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Research Article

Vertex colorings resulting from proper total domination of graphs

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Abstract

A vertex u in a graph G totally dominates a vertex v if v is adjacent to u. A set S of vertices in a graph G is a total dominating set for G if every vertex of G is totally dominated by at least one vertex of S. If S is a total dominating set of a graph G, then $\sigma_S(v)$ denotes the number of vertices in S that totally dominate v. A total dominating set S in a graph G is called a proper total dominating set if $\sigma_S(u) \neq \sigma_S(v)$ for every two adjacent vertices u and v of G. Not all graphs possess a proper total dominating set. For each graph G with a proper total dominating set S, the numbers in the set $\{\sigma_S(v) : v \in V(G)\}$ give rise to a proper vertex coloring of G. The number $|\{\sigma_S(v) : v \in V(G)\}|$ is called the proper total chromatic number $\chi_{pt}(G, S)$ of G with respect to S, while the proper total chromatic number $\chi_{pt}(G)$ of G is defined as the minimum value of $\chi_{pt}(G, S)$ among all proper total dominating sets S of G. Therefore, for every graph G with a proper total dominating set, $\chi_{pt}(G)$ is at least as large as the chromatic number $\chi(G)$ of G. The proper total chromatic number is investigated for trees and complete multipartite graphs. It is shown that for every pair a, b of positive integers with $2 \leq a \leq b$, there is a graph G with $\chi(G) = a$ and $\chi_{pt}(G) = b$.

Keywords: total domination; proper total domination; proper vertex coloring; proper total chromatic number.

2020 Mathematics Subject Classification: 05C15, 05C69.

1. Introduction

In recent decades, domination in graphs has grown in popularity in graph theory. While this area evidently began with the work of Berge [2] in 1958 and Ore [9] in 1962, it did not become an active area of research until 1977 with the appearance of a survey paper by Cockayne and Hedetniemi [6]. Since then, a large number of variations and applications of domination have surfaced. A vertex u in a graph G is said to *dominate* a vertex v if either $v = u$ or v is adjacent to u in G. That is, u dominates itself and all vertices in its neighborhood $N(u)$. A set S of vertices in G is a *dominating set* of G if every vertex of G is dominated by some vertex in S. In their 2023 book, Haynes, Hedetniemi, and Henning [7] presented the major results that have been obtained on what they refer to as the core concepts of graph domination. One of these core concepts is total domination, introduced by Cockayne, Dawes and Hedetniemi [5] in 1977. A vertex u in a graph G *totally dominates* a vertex v if v is adjacent to u. A set S of vertices in a graph G is a *total dominating set* for G if every vertex of G is totally dominated by at least one vertex of S. Therefore, a graph G has a total dominating set if and only if G contains no isolated vertices. The 2013 book by Henning and Yeo [8] deals exclusively with total domination in graphs.

For a total dominating set S and a vertex v of G, the number of vertices in S that totally dominate v is denoted by $\sigma_S(v)$. Thus, $1 \leq \sigma_S(v) \leq \deg v$ for each vertex v of G. While there is no total dominating set S for which $\sigma_S(u) \neq \sigma_S(v)$ for every two vertices u and v of G, it is possible that $\sigma_S(u) \neq \sigma_S(v)$ for every pair u, v of adjacent vertices of G (see [4]). A total dominating set with this property is a *proper total dominating set*. Not only is S a proper total dominating set of G but σ_S is a proper coloring of G. Thus, a proper total dominating set S of G gives rise to a σ_S-coloring of G. A σ_S-coloring of G is referred to as a *proper total coloring* or a *pt-coloring* of G. For every proper total dominating set S in a graph G, there is a resulting proper (vertex) coloring of G with colors in the set

$$\Sigma_S(G) = \{\sigma_S(v) : v \in V(G)\}.$$

If a graph G has maximum degree $\Delta(G)$, then $\Sigma_S(G) \subseteq \{1, 2, \ldots, \Delta(G)\}$. For example, the set $S = \{u, v, w, y\}$ of vertices the graph H of Figure 1.1 is a proper total dominating set for H, where $\sigma_S(u) = \sigma_S(w) = \sigma_S(y) = 1$, $\sigma_S(x) = \sigma_S(z) = 2$, and $\sigma_S(v) = 3$. For a graph G, let $\mathcal{D}(G) = \{\deg_G v : v \in V(G)\}$ be the *degree set* of G. Thus, for the proper total dominating set S given for the graph H of Figure 1.1, it follows that $\Sigma_S(H) = \mathcal{D}(H)$.

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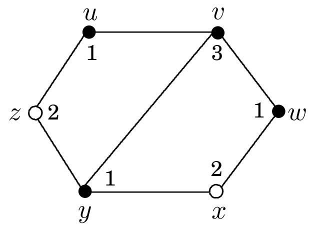

Figure 1.1: A graph with a proper total dominating set.

