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Research Article

Primitive ideals in (m,n)–near rings

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Abstract

An (m,n)-near ring is an algebraic structure similar to an (m,n)-ring but satisfying fewer axioms. More precisely, the notion of (m,n)-near rings generalizes the concepts of rings, near rings, and (m,n)-rings. In this article, we define the notions of i-(m,n)-near ring, N-group, N-ideal, η-primitive, constant (m,n)-near ring, modular j-ideal, and η-modular, and investigate their properties.

Keywords: i-(m,n)-near ring; N-group; N-ideal; η-primitive; constant (m,n)-near ring.

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1. Introduction

Let A be a non-empty set. The sequence $z_i, z_{i+1}, \ldots, z_m$ of elements of A is indicated by z_i^m where $1 \leq i \leq m$. For each $1 \leq i \leq j \leq m$, the phrase $h(z_1, z_2, \ldots, z_i, k_{i+1}, \ldots, k_j, l_{j+1}, \ldots, l_m)$ is represented as $h(z_1^i, k_{i+1}^j, l_{j+1}^m)$; in the case when $k_{i+1} = k_{i+2} = \cdots = k_j = k$, we simply write $h(z_1^i, k^{(j-i)}, l_{j+1}^m)$. An m-ary groupoid (A,h) is said to be an m-ary semigroup if h is associative; that is, if $h(z_1^{i-1}, h(z_i^{m+i-1}), z_{m+i}^{2m-1}) = h(z_1^{j-1}, h(z_j^{m+j-1}), z_{m+j}^{2m-1})$ for each $z_1, z_2, \ldots, z_{2m-1} \in A$, where $1 \leq i \leq j \leq m$. An m-ary semigroupoid (A,h) is said to be an m-ary group if for all $c_1^{i-1}, c_{i+1}^m, b \in A$, there exists $z_1^m \in A$, such that $h(c_1^{i-1}, z_i, c_{i+1}^m) = b$ for every $1 \leq i \leq m$. We say that h is commutative if $h(z_1, z_2, \ldots, z_m) = h(z_{\eta(1)}, z_{\eta(2)}, \ldots, z_{\eta(m)})$, for every permutation η of $\{1, 2, \ldots, m\}$ and $z_1, z_2, \ldots, z_m \in A$.

2. The (m,n)-near ring

We recommend that readers familiarize themselves with the basic concepts of near rings by consulting [2,4,5,7]; we do not define such notions in this article. In this section, we define the notion of (m,n)-near rings and provide some examples. We also present definitions of the N-group, N-ideal, η-primitive, and constant (m,n)-near ring. Moreover, we assert theorems related to these concepts.

Definition 2.1. *Assume that A is a non-empty set. Let h and k be the m-ary and n-ary operations on A, respectively. In this case, (A,h,k) is said to be an i-(m,n)-near ring if the following conditions are met:*

(1) (A,h) is an m-ary group (not necessarily abelian);

(2) (A,k) is an n-ary semigroup;

(3) The n-ary operation k is i-distributive with respect to the m-ary operation h,

where the definition of i-distributive condition is as follows: for every $c_1, c_2, \ldots, c_n$, $d_1, d_2 \ldots, d_m \in A$, if $i = n$, then

$$k(c_1^{n-1}, h(d_1, d_2, \ldots, d_m)) = h(k(c_1^{n-1}, d_1), k(c_1^{n-1}, d_2), \ldots, k(c_1^{n-1}, d_m)).$$

If $i = 1$ then

$$k(h(d_1, d_2, \ldots, d_m), c_2^n) = h(k(d_1, c_2^n), k(d_2, c_2^n), \ldots, k(d_m, c_2^n)).$$

If $1 < i < n$ then

$$k(c_1^{i-1}, h(d_1, d_2, \ldots, d_m), c_{i+1}^n) = h(k(c_1^{i-1}, d_1, c_{i+1}^n), k(c_1^{i-1}, d_2, c_{i+1}^n), \ldots, k(c_1^{i-1}, d_m, c_{i+1}^n)).$$

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In the rest of the paper, for simplicity, we write (m,n)-near ring instead of i-(m,n)-near ring. We remark here that every (m,n)-ring is an (m,n)-near ring.

Example 2.1. *Consider the additive group $\mathbb{Z}_{mn}$. Then $(\mathbb{Z}_{mn},h)$ is a group, where $h(s_1,s_2,\ldots,s_m)=s_1+s_2+\cdots+s_m$. We define k on $\mathbb{Z}_{mn}$ by $k(s_1,s_2,\ldots,s_n)=s_1$, for all $s_1,s_2,\ldots,s_n\in\mathbb{Z}_{mn}$. Note that $(\mathbb{Z}_{mn},h,k)$ is an (m,n)-near ring. For $1<i\leqslant n$ and $s_1^n,f_1^m\in\mathbb{Z}_{mn}$, we have $k(s_1,s_2,\ldots,s_{i-1},h(f_1,f_2,\ldots,f_m),s_{i+1},\ldots,s_n)=s_1$ and*

$$h(k(s_1,s_2,\ldots,s_{i-1},f_1,s_{i+1},\ldots,s_n),\ldots,k(s_1,s_2,\ldots,s_{i-1},f_m,s_{i+1},\ldots,s_n))=h(s_1^{(m)})=ms_1.$$

If $mn=m-1$, then $\overline{m}=\overline{1}\in\mathbb{Z}_{mn}$. Hence, for all $1<i\leqslant n$, $(\mathbb{Z}_{mn-1},h,k)$ is i-distributive. For $i=1$, we have

$$k(h(f_1,f_2,\ldots,f_m),s_2,\ldots,s_n)=h(f_1,f_2,\ldots,f_m)=f_1+f_2+\cdots+f_m \quad \text{and}$$
$$h(k(f_1,s_2,\ldots,s_n),k(f_2,s_2,\ldots,s_n),\ldots,k(f_m,s_2,\ldots,s_n))=h(f_1,f_2,\ldots,f_m)=f_1+f_2+\cdots+f_m.$$

Consequently, for $i=1$, $(\mathbb{Z}_{mn-1},h,k)$ is 1-distributive.

Assume that I is a non-empty subgroup of an (m,n)-near ring (A,h,k). Then I is said to be a normal subgroup of A if for each $a_i\in A$, $s_1^{i-1},s_{i+1}^m\in A$ and $1\le i,j\le m$ there is $b_j\in I$ such that $h(s_1^{i-1},a_i,s_{i+1}^m)=h(s_1^{j-1},b_j,s_{j+1}^m)$.

Definition 2.2. *Assume that I is a non-empty subset of an (m,n)-near ring (A,h,l). The set I is said to be an ideal of A if*

(1) I is a normal subgroup of the m-ary group (A,h), (I,h) is an m-ary group,

(2) for every $a_1,a_2,\ldots,a_n\in A$, $l(a_1^{i-1},I,a_{i+1}^n)\subseteq I$.

(3) For all $r_1^{k-1},r_{k+1}^m,w_1^{j-1},w_{j+1}^n\in A$ and $1\le k\le n$, $d\in I$, there exists $o\in I$ so that $l(w_1^{j-1},h(r_1^{k-1},d,r_{k+1}^m),w_{j+1}^n)$ equals $h(l(w_1^{j-1},r_1,w_{j+1}^n),l(w_1^{j-1},r_2,w_{j+1}^n),\ldots,l(w_1^{j-1},r_{k-1},w_{j+1}^n),o,l(w_1^{j-1},r_{k+1},w_{j+1}^n),\ldots,l(w_1^{j-1},r_m,w_{j+1}^n))$.

The set I is called an i-ideal of A if it satisfies (1) and (2). The set I is called a j-ideal of A for $j\neq i$ if it satisfies (1) and (3).

