_Y(H_Y(x)) \leq H_Y(y) = y$. As a consequence, $H_Y(H_Y(x))$ is a lower bound for the set $\{y \in Y : x \leq y\}$. By the definition of the infimum, $H_Y(x) \geq H_Y(H_Y(x))$.

Finally, suppose (in addition) that $0 \in Y$ and $Y$ is finitely sup-closed. Next, we prove that $H_Y$ is a closure operator. We already know that $H_Y$ is subadditive due to the increasingness. So, fix arbitrary elements $x_1, x_2 \in X$. Then,

$$H_Y(x_1) \vee H_Y(x_2) \leq H_Y(x_1 \vee x_2).$$

Now, by the assumption, $Y$ is finitely sup-closed, meaning that $H_Y(x_1) \vee H_Y(x_2) \in Y$. Notice that $x_1 \leq H_Y(x_1)$ and $x_2 \leq H_Y(x_2)$ by extensionality. Therefore, $x_1 \vee x_2 \leq H_Y(x_1) \vee H_Y(x_2)$. Since $H_Y(x_1) \vee H_Y(x_2) \in Y$, by construction of $H_Y$, we conclude that

$$H_Y(x_1 \vee x_2) \leq H_Y(x_1) \vee H_Y(x_2) \in Y,$$

and hence the reverse inequality is obtained. □

We are now in the right position to prove a classification theorem for hull operators.

**Theorem 2.2** (Classification)**.** *Let $X$ be a bounded poset satisfying the inf property. Let $H : X \to X$ be a hull operator. Let $Y_H := \{x \in X : H(x) = x\}$ be the set of fixed points of $H$. Then $H_{Y_H} = H$, where $H_{Y_H}$ is the hull operator induced by $Y$ in (1).*

**Proof.** Fix an arbitrary element $x \in X$. By following (1), one has

$$H_{Y_H}(x) := \inf \{y \in Y_H : x \leq y\}.$$

We will show that $H_{Y_H}(x) = H(x)$. First, if $y \in Y_H$ and $x \leq y$, then $H(x) \leq H(y) = y$ by monotonicity, meaning that $H(x)$ is a lower bound for the set $\{y \in Y_H : x \leq y\}$. Thus,

$$H(x) \leq H_{Y_H}(x).$$

Next, if $y \in Y_H$ and $H(x) \leq y$, then $x \leq H(x) \leq y$ by extensionality. Therefore, $H_{Y_H}(x) \leq y$, by monotonocity of $H_{Y_H}$ together with the fact that $Y_H$ is the set of fixed points of $H_{Y_H}$ (by the virtue of Theorem 2.1). This shows that $H_{Y_H}(x)$ is a lower bound for the set $\{y \in Y_H : H(x) \leq y\}$. As a consequence, $H_{Y_H}(x) \leq H_{Y_H}(H(x))$. However, $H(x) \in Y_H$ by idempotency. So, Theorem 2.1 assures that $Y_{H_{Y_H}} = Y_H$; that is, $H_{Y_H}(H(x)) = H(x)$, and hence

$$H_{Y_H}(x) \leq H_{Y_H}(H(x)) = H(x),$$

which gives the desired inequality. □

We end this section with two nontrivial examples. The first one is an example of a hull operator satisfying the nullity condition, which is not a closure operator.

**Example 2.2.** *Let $X$ be a real vector space with a dimension greater than or equal to 2. Let* span *denote the linear span. It is well known that the linear span is a hull operator which is neither additive nor nullitive. Then, simply consider the following hull operator:*

$$\begin{array}{rccc} H: & \mathcal{P}(X) & \to & \mathcal{P}(X) \\ & A & \mapsto & H(A) := \begin{cases} \varnothing & \text{if } A = \varnothing \\ \operatorname{span}(A) & \text{if } A \neq \varnothing. \end{cases} \end{array}$$

*By construction, $H$ is trivially null. Finally, if $x$ and $y$ are linearly independent vectors of $X$, then*

$$H(\{x, y\}) \neq H(\{x\}) \cup H(\{y\}).$$

The second and final example is about an additive hull operator that is not a closure operator.

**Example 2.3.** *Let $X$ be a topological space. Let*

$$O := \bigcap\{F : F \subseteq X \text{ is closed and nonempty}\} \neq \varnothing.$$

*Let* cl *denote the closure operator. Consider the following hull operator:*

$$\begin{aligned} H: \quad \mathcal{P}(X) &\to \mathcal{P}(X) \\ A &\mapsto H(A) := \begin{cases} O & \text{if } A = \varnothing \\ \operatorname{cl}(A) & \text{if } A \neq \varnothing. \end{cases} \end{aligned} \tag{2}$$

*By construction, $H$ is a hull operator that is additive but not null.*

Bounded posets satisfying the inf property is the least reasonable ambience to consider hull and closure operators. Theorems 2.1 and 2.2 provide a full characterization of hull and closure operators in the context of bounded posets satisfying the inf property. Finally, it is worth mentioning that if $X$ is a distributive complemented lattice satisfying the inf property, then a subset $Y$ of $X$ is inf-closed, finitely sup-closed, and contains $0$ and $1$, if and only if $\tau_Y := \{1 \setminus y : y \in Y\}$ satisfies the conditions for being a topology in the order setting.

## Acknowledgment

This research has been funded by Consejería de Universidad, Investigación e Innovación de la Junta de Andalucía: ProyExcel00780 (Operator Theory: An interdisciplinary approach) and ProyExcel01036 (Multifísica y optimización multiobjetivo de estimulación magnética transcraneal).

## References

[1] G. Castellini, *Categorical Closure Operators*, Birkhäuser, Boston, 2003.
[2] D. Dikranjan, W. Tholen, *Categorical Structure of Closure Operators: With Applications to Topology, Algebra and Discrete Mathematics*, Springer, Dordrecht, 1995.
[3] F. J. García-Pacheco, Chaos in topological modules, *Axioms* **11** (2022) #526.
[4] D. Lau, *Function Algebras on Finite Sets: Basic Course on Many-Valued Logic and Clone Theory*, Springer, Berlin, 2006, 45–49.
[5] M. H. Stone, The theory of representations for Boolean algebras, *Trans. Amer. Math. Soc.* **40** (1936) 37–111.
[6] D. A. Vladimirov, *Boolean Algebras in Analysis*, Springer, Dordrecht, 2002.
[7] L. R. J. Westermann, On the hull operator, *Indag. Math. Proc.* **79** (1976) 179–184.