For a graph G with a proper total dominating set S, the *proper total chromatic number* or *pt-chromatic number* $\chi_{pt}(G,S)$ of G *with respect to* S is defined as $\chi_{pt}(G,S) = |\Sigma_S(G)|$, while the *proper total chromatic number* or *pt-chromatic number* $\chi_{pt}(G)$ of G is defined as

$$\chi_{pt}(G) = \min\{\chi_{pt}(G,S) : S \text{ is a proper total dominating set of } G\}.$$

Consequently, $\chi(G) \leq \chi_{pt}(G)$ for every graph G with a proper total dominating set, where $\chi(G)$ is the chromatic number of G. The proper total dominating set S given for the graph H of Figure 1.1 shows that $\chi_{pt}(H) \leq 3$. Since $\chi(H) = 2$, it follows that $2 \leq \chi_{pt}(H) \leq 3$. We show that $\chi_{pt}(H) = 3$. Assume, to the contrary, that $\chi_{pt}(H) = 2$. Then there must be a proper total dominating set S such that $\Sigma_S(H) = \{1,2\}$ since u and z are adjacent vertices of degree 2 in H; in fact, it must occur that $\{\sigma_S(u), \sigma_S(z)\} = \{1,2\}$. Without loss of generosity, we can assume that $\sigma_S(u) = 2$ and $\sigma_S(z) = 1$. Since $\sigma_S(u) = 2$, it follows that $v, z \in S$. Necessarily, $\sigma_S(v) = 1$ and $\sigma_S(w) = 2$. Thus, $x \in S$. Since $\{v,x,z\} \subseteq S$, it follows that $\sigma_S(y) = 3$, which is impossible. Therefore, $\chi_{pt}(H) = 3$ and so $\chi_{pt}(H) > \chi(H)$.

In [4], all paths and cycles possessing a proper total dominating set were determined.

Proposition 1.1. [4] *For an integer $n \geq 2$, the path P_n of order n has a proper total dominating set if and only if*

$$n \equiv 3 \pmod 4.$$

Proposition 1.2. [4] *For an integer $n \geq 3$, the cycle C_n of order n has a proper total dominating set if and only if*

$$n \equiv 0 \pmod 4.$$

By Propositions 1.1 and 1.2, if $G = P_n$ where $n \equiv 3 \pmod 4$ or if $G = C_n$ where $n \equiv 0 \pmod 4$, then G has a proper total dominating set S with $\Sigma_S(G) = \{1,2\}$. Therefore, $\chi(G) = \chi_{pt}(G) = 2$.

The *clique number* $\omega(G)$ of a graph G is the maximum order of a complete subgraph of G. It is well known that $\chi(G) \geq \omega(G)$ for every graph G. The British mathematician Rowland Brooks proved that if G is a connected graph of order n, then $\chi(G) \leq \Delta(G)$ unless $G = K_n$ or $n \geq 3$ is odd and $G = C_n$. Furthermore, if S is a proper total dominating set of G, then $1 \leq \sigma_S(v) \leq \Delta(G)$ for every vertex v of G and so $\chi_{pt}(G,S) \leq \Delta(G)$. Hence, $\chi_{pt}(G) \leq \Delta(G)$. Neither odd cycles nor complete graphs of order at least 3 have a proper total dominating set (see [4]). Consequently, if G is a connected graph having a proper total dominating set, then

$$\omega(G) \leq \chi(G) \leq \chi_{pt}(G) \leq \Delta(G). \tag{1}$$

All graphs under consideration are nontrivial connected graphs. We refer to the books [3, 7] for notation and terminology not defined here.

2. The *pt*-chromatic number of a graph

Graphs in which every two adjacent vertices have different degrees have been referred to as *locally irregular graphs* by some (see [1], for example). The graphs $K_{1,2}$ and $K_{1,3}$ are the only locally irregular graphs of orders 3 and 4, respectively. While each graph $K_{s,t}$ with $s \neq t$ is locally irregular, so too are the graphs shown in Figure 2.1 where each vertex is labeled with its degree. If G is a locally irregular graph, then $S = V(G)$ is a proper total dominating set of G. In this case, $\Sigma_S(G)$ is the degree set $\mathcal{D}(G)$ of the graph G.

The following lemma will be useful in what follows. For a vertex u in a graph, let $N[u] = \{u\} \cup N(u)$ denote the *closed neighborhood* of u.

Lemma 2.1. *Let G be a graph containing an end-vertex adjacent to a vertex u of degree 2. If G has a proper total dominating set S, then $N[u] \subseteq S$.*

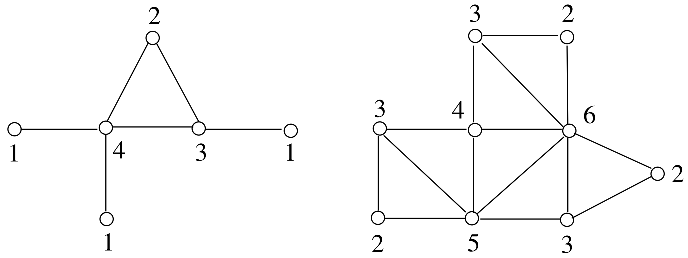

Figure 2.1: Locally irregular graphs.