If J is an i-ideal for each $1\le i\le n$, then J is referred to as an ideal of A.

Definition 2.3. *A proper ideal J of an (m,n)-near ring (A,h,l) is said to be prime if for any ideals $A_1,A_2,\ldots,A_n$ of A, $l(A_1,A_2,\ldots,A_n)\subseteq J$ implies $A_1\subseteq J$ or $A_2\subseteq J$ or ... or $A_n\subseteq J$.*

Definition 2.4. *Assume that (N,h,k) is an (m,n)-near ring, (G,h) is an m-group and*

$$f:\underbrace{G\times G\times\cdots\times G}_{i-1}\times N\times\underbrace{G\times G\times\cdots\times G}_{n-i}\longrightarrow G$$

is a mapping $(f(g_1^{i-1},n,g_{i+1}^n)=k(g_1^{i-1},n,g_{i+1}^n))$. In this case, (G,f) is an N-group if for all $g_1^n\in G$ and for all $n_1^m,a_1^n\in N$, the following conditions hold:

(1) $k(g_1^{i-1},h(n_1,n_2,\ldots,n_m),g_{i+1}^n)=h(k(g_1^{i-1},n_1,g_{i+1}^n),k(g_1^{i-1},n_2,g_{i+1}^n),\ldots,k(g_1^{i-1},n_m,g_{i+1}^n))$.

(2) For all $2\le j\le i-1$ and $2\le l\le n$,

$$k(g_1^{i-1},k(a_1^n),g_{i+1}^n)=k(g_1^{j-1},k(g_j^{i-1},a_1^{n-i+j}),a_{n-i+j+1}^n,g_{i+1}^n)=k(g_1^{i-1},a_1^{l-1},k(a_l^n,g_{i+1}^{l+i-1}),g_{l+i}^n).$$

Example 2.2. *In Example 2.1, if we let $N=\mathbb{Z}_{mn-1}$, $i\neq 1$ and $G=\mathbb{Z}_{m-1}$, then $(\mathbb{Z}_{m-1},k)$ is a $\mathbb{Z}_{mn-1}$-group. For all $z_1^n\in\mathbb{Z}_{m-1},z\in\mathbb{Z}_{mn-1}$, we have $k(z_1^{i-1},z,z_{i+1}^n)=z_1\in\mathbb{Z}_{m-1}=G$.*

(1) We note that $k(g_1^{i-1},h(n_1,n_2,\ldots,n_m),g_{i+1}^n)=g_1=mg_1=h(k(g_1^{i-1},n_1,g_{i+1}^n),\ldots,k(g_1^{i-1},n_m,g_{i+1}^n))$.

(2) For all $2\le j\le i-1$ and $2\le l\le n$, we have

$$g_1=k(g_1^{i-1},k(a_1^n),g_{i+1}^n)=k(g_1^{j-1},k(g_j^{i-1},a_1^{n-i+j}),a_{n-i+j+1}^n,g_{i+1}^n)=k(g_1^{i-1},a_1^{l-1},k(a_l^n,g_{i+1}^{l+i-1}),g_{l+i}^n).$$

Definition 2.5. *Assume that (N,h,k) is an (m,n)-near ring, G is an N-group, $\varnothing\neq A\subseteq G$ and $\varnothing\neq B\subseteq G$. We define*

$$(A:B)_N=\{n\in N\ \mid\ k(B^{(i-1)},n,B^{(n-i)})\subseteq A\}.$$

The set $(0:B)_N$ is called the i-annihilator of B in N.

Example 2.3. *In Example 2.1, if we let $N = \mathbb{Z}_{mn-1}$, $G = \mathbb{Z}$ and $A = 3\mathbb{Z}$, and $B = 6\mathbb{Z}$, then*

$$(A : B)_N = (3\mathbb{Z} : 6\mathbb{Z})_{\mathbb{Z}_{mn-1}} = \{s \in \mathbb{Z}_{mn-1} \mid k(6\mathbb{Z}^{(i-1)}, s, 6\mathbb{Z}^{(n-i)}) \subseteq 3\mathbb{Z}\} = \{s \in \mathbb{Z}_{mn-1} \mid 6\mathbb{Z} \subseteq 3\mathbb{Z}\} = \mathbb{Z}_{mn-1}.$$

For $i \neq 1$, we have $(0 : 6\mathbb{Z}) = \{s \in \mathbb{Z} \mid k(6\mathbb{Z}^{(i-1)}, s, 6\mathbb{Z}^{(n-i)}) = 0\} = \{s \in \mathbb{Z} \mid 6\mathbb{Z} = 0\}$, and for $i = 1$, we have

$$(0 : 6\mathbb{Z}) = \{s \in \mathbb{Z} \mid k(s, 6\mathbb{Z}^{(n-1)}) = 0\} = \{s \in \mathbb{Z} \mid s = 0\} = \{0\}.$$

Definition 2.6. *An N-group B is said to be faithful if $k(s_1^{i-1}, o, s_{i+1}^n) = 0$, for all $s_1^n \in B$ and $o \in N$, then $o = 0$.*

Example 2.4. *In Example 2.1, if $i = 1$, $N = \mathbb{Z}_{mn-1}$, and $B = \mathbb{Z}$, then*

$$(0 : B)_N = (0 : \mathbb{Z}) = \{s \in \mathbb{Z}_{mn-1} \mid k(s, \mathbb{Z}^{(n-1)}) = 0\} = \{s \in \mathbb{Z}_{mn-1} \mid s = 0\} = 0_{Z_{mn-1}}.$$

Definition 2.7. *Assume that M is an O-group and Q is a subgroup of M. In this case Q is said to be an O-subgroup of M if $k(Q^{(i-1)}, O, Q^{(n-i)}) \subseteq Q$ and M is said to be O-simple if the only O-subgroups of M are $k(0_M^{(i-1)}, O_c, 0_M^{(n-i)})$ and M.*

Example 2.5. *In Example 2.1, if we take $O = \mathbb{Z}_{mn-1}$, $i \neq 1$, and $M = \mathbb{Z}$, and let Q be a subgroup of $\mathbb{Z}$, then the relation $k(Q^{(i-1)}, \mathbb{Z}_{mn-1}, Q^{(n-i)}) = Q \subseteq Q$ holds. Thus, every subgroup of $\mathbb{Z}$ is a $\mathbb{Z}_{mn-1}$-subgroup of $\mathbb{Z}$.*

Definition 2.8. *Assume that (M, h) is an m-group, $H \subseteq M$, and that M is an O-group for the (m, n)-near ring (O, h, l). In this case, H is an O-ideal of M if the following conditions hold:*

(1) (H, h) is a normal subgroup of (M, h),

(2) for all $r_1^{j-1}, r_{j+1}^m \in M$, $s_1^{j-1}, s_{j+1}^n \in O$, $1 \leq k \neq i \leq m$, $1 \leq j \neq i \leq n$ and $d \in H$, there exists $z \in H$ such that

$$l(s_1^{j-1}, h(r_1^{k-1}, d, r_{k+1}^m), s_{j+1}^n)$$
$$= h(l(s_1^{j-1}, r_1, s_{j+1}^n), l(s_1^{j-1}, r_2, s_{j+1}^n), \ldots, l(s_1^{j-1}, r_{k-1}, s_{j+1}^n), z, l(s_1^{j-1}, r_{k+1}, s_{j+1}^n), \ldots, l(s_1^{j-1}, r_n, s_{j+1}^n)).$$

The O-group M is said to be a simple O-group if 0 and M are the only O-ideals of M.