Proof. Let $N[u] = \{u, v, w\}$ where v is an end-vertex of G. Since v is only totally dominated by u, it follows that $u \in S$. Since u is only totally dominated by either v or w, at least one of v and w must belong to S. However, if exactly one of v and w belongs to S, then $\sigma_S(u) = \sigma_S(v) = 1$, which is a contradiction. Therefore, $N[u] \subseteq S$. □

Let G be a nontrivial connected graph. For each vertex v and edge e of G, it is known that $\chi(G) - 1 \le \chi(G - v) \le \chi(G)$ and $\chi(G) - 1 \le \chi(G - e) \le \chi(G)$. This, however, is not true in general for the pt-chromatic number of a graph. For example, consider the graphs G and H in Figure 2.2 where $v \in V(G)$ and $e \in E(H)$. By Lemma 2.1, the vertex set is the only proper total dominating set for each of these four graphs G, $G - v$, H, and $H - e$. Hence, $\chi_{pt}(G) = \chi_{pt}(H) = 3$ and $\chi_{pt}(G - v) = \chi_{pt}(H - e) = 4$. Consequently, $\chi_{pt}(G - v) = \chi_{pt}(G) + 1$ and $\chi_{pt}(H - e) = \chi_{pt}(H) + 1$

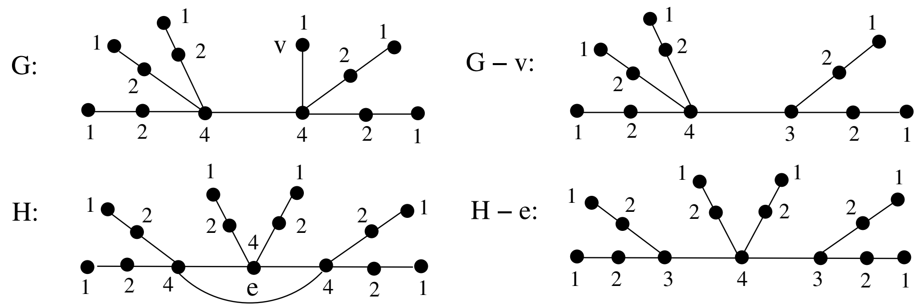

Figure 2.2: Graphs G and H such that $\chi_{pt}(G - v) = \chi_{pt}(G) + 1$ and $\chi_{pt}(H - e) = \chi_{pt}(H) + 1$.

While $\chi(F) \le \chi(G)$ for a subgraph F of a graph G, we saw that it is possible that $\chi_{pt}(F) = \chi_{pt}(G) + 1$. In fact, $\chi_{pt}(F) - \chi_{pt}(G)$ can be an arbitrarily large positive integer, which we show next. For a graph G, the *subdivision* $S(G)$ of G is obtained by subdividing each edge of G exactly once.

Theorem 2.1. *For each positive integer k, there exists a tree T containing a subtree T' such that $\chi_{pt}(T') = \chi_{pt}(T) + k$.*

Proof. For a given integer k, we first construct the tree T. We begin with the star $T_0 = K_{1,k+1}$ of size $k + 1$ with central vertex w and end-vertices $w_1, w_2, \ldots, w_{k+1}$. For each integer i with $1 \le i \le k + 1$, let $T_i = S(K_{1,4k})$ with central vertex v_i. The tree T is obtained from $T_0, T_1, T_2, \ldots, T_{k+1}$ by identifying the vertex w_i in T_0 and the vertex v_i in T_i for $1 \le i \le k + 1$. The tree T is shown in Figure 2.3 for $k = 1$ and $k = 2$.

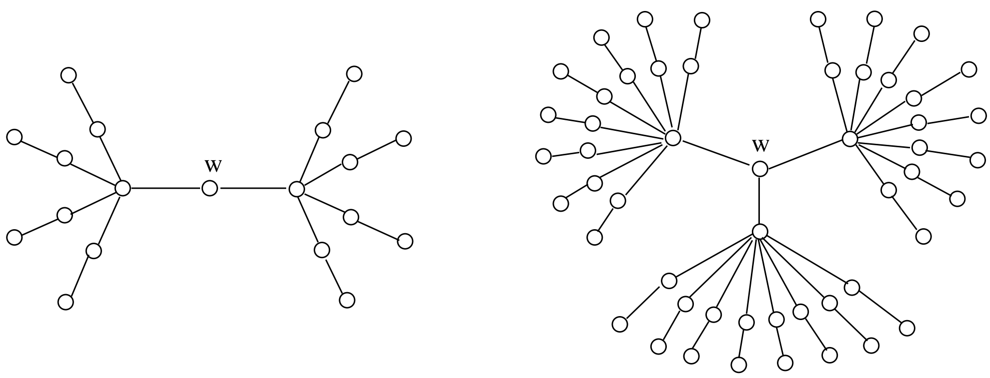

Figure 2.3: The trees T for $k = 1$ and $k = 2$.