Example 2.6. *In Example 2.1, if we let $H = 2\mathbb{Z}$, $O = \mathbb{Z}_{mn-1}$, $i = 1$, and $M = \mathbb{Z}$, then $(2\mathbb{Z}, h)$ is a normal subgroup of $(\mathbb{Z}, h)$, and for all $r_1^{j-1}, r_{j+1}^m \in \mathbb{Z}$, $s_1^{j-1}, s_{j+1}^n \in \mathbb{Z}_{mn-1}$, and $1 \leq k \leq n$, $d \in 2\mathbb{Z}$, the following relation holds:*

$$h(r_1^{k-1}, d, r_{k+1}^m) = k(h(r_1^{k-1}, d, r_{k+1}^m), s_2^n) = h(k(r_1, s_2^n), k(r_2, s_2^n), \ldots, k(r_{k-1}, s_2^n), l, k(r_{k+1}, s_2^n), \ldots, k(r_n, s_2^n)).$$

If we let $l = d$, then there exists $l \in 2\mathbb{Z}$ such that the second condition of the definition is valid. Now, we conclude that $2\mathbb{Z}$ is a $\mathbb{Z}_{mn-1}$-ideal of $\mathbb{Z}$

Definition 2.9. *Assume that (M, h, l) is an (m, n)-near ring and (W, l) is an M-group. In this case, W is said to be a monogenic M-group if there is $w \in W$ such that $l(w^{(i-1)}, M, w^{(n-i)}) = W$.*

Example 2.7. *In Example 2.1, if we let $M = \mathbb{Z}_{mn-1}$, $W = \{0\}$, and $w = 0$, then $k(0^{(i-1)}, \mathbb{Z}_{mn-1}, 0^{(n-i)}) = 0 = W$. Hence, $W = \{0\}$ is a monogenic $\mathbb{Z}_{mn-1}$-group.*

Example 2.8. *In Example 2.1, if we let $M = W = \mathbb{Z}_{mn-1}$, then for all $w \in W$, $k(\mathbb{Z}_{mn-1}, w^{n-1}) = \mathbb{Z}_{mn-1} = W$. So, we conclude that $\mathbb{Z}_{mn-1}$ is a monogenic $\mathbb{Z}_{mn-1}$-group.*

Assume that (A, h, k) is an (m, n)-near ring, I is an ideal, (A, h) is a group, and I is a normal subgroup. Then the quotient group $(\frac{A}{I}, H, K)$ is defined. An m-ary operation H on the cosets is defined by the m-ary operation h given below:

$$\begin{aligned} &H(h(d_{1_1}, d_{1_2}, \ldots, d_{1_{m-1}}, I), h(d_{2_1}, d_{2_2}, \ldots, d_{2_{m-1}}, I), \ldots, h(d_{m_1}, d_{m_2}, \ldots, d_{m_{m-1}}, I)) \\ &= h(h(d_{1_1}, d_{1_2}, \ldots, d_{1_{m-1}}, h(d_{2_1}, d_{2_2}, \ldots, d_{2_{m-1}}, h(d_{3_1}, d_{3_2}, \ldots, d_{3_{m-1}}, \ldots \\ &\quad h(d_{(m-1)_1}, d_{(m-1)_2}, \ldots, d_{(m-1)_{m-1}} h(d_{m_1}, d_{m_2}, \ldots, d_{m_{m-1}}, I) \ldots)). \end{aligned}$$

An n-ary operation K on cosets is defined by the n-ary operation k given below:

$$\begin{aligned} &K(h(d_{1_1}, d_{1_2}, \ldots, d_{1_{m-1}}, I), h(d_{2_1}, d_{2_2}, \ldots, d_{2_{m-1}}, I), \ldots, h(d_{n_1}, d_{n_2}, \ldots, d_{n_{m-1}}, I)) \\ &= h(k(h(d_{1_1}, d_{1_2}, \ldots, d_{1_{m-1}}, I), \ldots, h(d_{(i-1)_1}, d_{(i-1)_2}, \ldots, d_{(i-1)_{(m-1)}}, I), d_{i_1}, \\ &h(d_{(i+1)_1}, d_{(i+1)_2}, \ldots, d_{(i+1)_{m-1}}, I)) \ldots, h(d_{n_1}, d_{n_2}, \ldots, d_{n_{m-1}}, I)), \ldots, k(h(d_{1_1}, d_{1_2}, \ldots, \\ &d_{1_{m-1}}, I), \ldots, h(d_{(i-1)_1}, d_{(i-1)_2}, \ldots, d_{(i-1)_{m-1}}, I), d_{i_{m-1}}, h(d_{(i+1)_1}, d_{(i+1)_2} \ldots, d_{(i+1)_{m-1}}, I)) \ldots, h(d_{n_1}, d_{n_2}, \ldots, d_{n_{m-1}}, I)), I). \end{aligned}$$

Theorem 2.1. *If I is an ideal in an (m,n)-near ring (A,h,k), then $(\frac{A}{I},H,K)$ has the structure of an (m,n)-near ring, where the operations H and K are defined after Example 2.8.*

Proof. We prove that H is well defined. Assume that $h(d_{i_1},d_{i_2},\ldots,d_{i_{m-1}},I)=h(w_{i_1},w_{i_2},\ldots,w_{i_{m-1}},I)$, for $1\le i\le m$. Then

$$H(h(d_{1_1},d_{1_2},\ldots,d_{1_{m-1}},I),h(d_{2_1},d_{2_2},\ldots,d_{2_{m-1}},I),\ldots,h(d_{m_1},d_{m_2},\ldots,d_{m_{m-1}},I))$$

$$=h(h(d_{1_1},d_{1_2}\ldots,d_{1_{m-1}},h(d_{2_1},d_{2_2},\ldots,d_{2_{m-1}},h(d_{3_1},d_{3_2},\ldots,d_{3_{m-1}},$$
$$\ldots h(d_{(m-1)_1},d_{(m-1)_2},\ldots,d_{(m-1)_{m-1}}h(d_{m_1},\ldots,d_{m_{m-1}},I)\ldots))$$

$$=h(d_{1_1},d_{1_2},\ldots,d_{1_{m-1}},h(d_{2_1},d_{2_2},\ldots,d_{2_{m-1}},h(d_{3_1},d_{3_2},\ldots,d_{3_{m-1}},\ldots$$
$$h(d_{(m-1)_1},d_{(m-1)_2},\ldots,d_{(m-1)_{m-1}},h(w_{m_1},w_{m_2},\ldots,w_{m_{m-1}},I)\ldots))$$

$$=h(d_{1_1},d_{2_2},\ldots,d_{1_{m-1}},h(d_{2_1},d_{2_2},\ldots,d_{2_{m-1}},h(d_{3_1},d_{3_2},\ldots,d_{3_{m-1}},\ldots$$
$$h(d_{(m-1)_1},d_{(m-1)_2},\ldots,d_{(m-1)_{m-1}},h(I,w_{m_1},w_{m_2},\ldots,w_{m_{m-1}})\ldots))$$

$$=h(d_{1_1},d_{1_2},\ldots,h(d_{1_{(m-1)}},h(d_{2_1},d_{2_2},\ldots,d_{2_{m-1}},h(d_{3_1},d_{3_2},\ldots,d_{3_{m-1}},\ldots$$
$$h(h(d_{(m-1)_1},d_{(m-1)_2},\ldots,d_{(m-1)_{m-1}},I),h(w_{m_1},w_{m_2},\ldots,w_{m_{m-1}})\ldots))$$

$$=h(d_{1_1},d_{1_2},\ldots,d_{1_{m-1}},h(d_{2_1},d_{2_2},\ldots,d_{2_{m-1}},h(d_{3_1},d_{3_2},\ldots,d_{3_{m-1}},\ldots$$
$$h(h(w_{(m-1)_1},w_{(m-1)_2},\ldots,w_{(m-1)_{m-1}},I),w_{m_1},w_{m_2},\ldots,w_{m_{(m-1)}})\ldots))$$

$$=\cdots=h(d_{1_1},d_{1_2},\ldots,d_{1_{m-1}},h(h(d_{2_1},d_{2_2},\ldots,d_{2_{m-1}},I),w_{3_1},w_{3_2},\ldots,w_{3_{m-1}}),\ldots$$
$$h(w_{(m-1)_1},w_{(m-1)_2},\ldots,w_{(m-1)_{m-1}},h(w_{m_1},w_{m_2},\ldots,w_{m_{m-1}},I)\ldots))$$

$$=h(d_{1_1},d_{1_2},\ldots,d_{1_{m-1}},h(h(w_{2_1},w_{2_2},\ldots,w_{2_{m-1}},I),w_{3_1},w_{3_2},\ldots,w_{3_{m-1}}),$$
$$\ldots h(w_{(m-1)_1},w_{(m-1)_2},\ldots,w_{(m-1)_{m-1}},h(w_{m_1},w_{m_2},\ldots,w_{m_{m-1}},I)\ldots))$$

$$=h(d_{1_1},d_{1_2},\ldots,d_{1_{m-1}},h(h(I,w_{2_1},w_{2_2},\ldots,w_{2_{m-1}}),w_{3_1},\ldots,w_{3_{m-1}}),\ldots,$$
$$h(w_{(m-1)_1},w_{(m-1)_2},\ldots,w_{(m-1)_{m-1}},h(w_{m_1},w_{m_2},\ldots,w_{m_{m-1}},I)\ldots))$$

$$=h(h(d_{1_1},d_{1_2},\ldots,d_{1_{m-1}},I),w_{2_1},w_{2_2},\ldots,w_{2_{m-1}}),h(w_{3_1},w_{3_2},\ldots,w_{3_{m-1}},\ldots$$
$$h(w_{(m-1)_1},w_{(m-1)_2},\ldots,w_{(m-1)_{m-1}},h(w_{m_1},w_{m_2},\ldots,w_{m_{m-1}},I)\ldots))$$

$$=h(w_{1_1},w_{1_2},\ldots,w_{1_{m-1}},h(w_{2_1},w_{2_2},\ldots,w_{2_{m-1}},h(w_{3_1},w_{3_2},\ldots,w_{3_{m-1}},\ldots$$
$$h(w_{(m-1)_1},w_{(m-1)_2},\ldots,w_{(m-1)_{m-1}},h(w_{m_1},w_{m_2},\ldots,w_{m_{m-1}},I)\ldots))$$

$$=H(h(w_{1_1},w_{1_2},\ldots,w_{1_{m-1}},I),\ldots,h(w_{m_1},w_{m_2},\ldots,w_{m_{m-1}},I)).$$

Since I is an ideal, it follows that the operator K is well defined and since (A,h) is an m-ary group so $(\frac{A}{I},H)$ is an m-ary group. Furthermore, since (A,k) is an n-ary semigroup, it follows that $(\frac{A}{I},K)$ is an n-ary semigroup. The n-ary operation k is i-distributive with respect to the m-ary operation h. Thus, the n-ary operation K is i-distributive with respect to the m-ary operation H. □

Definition 2.10. *Assume that (M,h,k) is an (m,n)-near ring. In this case, a monogenic M-group W $(W\neq 0)$ is of*

(1) *type 0 if W has no M-ideals except 0 and W,*

(2) *type 1 if W is of type 0 and for all $w\in W$ either $k(w^{(i-1)},M,w^{(n-i)})=W$ or $k(w^{(i-1)},M,w^{(n-i)})=k(0_W^{(i-1)},M_c,0_W^{(n-i)})$,*

(3) *type 2 if W has no M-subgroups except 0 and W (particularly, W is M_0-simple),*

(4) *type 3 if W is of type 2 and for all $s\in M$, $k(a_1^{i-1},s,a_{i+1}^n)=k(w_1^{i-1},s,w_{i+1}^n)$ implies $a_i=w_i$ for all $i\in\{1,2,\ldots,n\}$.*

Example 2.9. *In Example 2.1, if we let $M=\mathbb{Z}_{mn-1}$ and $W=\{0\}$, then $\mathbb{Z}_{mn-1}$-group $\{0\}$ is of type* 0.

Example 2.10. *In Example 2.1, if we let $M=W=\mathbb{Z}_{mn-1}$ and $i=1$, then $\mathbb{Z}_{mn-1}$-group $\mathbb{Z}_{mn-1}$ is of type* 1.

Primitive near rings play a particular role in the structure theory of near rings because of the applications of primitive rings in ring theory [3, 8–11]. In the case of (m,n)-near rings, we can consider several kinds of primitives.

Definition 2.11. *Let (W,h,k) be an (m,n)-near ring. Then W is said to be η-primitive, for $\eta \in \{0,1,2,3\}$, if there exists a group S such that S is a faithful W-group of type η.*

An ideal I of an (m,n)-near ring (M,h,k) is an η-primitive ideal of M if and only if $\frac{M}{I}$ is an η-primitive.

Definition 2.12. *Γ is said to be strongly monogenic if $k(\Gamma^{(i-1)},N,\Gamma^{(n-i)}) \neq \{0\}$ and for all $s \in \Gamma$ either $k(s^{(i-1)},N,s^{(n-i)}) = \Gamma$ or $k(s^{(i-1)},N,s^{(n-i)}) = \{0\}$.*

Example 2.11. *In Example 2.7, $W = \{0\}$ is a strongly monogenic $\mathbb{Z}_{mn-1}$-group.*

Definition 2.13. *Assume that (W,h,k) is an (m,n)-near ring and 0 is the identity element of (W,h). Then*

$$W_0 = \{w \in W \mid k(0^{(s-1)},w,0^{(n-s)}) = 0,\ 1 \le s \le n\}$$

is called the zero symmetric part of W. In addition, $W_c = \{w \in W \mid k(0^{(s-1)},w,0^{(n-s)}) = w,\ 1 \le s \le n\}$ is called the resistant part of W. An (m,n)-near ring W is said to be a zero symmetric near ring if $W = W_0$. An (m,n)-near ring W is said to be a constant (m,n)-near ring if $W = W_c$.

Example 2.12. *In Example 2.1, if we let $W = \mathbb{Z}_{mn-1}$, $i \neq 1$, then*

$$\mathbb{Z}_{(mn-1)_0} = \{w \in \mathbb{Z}_{mn-1} \mid k(0^{(s-1)},w,0^{(n-s)}) = 0,\ 1 \le s \le n\} = \mathbb{Z}_{mn-1},$$

which implies that $\mathbb{Z}_{(mn-1)_c} = \{w \in \mathbb{Z}_{mn-1} \mid k(w,0^{(n-1)}) = w,\ 1 \le s \le n\} = \mathbb{Z}_{mn-1}$. Therefore, $\mathbb{Z}_{mn-1}$ is a constant (m,n)-near ring.