We show that $\chi_{pt}(T) = 3$ if $k = 1$ and $\chi_{pt}(T) = 4$ if $k \geq 2$. By Lemma 2.1, the tree T has exactly two proper total dominating sets, namely $S_1 = V(T) - \{w\}$ and $S_2 = V(T)$. Since $\Sigma_{S_1}(T) = \{1, 2, k+1, 4k\}$ and $\Sigma_{S_2}(T) = \{1, 2, k+1, 4k+1\}$, it follows that $\chi_{pt}(T) = 3$ if $k = 1$ and $\chi_{pt}(T) = 4$ if $k \geq 2$.

Next, we construct a subtree T' of T. For each integer i with $1 \leq i \leq k+1$, let $T'_i = S(K_{1,4k-i+1})$ with central vertex v'_i. The tree T' is obtained from $T_0, T'_1, T'_2, \ldots, T'_{k+1}$ by identifying the vertex w_i in T_0 and the vertex v'_i in T'_i for $1 \leq i \leq k+1$. Then T' is a subtree of T. This tree T' is shown in Figure 2.4 for $k = 1$ and $k = 2$, respectively.

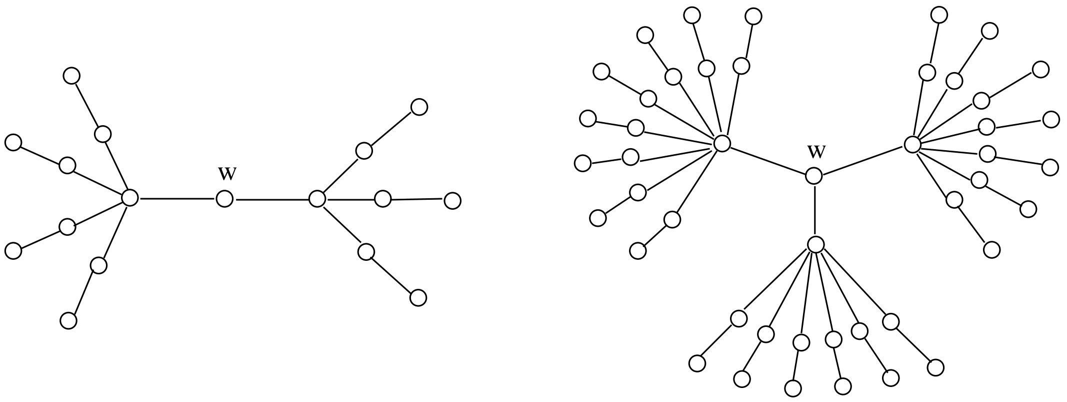

Figure 2.4: The trees T' for $k = 1$ and $k = 2$.

We show that $\chi_{pt}(T') = 3 + k$ if $k = 1$ and $\chi_{pt}(T') = 4 + k$ if $k \geq 2$. By Lemma 2.1, the tree T' has exactly two proper total dominating sets, namely $S_1 = V(T') - \{w\}$ and $S_2 = V(T')$. Then $\Sigma_{S_1}(T') = \{1, 2, k+1, 4k, 4k-1, 4k-2, \ldots, 3k\}$ and $\Sigma_{S_2}(T') = \{1, 2, k+1, 4k+1, 4k, 4k-1, \ldots, 3k+1\}$. Therefore, $\chi_{pt}(T') = 3 + k$ if $k = 1$ and $\chi_{pt}(T') = 4 + k$ if $k \geq 2$. Consequently, $\chi_{pt}(T') = \chi_{pt}(T) + k$. □

By (1), if G is a connected graph with $\chi(G) = a$ and $\chi_{pt}(G) = b$, then $2 \leq a \leq b$. The difference of these two numbers can be arbitrarily large. For example, while the chromatic number of every nontrivial tree is 2, the proper total chromatic number of a tree can be any integer 2 or more.

Theorem 2.2. *For each integer $k \geq 2$, there is a connected graph G such that*

$$\chi(G) = 2 \text{ and } \chi_{pt}(G) = k.$$

Proof. Every nontrivial tree has chromatic number 2. Thus, we show that for each integer $k \geq 2$ there exists a tree T with $\chi_{pt}(T) = k$. We have seen that if $T = P_n$ where $n \equiv 3 \pmod 4$, then T has a proper total dominating set and $\chi_{pt}(T) = 2$. Thus, we may assume that $k \geq 3$. Suppose first that $k = 3$. Let $T = S(K_{1,3})$ be the tree obtained by subdividing each edge of $K_{1,3}$ exactly once. Since T is locally irregular, $V(T)$ is a proper total dominating set. By Lemma 2.1, every proper total dominating set of T must contain $V(T)$ and so $V(T)$ is the unique proper total dominating set of T. Since the degree set of T is $\mathcal{D}(T) = \{1, 2, 3\}$, it follows that $\chi_{pt}(T) = 3$.

We may now assume that $k \geq 4$. Suppose first that $k = 4$. Let T be the tree of order 13 obtained from two copies T_1 and T_2 of the tree $S(K_{1,3})$ by identifying an end-vertex of T_1 with the central vertex of T_2. Since T is locally irregular, T has a proper total dominating set S. By Lemma 2.1, a proper total dominating set of T must contain every vertex of T except possibly the vertex v of degree 2 whose neighbors have degrees 3 and 4. However, if $v \notin S$, then the vertex $u \in N(v)$ of degree 3 has $\sigma_S(u) = \sigma_S(v) = 2$, which is impossible. Thus, $v \in S$ and so $S = V(T)$ is the unique proper total dominating set of T. Since $\Sigma_S(G) = \mathcal{D}(G) = \{1, 2, 3, 4\}$, it follows that $\chi_{pt}(T) = 4$. This is illustrated in Figure 2.5 where each vertex v of T is labeled by $\sigma_S(v)$.

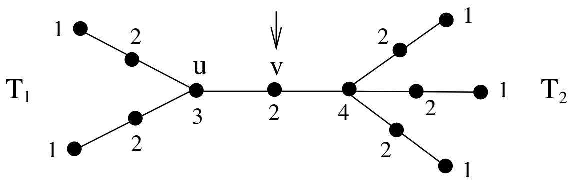

Figure 2.5: The tree in the proof of Theorem 2.2 for $k = 4$.

Next, let $k \geq 5$. Let $T' = K_{1,k-2}$ where v is the central vertex of T' and $v_1, v_2, \ldots, v_{k-2}$ are the end-vertices of T'. For each integer i with $2 \leq i \leq k-2$, let $T_i = S(K_{1,i+1})$ if $i+3 \geq k$ and let $T_i = S(K_{1,i})$ if $i+3 < k$. The tree T is constructed by placing a pendant edge at v_1 and identifying the central vertex of T_i and the vertex v_i of T' for $2 \leq i \leq k-2$. Then T is a locally irregular tree with degree set $\mathcal{D}(T) = \{1, 2, \ldots, k\}$. By Lemma 2.1, the unique proper total dominating set of T is $S = V(T)$. Hence, $\Sigma_S(T) = \mathcal{D}(T)$ and so $\chi_{pt}(T) = k$. The tree T is illustrated in Figure 2.6 for $k = 5, 6, 7$, where some vertices v of T are labeled by $\sigma_S(v)$. □

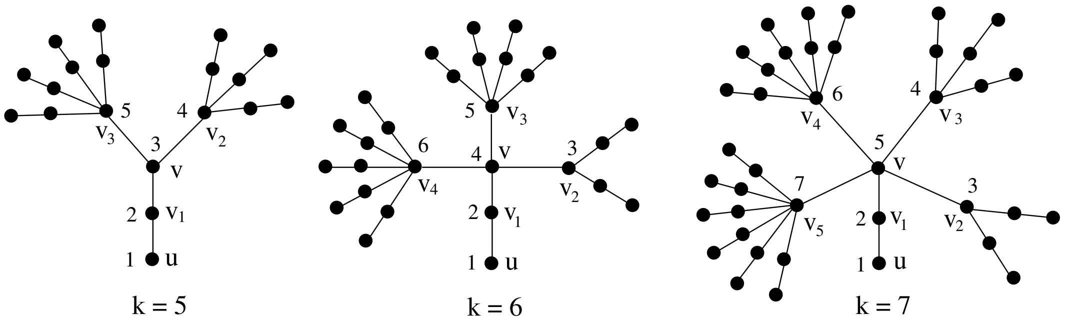

Figure 2.6: The trees in the proof of Theorem 2.2 for $k = 5, 6, 7$.

By (1), if G is a connected graph possessing a proper total dominating set, then $\chi_{pt}(G) \leq \Delta(G)$. The trees T constructed in the proof of Theorem 2.2 have the property that $\chi_{pt}(T) = \Delta(T)$. Hence, the following is a consequence of the proof of Theorem 2.2.

Corollary 2.1. *For each integer $k \geq 2$, there is a connected graph G such that*

$$\chi_{pt}(G) = \Delta(G) = k.$$

While Theorem 2.2 states that there are connected graphs G for which $\chi_{pt}(G) - \chi(G)$ can be arbitrarily large, there are connected graphs G for which $\chi_{pt}(G)$ and $\chi(G)$ are the same prescribed number at least 2. To illustrate this, we determine those complete multipartite graphs that possess a proper total dominating set.

Theorem 2.3. *For an integer $k \geq 2$, let $G = K_{n_1, n_2, \ldots, n_k}$ be a complete k-partite graph of order 3 or more where*

$$n_1 \leq n_2 \leq \cdots \leq n_k.$$

Then G has a proper total dominating set if and only if $n_i \geq i-1$ for $2 \leq i \leq k$. Furthermore, if G has a proper total dominating set, then $\chi_{pt}(G) = \chi(G) = k$.