Lemma 2.1. *A_0 and A_c are i-(m,n)-subnear rings of the i-(m,n)-near ring (A,c,l) for $i = 1,n$.*

Proof. We show that A_0 is a subgroup of A. If $x_1,x_2,\ldots,x_m \in A_0$, then $l(0^{(i-1)},x_j,0^{(n-i)}) = 0$ for $1 \le j \le m$ and $1 \le i \le n$. Now, if $i = n$, then $l(0^{(i-1)},c(x_1,x_2,\ldots,x_m),0^{(n-i)}) = c(l(0^{(i-1)},x_1,0^{(n-i)}),l(0^{(i-1)},x_2,0^{(n-i)}),\ldots,l(0^{(i-1)},x_m,0^{(n-i)})) = 0$. Therefore, $c(x_1,x_2,\ldots x_m) \in A_0$. Since (A,h) is an m-group, for all $x_1^m, y \in A_0$ there is $s \in A$ such that for all $1 \le j \le m$, $c(x_1^{j-1},s,x_{j+1}^m) = y$. It is enough to show $s \in A_0$. Here, we have

$$\begin{aligned} 0 &= l(0^{(i-1)},y,0^{(n-i)}) = l(0^{(i-1)},c(x_1^{j-1},s,x_{j+1}^m),0^{(n-i)}) \\ &= c(l(0^{(i-1)},x_1,0^{(n-i)}),\ldots,l(0^{(i-1)},x_{i-1},0^{(n-i)}),l(0^{(i-1)},s,0^{(n-i)}),l(0^{(i-1)},x_{i+1},0^{(n-i)}),\ldots,l(0^{(i-1)},x_m,0^{(n-i)})) \\ &= c(0^{(j-1)},l(0^{(i-1)},s,0^{(n-i)}),0^{(m-j)}). \end{aligned}$$

Hence, $0 = c(0^{(j-1)},l(0^{(i-1)},s,0^{(n-i)}),0^{(m-j)})$. Since (A,c) is an m-group, it follows that $0 = l(0^{(i-1)},s,0^{(n-i)})$. Thus, $s \in A_0$, which implies that (A_0,c) is a subgroup of (A,c,l). Next, if we take $s_1,s_2,\ldots,s_n \in A_0$, then for all $1 \le i \le n$ and $1 \le j \le n$, we have $l(0^{(i-1)},s_j,0^{(n-i)}) = 0$. If $i = n$, then $l(0^{(n-1)},l(s_1,s_2,\ldots,s_n))$ equals $l(l(0^{(n-1)},s_1),s_2,\ldots,s_n) = l(0,s_2,s_3,\ldots,s_n) = l(l(0^{(n)}),s_2,s_3,\ldots,s_n) = l(0,l(0^{(n-1)},s_2),s_3,\ldots,s_n) = l(0,0,s_3,s_4,\ldots,s_n) = \cdots = l(0^{(n-1)},s_n) = 0$. If $i = 1$, then

$$\begin{aligned} l(l(s_1,s_2,\ldots,s_n),0^{(n-1)}) &= l(s_1,s_2,\ldots,s_{n-1},l(s_n,0^{(n-1)})) = l(s_1,s_2,\ldots,s_{n-1},0) = l(s_1,s_2,\ldots,s_{n-1},l(0^{(n)})) \\ &= l(s_1,s_2,\ldots,s_{n-2},l(s_2,0^{(n-1)}),0) = l(s_1,s_2,\ldots,s_{n-2},0,0) = \cdots = l(s_1,0^{(n-1)}) = 0. \end{aligned}$$

Therefore, $l(s_1,s_2,\ldots,s_n) \in A_0$. So, we get $l(A_0^{(n)}) \subseteq A_0$. This indicates that (A_0,c,l) is an (m,n)-subnear ring of (m,n)-near ring (A,c,l). We show that A_c is a subgroup of A. Let $x_1,x_2,\ldots,x_m \in A_0$. Then, we have $l(0^{(i-1)},x_j,0^{(n-i)}) = x_j$ for $1 \le j \le m$ and $1 \le i \le n$. Now, we obtain

$$l(0^{(i-1)},c(x_1,x_2,\ldots x_m),0^{(n-i)}) = c(l(0^{(i-1)},x_1,0^{(n-i)}),l(0^{(i-1)},x_2,0^{(n-i)}),\ldots,l(0^{(i-1)},x_m,0^{(n-i)})) = l(x_1,x_2,\ldots,x_m).$$

This yields that $c(x_1,x_2,\ldots x_m) \in A_c$. as (A,h) is an m-group, for all $x_1^m, y \in A_c$ there is $s \in A$ such that for all $1 \le j \le m$, $c(x_1^{j-1},s,x_{j+1}^m) = y$. It is enough to show that $s \in A_c$. We know that

$$\begin{aligned} c(x_1^{j-1},s,x_{j+1}^m) &= y = l(0^{(i-1)},y,0^{(n-i)}) = l(0^{(i-1)},c(x_1^{j-1},s,x_{j+1}^m),0^{(n-i)}) \\ &= c(l(0^{(i-1)},x_1,0^{(n-i)}),...,l(0^{(i-1)},x_{i-1},0^{(n-i)}),l(0^{(i-1)},s,0^{(n-i)}),l(0^{(i-1)},x_{i+1},0^{(n-i)}),...,l(0^{(i-1)},x_m,0^{(n-i)})) \\ &= c(x_1^{j-1},l(0^{(i-1)},s,0^{(n-i)}),x_{j+1}^m). \end{aligned}$$

So, we obtain $c(x_1^{j-1},s,x_{j+1}^m) = c(x_1^{j-1},l(0^{(i-1)},s,0^{(n-i)}),x_{j+1}^m)$. As (A,c) is an m-group, we get $s = l(0^{(i-1)},s,0^{(n-i)})$, and so $s \in A_c$. Hence, (A_c,c) is a subgroup of (A,c,l). Next, if $s_1,s_2,\ldots,s_n \in A_c$, then $l(0^{(i-1)},s_j,0^{(n-i)}) = s_j$, for all $1 \le i \le n$, $1 \le j \le n$. This gives that if $i = n$, then $l(0^{(n-1)},l(s_1,s_2,\ldots,s_n)) = l(l(0^{(n-1)},s_1),s_2,\ldots,s_n) = l(s_1,s_2,\ldots,s_n)$. On the other hand, if $i = 1$, then we have $l(l(s_1,s_2,\ldots,s_n),0^{(n-1)}) = l(s_1,s_2,\ldots,l(s_n,0^{(n-1)})) = l(s_1,s_2,\ldots,s_n)$.

Therefore, $l(s_1,s_2,\ldots,s_n) \in A_c$ and $l(A_c^{(n)}) \subseteq A_c$. Hence, (A_c,c,l) is an (m,n)-subnear ring of (m,n)-near ring (A,c,l). □

Theorem 2.2. *If Γ is a monogenic M-group (by y_0) and an M_0-simple (Γ can be considered as an M_0-group), then either $k(\Gamma^{(i-1)}, M, \Gamma^{(n-i)}) = 0$ or Γ is an M-group strongly monogenic.*

Proof. For all $s \in \Gamma$, $k(s^{(i-1)}, M, s^{(n-i)}) \leq_{M_0} \Gamma$ implies $k(s^{(i-1)}, M, s^{(n-i)}) = \{0\}$ or $k(s^{(i-1)}, M, s^{(n-i)}) = \Gamma$. □

Theorem 2.3. *Assume that I is an ideal of $(2,2)$-near ring (S, h, z), Γ is a group, and $\eta \in \{1,2,3\}$ (see [6]).*

(1) If Γ is an S-group with $I \subseteq (0 : \Gamma)$ then $z(h(n_1, I), \gamma_1) = z(n_1, \gamma_1)$ makes Γ an $\frac{S}{I}$-group.

If Γ is an S-group of type η, then Γ is an $\frac{S}{I}$-group of type η.

If Γ is a faithful S-group, then Γ is a faithful $\frac{S}{I}$-group.

(2) If Γ is an $\frac{S}{I}$-group, then $z(h(n_1, I), \gamma_1) = z(n_1, \gamma_1)$, so Γ is an S-group with $I \subseteq (0 : \Gamma)$.

If Γ is an $\frac{S}{I}$-group of type η, then Γ is an S-group of type η.

If Γ is a faithful $\frac{S}{I}$-group, then Γ is a faithful S-group.