Proof. Let $U_1, U_2, \ldots, U_k$ be the partite sets of G where $|U_i| = n_i$ for $1 \leq i \leq k$. First, suppose that $n_i \geq i-1$ for each integer i with $2 \leq i \leq k$. Let $S_i \subseteq U_i$ such that $|S_i| = i-1$ for $2 \leq i \leq k$. Then

$$\sum_{i=2}^{k} |S_i| = \binom{k}{2}.$$

Let $S = \bigcup_{i=1}^{k} S_i$. For each vertex $v \in U_i$ ($1 \leq i \leq k$), it follows that

$$\sigma_S(v) = \binom{k}{2} - (i-1).$$

Therefore, S is a proper total dominating set of G. This is shown in Figure 2.7 for the complete 4-partite graph $K_{1,1,2,3}$ where each vertex in S is indicated by a solid vertex and each vertex v is labeled by $\sigma_S(v)$. Therefore, $\chi_{pt}(G, S) = k$ and so $\chi_{pt}(G) \leq k$. Since $k = \chi(G) \leq \chi_{pt}(G)$, it follows that $\chi_{pt}(G) = k$.

For the converse, assume, to the contrary, that G has a proper total dominating set S and there is an integer j with $2 \leq j \leq k$ such that $n_j \leq j-2$. Then $k \geq 3$. Suppose that r_i vertices of U_i belong to S for $1 \leq i \leq k$. Thus, $0 \leq r_i \leq n_i$ for $1 \leq i \leq k$. If $1 \leq i \leq j$, then $r_i \leq n_i \leq n_j \leq j-2$. Consequently, $r_i \in \{0, 1, \ldots, j-2\}$ for $1 \leq i \leq j$. Hence, there are two distinct integers s and t with $1 \leq s, t \leq j$ such that $r_s = r_t$. Let $x \in U_s$ and $y \in U_t$. Then

$$\sigma_S(x) = \left(\sum_{i=1}^{k} r_i\right) - r_s = \left(\sum_{i=1}^{k} r_i\right) - r_t = \sigma_S(y).$$

Since $xy \in E(G)$, it follows that S is not a proper total dominating set of G, a contradiction. □

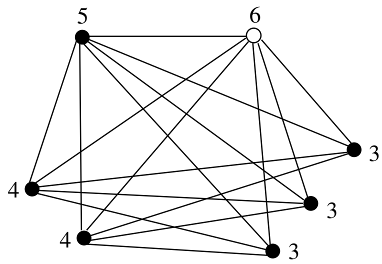

Figure 2.7: A proper total dominating set of $K_{1,1,2,3}$.

By (1), for every connected graph possessing a proper total dominating set, it follows that $\omega(G) \le \chi_{pt}(G)$. If G is a complete k-partite graph of order 3 or more, then $\omega(G) = k$. Thus, the following is a consequence of Theorem 2.3.

Corollary 2.2. *For each integer $k \ge 2$, there is a connected graph G such that*

$$\omega(G) = \chi_{pt}(G) = k.$$

Another consequence of Theorem 2.3 and its proof is the following.

Corollary 2.3. *For every integer $k \ge 4$, there exists a connected graph G with $\chi(G) = \chi_{pt}(G) = k$ where there is a proper coloring of G using the colors $1, 2, \dots, k$, of course, but no pt-coloring of G that uses any of the colors $1, 2, \dots, k$.*

3. Connected graphs with prescribed chromatic and *pt*-chromatic numbers

We are now prepared to show that every pair a, b of integers with $2 \le a \le b$ can be realized as the chromatic number and proper total chromatic number, respectively, of some connected graph.

Theorem 3.1. *For every pair a, b of integers with $2 \le a \le b$, there exists a connected graph G such that $\chi(G) = a$ and $\chi_{pt}(G) = b$.*

Proof. By Theorem 2.3, we may assume that $2 \le a < b$. We consider three cases.

Case 1. $a = 2$. By Theorem 2.2, there is a tree T with $\chi(T) = 2$ and $\chi_{pt}(T) = b$.

Case 2. $b = a + 1 \ge 4$. Let G be the graph obtained from K_a where $V(K_a) = \{v_1, v_2, \dots, v_a\}$ by adding i pendant edges at the vertex v_i of K_a for $1 \le i \le a$. Then $\chi(G) = a$. It remains to show that $\chi_{pt}(G) = a + 1$. Since G is locally irregular, $S_0 = V(G)$ is a proper total dominating set of G with $|\Sigma_{S_0}(G)| = a + 1$. Thus, $\chi_{pt}(G) \le a + 1$. Next, let S be any proper total dominating set of G. Since $\sigma_S(u) = 1$ for each end-vertex u of G and each vertex of K_a is adjacent to at least one end-vertex of G, it follows that $\sigma_S(v_i) \ge 2$ for $1 \le i \le a$. Furthermore, $\sigma_S(v_i) \ne \sigma_S(v_j)$ for each pair i, j of integers with $1 \le i < j \le a$. Hence, $\chi_{pt}(G, S) \ge a + 1$ for every proper total dominating set S of G. Therefore, $\chi_{pt}(G) \ge a + 1$ and so $\chi_{pt}(G) = a + 1$.

Case 3. $b \ge a + 2 \ge 5$. We construct a connected graph G with $\chi(G) = a$ and $\chi_{pt}(G) = b$ by first constructing two graphs G_1 and G_2.