Proof. We prove the result for $i = 1$. The proof for $i = 2$ is similar to the proof concerning $i = 1$. We have $z(h(n, I), \gamma) = h(z(n, \gamma), z(I, \gamma)) = z(n, \gamma) \in \Gamma$, and so $z(h(n, I), \gamma) \in \Gamma$. Hence, Γ is an $\frac{S}{I}$-group. Let Γ be an S-group of type 0. So, Γ has no S-ideals except 0 and Γ. Assume that L is an $\frac{S}{I}$-ideal of Γ, for all $d \in L$, $r \in \Gamma$, and $h(l, I) \in \frac{S}{I}$; so, there is $s \in L$ such that

$$z(h(d,r), h(l,I)) = h(s, z(r, h(l,I))), z(l, h(d,r)) = z(h(l,I), h(d,r)) = h(s, z(h(l,I), r)) = h(s, z(l,r)).$$

This implies that L is an S-ideal of Γ. Thus, $L = 0$ or $L = \Gamma$, and consequently, Γ is an $\frac{S}{I}$-group of type 0.

If Γ is an S-group of type 1, then for all $g \in \Gamma$, $z(S, g) = \Gamma$ or $z(S, g) = z(S_c, 0_\Gamma)$. Also, $z(\frac{S}{I}, g) = z(h(S, I), g) = z(S, g)$. Thus, $z(\frac{S}{I}, g) = \Gamma$ or $z(\frac{S}{I}, g) = z(S_c, 0_\Gamma) = z(h(S_c, I), 0_\Gamma) = z(\frac{S_c}{I}, 0_\Gamma)$. Therefore, Γ is an $\frac{S}{I}$-group of type 1.

If Γ is an S-group of type 2, then Γ has no S-subgroups except 0 and Γ. Assume that H is an $\frac{S}{I}$-subgroup of Γ. So, $z(\frac{S}{I}, H) \subseteq H$ and $z(\frac{S}{I}, H) = z(h(S, I), H) = h(z(S, H), z(I, H)) = z(S, H)$. It means that $z(S, H) \subseteq H$, which implies that H is an S-subgroup of Γ. Therefore, $H = 0$ or $H = \Gamma$. Consequently, Γ is an $\frac{S}{I}$-group of type 2.

Assume that Γ is a faithful S-group, $(0 : \Gamma)_S = 0$. If $h(n, I) \in (0 : \Gamma)_{\frac{S}{I}}$, then $0 = z(h(n, I), \gamma) = z(n, \gamma)$. Therefore, $n \in (0 : \gamma) = 0$, which implies that $n = 0$, and hence $(0 : \gamma) = 0_{\frac{S}{I}}$. So, we deduce that Γ is a faithful $\frac{S}{I}$-group.

Next, we prove (2). If Γ is an $\frac{S}{I}$-group and $z(n_1, \gamma_1) = z(h(n_1, I), \gamma_1) \in \Gamma$, then Γ is an S-group.

Assume that Γ is an $\frac{S}{I}$-group of type 0 so Γ has no $\frac{S}{I}$-ideals except 0 and Γ. Assume that S is an S-ideal of Γ. Then, for all $d \in L$, $r \in \Gamma$ and $l \in S$ there is $s \in L$ such that

$$z(l, h(d,r)) = h(s, z(l,r)), z(h(l,I), h(d,r)) = z(l, h(d,r)) = h(s, z(l,r)) = h(s, z(h(l,I), r)).$$

This yields that L is an $\frac{S}{I}$-ideal of Γ, and hence $L = 0$ or $L = \Gamma$. Therefore, Γ is an S-group of type 0.

If Γ is an $\frac{S}{I}$-group of type 1, then for all $g \in \Gamma$, we have $z(\frac{S}{I}, g) = \Gamma$ or $z(\frac{S}{I}, g) = z(\frac{S_c}{I}, 0_\Gamma)$, $z(\frac{S}{I}, g) = z(h(S, I), g) = z(S, g)$. Thus, $z(S, g) = \Gamma$ or $z(S, g) = z(S_c, 0_\Gamma) = z(h(S_c, I), 0_\Gamma) = z(\frac{S_c}{I}, 0_\Gamma)$. Hence, we conclude that Γ is an S-group of type 1

If Γ is an $\frac{S}{I}$-group of type 2, then Γ has no $\frac{S}{I}$-subgroups except 0 and Γ. Assume that H is an S-subgroup of Γ, so $z(S, H) \subseteq H$ and $z(\frac{S}{I}, H) = z(h(S, I), H) = h(z(S, H), z(I, H)) = z(S, H)$. Thus, $z(\frac{S}{I}, H) \subseteq H$, which implies that H is an $\frac{S}{I}$-subgroup of Γ. Therefore, we have $H = 0$ or $H = \Gamma$. Consequently, Γ is an S-group of type 2.

Assume that Γ is a faithful $\frac{S}{I}$-group, so $(0 : \Gamma)_{\frac{S}{I}} = 0$. Assume that $n \notin I$ and $n \in (0 : \Gamma)_S$. So, $z(h(n, I), \gamma) = z(n, \gamma) = 0$. This implies that $h(n, I) \in (0 : \gamma)_{\frac{S}{I}} = 0$. Hence, $h(n, I) = 0_{\frac{S}{I}}$ and so $n = 0$. This yields that $(0 : \gamma)_S = 0$. Therefore, we conclude that Γ is a faithful S-group. □

We note that $(\frac{S}{I})_0 = \frac{S_0}{I}$. Each S-group Γ can be viewed as an S_0-group as well as an S_c-group [6].

Theorem 2.4. *If Γ is a faithful N-group, then N_c and N_0 are faithful N-groups (see [6])*

Theorem 2.5. *Assume that I is an ideal of a $(2,2)$-near ring (N, h, k). The following statements are equivalent:*

(1) I is an η-primitive.

(2) there is an $\frac{N}{I}$-group Γ such that Γ is faithful and of type η.

Proof. $(1) \iff (2)$ By the definition of η-primitive ideal, I is η-primitive if and only if $\frac{N}{I}$ is an η-primitive near ring if and only if there exists an $\frac{N}{I}$-group Γ such that $\frac{N}{I}$ is η-primitive on $\frac{N}{I}$-group Γ if and only if there exists an $\frac{N}{I}$-group Γ such that Γ is faithful and of type η. □

Theorem 2.6. *If (M, h, k) is a simple (m, n)-near ring and Γ is an M-group of type η, then M is an η-primitive on Γ.*

Proof. We only need to prove that Γ is faithful. Since $(0 : \Gamma)$ is a normal subgroup of M, we have $(0 : \Gamma) = \{0\}$. Hence M is an η-primitive on Γ (see [6]). □

3. Modular j-ideals

Definition 3.1. *For $i \neq j$, a j-ideal of the (m,n)-near ring (N,h,k) is modular if there are some $e_1^{i-1}, e_{i+1}^n \in N$ such that for all $l \in N$, there are $j_1^m \in J$ satisfying $l = h(j_1^{j-1}, k(e_1^{i-1}, l, e_{i+1}^n), j_{j+1}^m)$.*

Assume that (A,h,k) is an (m,n)-near ring. An element $e \in A$ is said to be an i-identity element if $k(e^{(i-1)}, z, e^{(n-i)}) = z$. If for all $1 \leqslant i \leqslant n$, $e \in A$ is an i-identity element, then e is called an identity element.

Remark 3.1. *(1) If A_1 and A_2 are j-ideals $(i \neq j)$ of N with $A_1 \subseteq A_2$ and A_1 is modular by e_1^{i-1}, e_{i+1}^n then A_2 is modular by e_1^{i-1}, e_{i+1}^n.*

(2) $\{0\}$ is modular if N contains an identity element.

(3) Every normal subgroup of (N_c, h) is a modular j-ideal $(i \neq j)$ of N_c (by any element of N_c).

(4) If L is modular by e_1^{i-1}, e_{i+1}^n in an (m,n)-near ring N, then $e_1^{i-1}, e_{i+1}^n \in L$ if and only if $L = N$.

Theorem 3.1. *Each modular j-ideal $(i \neq j)$ $S \neq N = N_0$ is contained in a maximal one (which is modular, too).*

Proof. Let S be modular by e_1^{i-1}, e_{i+1}^n. By applying Zorn's Lemma to the set of all j-ideals $S \subseteq I$ with $e_1^{i-1}, e_{i+1}^n \notin I$ and using Remark 3.1(1), we obtain the desired conclusion. □

Theorem 3.2. *Assume that (N,h,k) is an i-(m,n)-near ring and L is an j-ideal $(i \neq j)$ of N. If L is modular, then $(L:N) \subseteq L$.*

Proof. We consider a monogenic N-group Γ (by γ) with $L = (0:\gamma)$. Then $(L:N) = (0:\frac{N}{L}) = (0:\Gamma) \subseteq (0:\gamma) = L$. □

Theorem 3.3. *If (M,h,l) is an $(m,2)$-near ring and H is modular by e_1, then*

$$(H:M) = (H: l(e_1^{i-1}, M, e_{i+1}^n))$$

and this is the greatest ideal of M contained in H.

Proof. If $n \in (H:M)$ and $i=1$, then $l(n,M) \subseteq H$. So, we have

$$l(n, l(M, e_1)) = l(l(n,M), e_1) \subseteq l(H, e_1) \subseteq H.$$

Hence, we get $(H:M) \subseteq (H: l(M,e_1)$. If $n \in (H: l(M,e_1))$, then $l(n, l(M,e_1) \subseteq H$. So, for all $m \in M$, we have $l(n, l(m,e_1)) \in H$. On the other hand, since H is modular, it follows that

$$l(n,m) = h(l_1^{j-1}, l(l(n,m),e_1), l_{j+1}^m) = h(l_1^{j-1}, l(n, l(m,e_1)), l_{j+1}^m) \in H.$$

Thus, $n \in (H:M)$, which implies that $(H: l(M,e_1)) \subseteq (H:M)$, and so $(H: l(M,e_1)) = (H:M)$.

If $n \in (H:M)$ and $i=2$, then $l(M,n) \subseteq H$. So, we have

$$l(l(e_1,M),n) = l(e_1, l(M,n)) \subseteq l(e_1,H) \subseteq H.$$

Hence, we obtain

$$(H:M) \subseteq (H: l(e_1,M)).$$

If $n \in (H: l(e_1,M))$, then $l(l(e_1,M),n) \subseteq H$. So, for all $m \in M$, we have $l(l(e_1,m),n) \in H$. On the other hand, since H is modular, it follows that

$$l(m,n) = h(l_1^{j-1}, l(e_1, l(m,n)), l_{j+1}^m) = h(l_1^{j-1}, l(l(e_1,m),n), l_{j+1}^m) \in H.$$

So, we get $n \in (H:M)$, and hence $(H: l(e_1,M)) \subseteq (H:M)$, which implies that $(H: l(e_1,M)) = (H:M)$.

Since H is a j-ideal, it follows that $(H:M)$ is an ideal of M and hence $(H:M) \subseteq H$ by Theorem 3.2. If J is a normal subgroup of M with $J \subseteq H$ then trivially $J \subseteq (H:M)$. □

Definition 3.2. *Assume that $\eta \in \{1,2,3\}$ and $i \neq j$. A j-ideal L of (m,n)-near ring S is said to be η-modular if L is modular and $\frac{S}{I}$ is an S-group of type η.*

The next theorem is stated for $(2,2)$-near ring $(R,+,.)$ in [1].

Theorem 3.4. *Let (M,h,k) be an i-(m,n)-near ring and S be an M-group.*

(1) If I is a j-ideal of M such that $j \neq i$, then $k(M_0^{(j-1)}, I, M_0^{(n-j)}) \subseteq I$.

(2) If B is an M-ideal of S, then B is an M_0-subgroup of S.

Proof. The result follows straightforwardly from the definitions of the ideal and M-ideal. □

Lemma 3.1. *Assume that (M,h,k) is an (m,n)-near ring. The following conditions are equivalent:*

(1) M is a zero symmetric near ring.

(2) Every j-ideal of M, $j \neq i$, is an M-subgroup of M.

Proof. $(1) \Rightarrow (2)$. It follows from Lemma 3.4.

$(2) \Rightarrow (1)$. Suppose that every j-ideal of M is an M-subgroup of M. Since 0 is clearly a j-ideal of M, it follows that 0 is an M-subgroup of M. Consequently, $k(0^{(j-1)}, M, 0^{(n-j)}) = 0$. This shows that $M = M_0$. □

Lemma 3.2. *If S is an M-group, then for all $x \in S$ and $n = 2$, $k(x^{(i-1)}, M, x^{(n-i)})$ is an M-subgroup of S.*

Proof. If $i = 1$, then

$$k(M, k(M,x)) = k(k(M,M),x) = k(M,x),$$

so $k(M,x)$ is an N-subgroup of S.

If $i = 2$, then

$$k(k(x,M),M) = k(x,k(M,M)) = k(x,M),$$

so $k(x,M)$ is an M-subgroup of S. □

Definition 3.3. *Assume that (M,h,k) is an (m,n)-near ring. M is said to have the i-cancellation property if and only if whenever $r, s, r_1^{i-1}, r_{i+1}^n \in M$, $r_j \neq 0$ for $i \in \{1,2,\ldots,n\}$, and $k(r_1^{i-1}, r, r_{i+1}^n) = k(r_1^{i-1}, r, r_{i+1}^n)$, then $r = s$.*

Lemma 3.3. *Assume that S is an M-group and O is a subgroup of S.*

(1) If O is an M-ideal of S, then O is an M_0-ideal of S.

(2) If O is an M-subgroup of S, then O is an M_0-subgroup of S.

Proof. (1) Assume that O is an M-ideal of S. Since $M_0 \subseteq M$, it follows that O is an M_0-ideal of S.

(2) Assume that O is an M-subgroup of S. then, $k(O^{(i-1)}, M, O^{(n-i)}) \subseteq O$. Thus, we get

$$k(O^{(i-1)}, M_0, O^{(n-i)}) \subseteq k(O^{(i-1)}, M, O^{(n-i)}) \subseteq O.$$

This yields that O is an M_0-subgroup of S. □

Lemma 3.4. *Assume that (M,h,l) is an i-(m,n)-near ring, S is an M-group, and H_1, H_2 are subsets of S.*

(1) If H_1 be a normal subgroup of S, then $(H_1 : H_2)$ is a normal subgroup of the (m,n)-near ring M.

(2) If $n = 2$ and H_1 is an M-subgroup of S, then $(H_1 : H_2)$ is an M-subgroup of M. (It is also valid for an M-group.)

(3) If $n = 2$, H_1 is an M-ideal of S, and H_2 is an M-subset of S, then $(H_1 : H_2)$ is an i-ideal of M.