- ⋆ To construct the graph G_1, we begin with $H = K_a$ where $V(H) = \{u_1, u_2, \dots, u_a\}$. For $1 \le i \le a$, let $H_1 = S(K_{1,3})$ and let $H_i = iP_3$ for $2 \le i \le a$. The graph G_1 is obtained from H and the graphs H_i $(1 \le i \le a)$ by (1) joining u_1 to the central vertex of H_1 and (2) joining u_i to exactly one end-vertex in each copy of P_3 in H_i for $2 \le i \le a$. For $a = 3, 4$, the graph G_1 is shown in Figure 3.1.

  Let $S_1 = V(G_1) - V(H)$ and $S_1' = [V(G_1) - (V(H)] \cup \{u_1\}$. Then S_1 and S_1' are proper total dominating sets of G_1 such that

  $$\Sigma_{S_1}(G_1) = \{1, 2, \dots, a\} \text{ and } \Sigma_{S_1'}(G_1) = \{1, 2, \dots, a+1\}.$$

  We claim that S_1 and S_1' are only proper total dominating sets of G_1. Let S be any proper total dominating set of G_1. By Lemma 2.1, it follows that $S_1 \subseteq S$. Furthermore, $u_i \notin S$ for every integer i with $2 \le i \le a$, for otherwise, say $u_j \in S$ where $2 \le j \le a$. Let x be the neighbor of u_j in H_j and let y be the neighbor of x in H_j. Then $\sigma_S(x) = \sigma(y) = 2$, which is impossible. Therefore, as claimed, S_1 and S_1' are only proper total dominating sets of G_1. This also shows that $\chi_{pt}(G_1) = \chi(G_1) = a$.

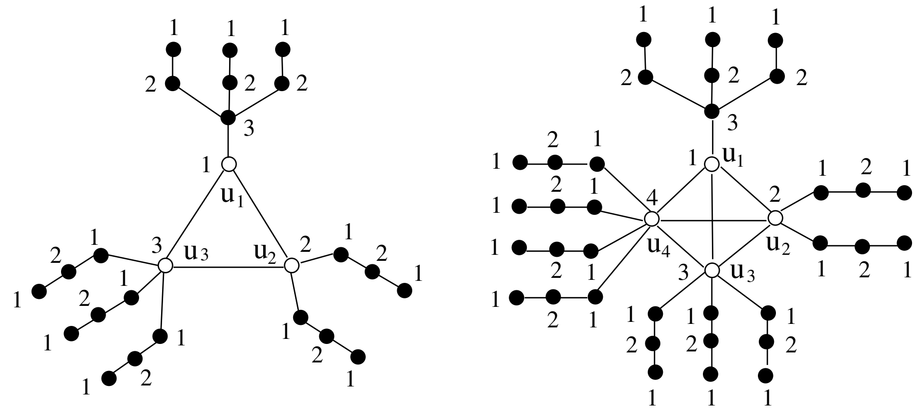

Figure 3.1: Graphs G_1 in the proof of Theorem 3.1 for $a = 3, 4$.

- ⋆ Let G_2 be the tree constructed in the proof of Theorem 2.2 for $k = b \geq 5$. Thus, $\chi_{pt}(G_2) = b$ and $S_2 = V(G_2)$ is the unique proper total dominating set of G_2 with $\Sigma_{S_2}(G_2) = \{1, 2, \ldots, b\}$.

We now construct the graph G from G_1 and G_2. Using the vertex labeling of G_2 as in the proof of of Theorem 2.2, let u be the end-vertex of G_2 that is adjacent to v_1. The graph G is constructed by G_1 and G_2 by adding the edge uu_a. Then $\chi(G) = a$. It remains to show that $\chi_{pt}(G) = b$.

First, let $S = S_1 \cup S_2 = (V(G_1) - V(K_a)) \cup V(G_2)$. Then $\sigma_S(x) = \sigma_{S_1}(x)$ if $x \in V(G_1) - \{u_a\}$, $\sigma_S(u_a) = \sigma_{S_1}(u_a) + 1 = a + 1$, and $\sigma_S(x) = \sigma_{S_2}(x)$ if $x \in V(G_2)$. Furthermore, if $y \in N_G(u_a)$, then $\sigma_S(y) \in \{1, 2, \ldots, a-1\}$. It follows that S is a proper total dominating set of G with $\Sigma_S(G) = \{1, 2, \ldots, b\}$. For $(a, b) \in \{(3, 5), (4, 6)\}$, the graphs G are shown in Figure 3.2, where each vertex of S is indicated by a solid vertex and a vertex v of G is labeled by $\sigma_S(v)$. Similarly, it can be shown that $S' = S \cup \{u_1\}$ is also a proper total dominating sets of G and $\Sigma_{S'}(G) = \{1, 2, \ldots, b\}$.

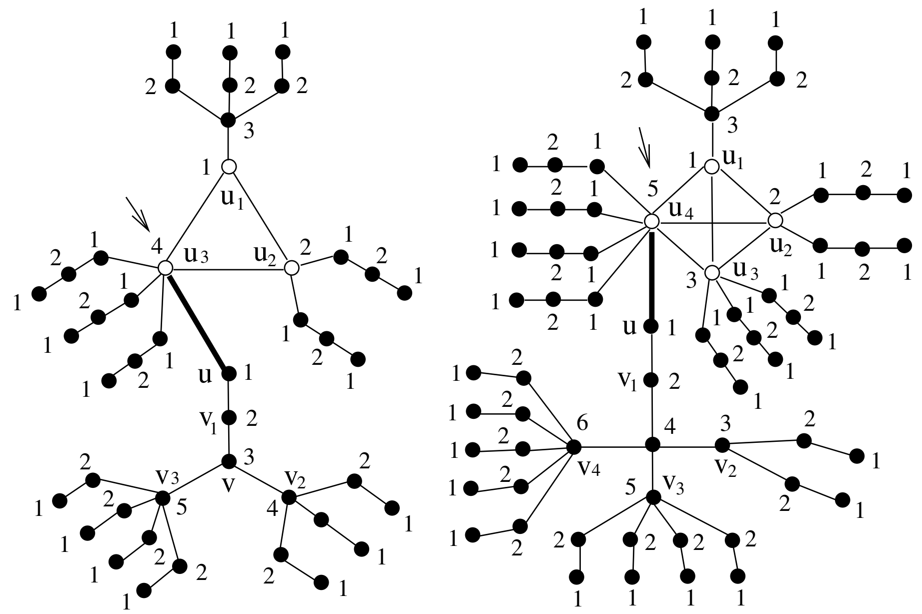

Figure 3.2: The graphs in the proof of Theorem 3.1 for $(a, b) \in \{(3, 5), (4, 6)\}$.

Next, we show that that S and S' are the only proper total dominating sets of G. Let X be any proper total dominating set of G. By Lemma 2.1, it follows that $V(G_1) - V(K_a) \subseteq X$ and $V(G_2) - \{u, v_1, v\} \subseteq S'$. Furthermore, as described above, $u_i \notin S$ for every integer i with $2 \leq i \leq a$. We show that $\{u, v_1, v\} \subseteq X$.

- ⋆ Since u is only totally dominated by u_a and v_1 and $u_a \notin X$, it follows that $v_1 \in X$.

- ⋆ Since v_1 is only totally dominated by u and v, at least one of u and v must belong to S. If $u \notin X$, then $v \in X$ and $\sigma_X(u) = \sigma_X(v_1) = 1$, a contradiction. If $v \notin X$, then $u \in X$ and $\sigma_X(u) = \sigma_X(v_1) = 1$, a contradiction.

Thus, $\{u, v_1, v\} \subseteq X$, implying that $X = S$ or $X = S'$. Therefore, $\chi_{pt}(G) = b$. □

By (1), for every connected graph possessing a proper total dominating set $\omega(G) \leq \chi(G) \leq \chi_{pt}(G) \leq \Delta(G)$. The graphs G_1 constructed in Case 3 of the proof of Theorem 3.1 have the property that $\omega(G) = \chi(G) = \chi_{pt}(G) = \Delta(G) \geq 3$. Hence, the following is a consequence of Theorem 2.2 and the proof of Theorem 3.1.

Corollary 3.1. *For each integer $k \geq 2$, there is a connected graph G such that*

$$\omega(G) = \chi(G) = \chi_{pt}(G) = \Delta(G) = k.$$

We conclude with the following two natural questions.

Problem 3.1. *Is there a characterization of connected graphs G (having proper total dominating sets) such that $\omega(G) = \chi(G) = \chi_{pt}(G) = \Delta(G)$?*

Problem 3.2. *Is there a characterization of connected graphs G (having proper total dominating sets) such that $\chi(G) = \chi_{pt}(G)$?*

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References

[1] J. Bensmail, F. Dross, N. Nisse, Decomposing degenerate graphs into locally irregular subgraphs, *Graphs Combin.* **36** (2020) 1869–1889.
[2] C. Berge, Sur le couplage maximum d'un graphe, *C. R. Acad. Sci. Paris* **247** (1958) 258–259.
[3] G. Chartrand, P. Zhang, *Chromatic Graph Theory*, Second Edition, Chapman & Hall/CRC Press, Boca Raton, 2020.
[4] R. Chatterjee, E. Jent, S. Osborn, P. Zhang, Proper total domination in graphs, *Electron. J. Math.* **7** (2024) 58–68.
[5] E. J. Cockayne, R. M. Dawes, S. T. Hedetniemi, Total domination in graphs, *Networks* **10** (1977) 211–219.
[6] E. J. Cockayne, S. T. Hedetniemi, Towards a theory of domination in graphs, *Networks* **7** (1977) 247–261.
[7] T. W. Haynes, S. T. Hedetniemi, M. A. Henning, *Domination in Graphs: Core Concepts*, Springer, New York, 2023.
[8] M. A. Henning, A. Yeo, *Total Domination in Graphs*, Springer, New York, 2013.
[9] O. Ore, *Theory of Graphs*, Amer. Math. Soc., Providence, 1962.