Proof. (1) Since H_1 is a normal subgroup of S, it follows that for all $a_i \in H_1$ and $s_1^{k-1}, s_{k+1}^m \in S$, $1 \leq k,j \leq m$, there is $b_j \in H_1$ such that $h(s_1^{k-1}, a_k, s_{k+1}^m) = h(s_1^{j-1}, b_j, s_{j+1}^m)$. Since (M,h) is an m-group of S, it follows that for all $z_i \in M$ and $d_1^{i-1}, d_{i+1}^m \in N$, $1 \leq i,j \leq m$, there is $q_j \in M$ such that $h(d_1^{i-1}, z_i, d_{i+1}^m) = h(d_1^{j-1}, q_j, d_{j+1}^m)$. It is enough to prove that $q_j \in (H_1 : H_2)$. We have

$$l(H_2^{(i-1)}, h(d_1^{k-1}, z_i, d_{k+1}^m), H_2^{(n-i)}) = l(H_2^{(i-1)}, h(d_1^{j-1}, q_j, d_{j+1}^m), H_2^{(n-i)}).$$

Hence, we have

$$\begin{aligned}&h(l(H_2^{(i-1)},d_1,H_2^{(n-i)}),\dots,l(H_2^{(i-1)},d_{k-1},H_2^{(n-i)}),l(H_2^{(i-1)},z_i,H_2^{(n-i)}),l(H_2^{(i-1)},d_{k+1},H_2^{(n-i)}),\dots,l(H_2^{(i-1)},d_m,H_2^{(n-i)}))\\&=h(l(H_2^{(i-1)},d_1,H_2^{(n-i)}),\dots,l(H_2^{(i-1)},d_{j-1},H_2^{(n-i)}),l(H_2^{(i-1)},q_j,H_2^{(n-i)}),l(H_2^{(i-1)},d_{j+1},H_2^{(n-i)}),\dots,l(H_2^{(i-1)},d_m,H_2^{(n-i)})).\end{aligned}$$

We know that $l(H_2^{(i-1)},d_f,H_2^{n-i)})\subseteq S$ for all $f\in\{1,2,\dots,m\}$ and $l(H_2^{(i-1)},z_i,H_2^{(n-i)})\subseteq H_1$. Since H_1 is a normal subgroup of S, it follows that $l(H_2^{(i-1)},q_j,H_2^{(n-i)})\subseteq H_1$. Thus, $q_j\in(H_1:H_2)$, which implies that $(H_1:H_2)$ is a normal subgroup of the (m,n)-near ring M.

(2) If $i=2$ then $l(H_1,N)\subseteq H_1$ since H_1 is an M-subgroup of S. Assume that $x\in(H_1:H_2)$. Then, $l(H_2,x)\subseteq H_1$. We have $l(H_2,l((H_1:H_2),N))\subseteq l(H_1,N)\subseteq H_1$, and so H_2 is an M-subgroup of S.

If $i=1$ then $l(M,H_1)\subseteq H_1$ since H_1 is an M-subgroup of S. Assume that $x\in(H_1:H_2)$. Then $l(x,H_2)\subseteq H_1$. We have

$$l(l(M,(H_1:H_2)),H_2)\subseteq l(M,H_1)\subseteq H_1.$$

Consequently, H_2 is an M-subgroup of S.

Recall that H_1 is an M-group of S. If $i=2$ then for all $g_1,g_2\in H_1$, and $n_1,n_2,\dots,n_m\in M$, we have

(I) $l(g_1,h(n_1,n_2,\dots,n_m))=h(l(g_1,n_1),l(g_1,n_2),\dots,l(g_1,n_m))$.

(II) $l(g_1,l(a_1,a_2))=l(l(g_1,a_1),a_2)$.

Hence, for all $s_1,s_2\in(H_1:H_2)$, $t\in H_2$, $n_1,n_2,\dots,n_m\in M$, $andk(H_2,s_i)\subseteq H_1$, we have

(i)

$$\begin{aligned}l(t,l(s_1,h(n_1,n_2,\dots,n_m)))&=l(l(t,s_1),h(n_1,n_2,\dots,n_m))\\&=h(l(l(t,s_1),n_1),l(l(t,s_1),n_2),\dots,l(l(t,s_1),n_m))\\&=h(l(t,l(s_1,n_1)),l(t,l(s_1,n_2),\dots,l(t,l(s_1,n_m))\\&=l(t,h(l(s_1,n_1),l(s_1,n_2),\dots,l(s_1,n_m)).\end{aligned}$$

Consequently, we obtain $l(s_1,h(n_1,n_2,\dots,n_m))=h(l(s_1,n_1),l(s_1,n_2),\dots,l(s_1,n_m))$. Also,

(ii)

$$l(t,l(s_1,l(a_1,a_2)))=l(l(t,s_1),l(a_1,a_2))=l(l(l(t,s_1),a_1),a_2),$$

which gives $l(s_1,l(a_1,a_2))=l(l(s_1,a_1),a_2)$.

Thus, we deduce that $(H_1:H_2)$ is an M-group of S.

Again, we recall that H_1 is an M-group of S. So, if $i=1$ then for all $g_1,g_2\in H_1$ and all $n_1,n_2,\dots,n_m\in M$, we have

(I) $l(h(n_1,n_2,\dots,n_m),g_1)=h(l(n_1,g_1),l(n_2,g_1),\dots,l(n_m,g_1))$,

(II) $l(l(a_1,a_2),g_2)=l(a_1,l(a_2,g_2))$.

Thus, for all $s_1,s_2\in(H_1:H_2)$, $t\in H_2$, $n_1,n_2,\dots,n_m\in M$, and $l(s_i,H_2)\subseteq H_1$, we have

(i)

$$\begin{aligned}l(l(h(n_1,n_2,\dots,n_m),s_1),t)&=l(h(n_1,n_2,\dots,n_m),l(s_1,t))\\&=h(l(n_1,l(s_1,t)),l(n_2,l(s_1,t)),\dots,l(n_m,l(s_1,t)))\\&=h(l(l(n_1,s_1),t),l(l(n_2,s_1),t),\dots,l(l(n_m,s_1),t))\\&=l(h(l(n_1,s_1),l(n_2,s_2),\dots,l(n_m,s_m)),t).\end{aligned}$$

Hence, we obtain $l(h(n_1,n_2,\dots,n_m),s_1)=h(l(n_1,s_1),l(n_2,s_1),\dots,l(n_m,s_1))$. Also, we have

(ii)

$$l(t,l(s_1,l(a_1,a_2)))=l(l(t,s_1),l(a_1,a_2))=l(l(l(t,s_1),a_1),a_2),$$

which implies that $l(s_1,l(a_1,a_2))=l(l(s_1,a_1),a_2)$. Thus, we deduce that $(H_1:H_2)$ is an M-group of S.

(3) If $n=2$ and $i=1$, then using the statement (1), we conclude that $(H_1:H_2)$ is a normal subgroup of M. If $a\in(H_1:H_2)$ then $l(a,H_2)\subseteq H_1$.

If $i=1$, then for every $a_2\in M$, we have $l(l((H_1:H_2),a_2),H_2)=l((H_1:H_2),l(a_2,H_2))=l((H_1:H_2),H_2)\subseteq H_1$.

Thus, we have $l((H_1 : H_2), a_2) \subseteq (H_1 : H_2)$. If $i = 2$, then $a \in (H_1 : H_2)$, and hence $l(H_2, a) \subseteq H_1$; for every $a_1 \in M$, we have $l(H_2, (l(a_1, (H_1 : H_2)) = l(l(H_2, a_1), (H_1 : H_2)) = l(H_2, (H_1 : H_2),) \subseteq H_1$. Therefore, $l(a_1, (H_1 : H_2)) \subseteq (H_1 : H_2)$. □

Remark 3.2. *For any N-group homomorphism $f : G \longrightarrow T$, it holds that $(0 : G) \subseteq (0 : f(G))$. Hence, every monomorphism image of a faithful N-group is also faithful. Moreover, for any N-group isomorphism $f : G \longrightarrow T$, we have $(0 : G) = (0 : T)$. Therefore, G is faithful if and only if T is faithful.*

Theorem 3.5. *Assume that (R, h, k) is an (m, n)-near ring. If the R_0-group G is monogenic by s, then the R-group G is monogenic by s.*

Proof. Since the R_0-group G is monogenic by s, it follows that $G = k(s^{(i-1)}, R_0, s^{(n-i)}) \subseteq k(s^{(i-1)}, R, s^{(n-i)}) \subseteq G$. Thus, $k(s^{(i-1)}, R, s^{(n-i)}) = G$, which implies that the R-group G is monogenic by s. □

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