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*Research Article*

# Largest and smallest eigenvalues of matrices and some Hamiltonian properties of graphs

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**Abstract**

Let $G = (V, E)$ be a graph. Define $M(G; \alpha, \beta) := \alpha D + \beta A$, where $D$ and $A$ are the diagonal matrix and adjacency matrix of $G$, respectively, and $\alpha$, $\beta$, are real numbers such that $(\alpha, \beta) \neq (0, 0)$. Using the largest and smallest eigenvalues of $M(G; \alpha, \beta)$ with $\alpha \geq \beta > 0$, sufficient conditions for the Hamiltonian and traceable graphs are presented.

**Keywords:** matrix; largest eigenvalue; Hamiltonian graph; traceable graph.

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## 1. Introduction

We consider only finite undirected graphs without loops and multiple edges. Notation and terminology not defined here follow those in [1]. For a graph $G = (V(G), E(G))$, we use $n$ and $e$ to denote its order and size, respectively. The minimum degree and maximum degree of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We use $N(u)$ to denote the set of all vertices adjacent to $u$ in $G$. A set of vertices in a graph $G$ is independent if the vertices in the set are pairwise nonadjacent. A maximum independent set in a graph $G$ is an independent set with the largest possible size. The independence number, denoted as $\gamma(G)$, of a graph $G$ is the cardinality of a maximum independent set in $G$. For disjoint vertex subsets $X$ and $Y$ of $V(G)$, we define $E(X, Y)$ as $\{ f : f = xy \in E, x \in X, y \in Y \}$. A cycle $C$ in a graph $G$ is said to be a Hamiltonian cycle of $G$ if $C$ contains all the vertices of $G$. A graph $G$ is called Hamiltonian if $G$ has a Hamiltonian cycle. A path $P$ in a graph $G$ is said to be a Hamiltonian path of $G$ if $P$ contains all the vertices of $G$. A graph $G$ is called traceable if $G$ has a Hamiltonian path.

For a graph $G$, we define $M(G; \alpha, \beta) := \alpha D + \beta A$, where $D$ and $A$ are the diagonal matrix and adjacency matrix of $G$, respectively, and $\alpha$, $\beta$, are real numbers such that $(\alpha, \beta) \neq (0, 0)$. If $\alpha = 0$ and $\beta = 1$ (respectively, $\alpha = 1$ and $\beta = 1$), then $M(G; \alpha, \beta)$ is the same as the adjacency matrix (respectively, the signless Laplacian matrix) of $G$. Thus, $M(G; \alpha, \beta)$ is a generalization of both adjacency matrix and signless Laplacian matrix of $G$. We use $\lambda_{\alpha,\beta;\,1}, \lambda_{\alpha,\beta;\,2}, \cdots, \lambda_{\alpha,\beta;\,n}$ to denote the eigenvalues of $M(G; \alpha, \beta)$ and assume that $\lambda_{\alpha,\beta;\,1} \geq \lambda_{\alpha,\beta;\,2} \geq \cdots \geq \lambda_{\alpha,\beta;\,n}$. Since $M(G; \alpha, \beta)$ is symmetric, its eigenvalues $\lambda_{\alpha,\beta;\,1}, \lambda_{\alpha,\beta;\,2}, \cdots, \lambda_{\alpha,\beta;\,n}$ are real numbers. In this article, using the largest and smallest eigenvalues of $M(G; \alpha, \beta)$ with $\alpha \geq \beta > 0$, we present sufficient conditions for the Hamiltonian and traceable graphs. Now, we state the main results of the present article.

**Theorem 1.1.** *Let $G$ be a $k$-connected graph with $n \geq 3$ vertices and $e$ edges, where $k \geq 2$. Let $\alpha \geq \beta > 0$. Set $\lambda_1 := \lambda_{\alpha,\beta;\,1}$ and $\lambda_n := \lambda_{\alpha,\beta;\,n}$.*

**(i).** *If the inequality*

$$\lambda_1 \leq (\alpha + \beta)\sqrt{\frac{(k+1)\delta^2}{n} + \frac{e^2}{n(n-k-1)}}$$

*holds then $G$ is Hamiltonian or $G$ is $K_{k,\,k+1}$.*

**(ii).** *If the inequality*

$$\lambda_n \geq (\alpha + \beta)\sqrt{\frac{(n-k-1)\Delta^2}{n} + \frac{e^2}{n(k+1)}}$$

*holds then $G$ is Hamiltonian or $G$ is $K_{k,\,k+1}$.*

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**Theorem 1.2.** *Let $G$ be a $k$-connected graph with $n \geq 9$ vertices and $e$ edges, where $k \geq 1$. Let $\alpha \geq \beta > 0$. Set $\lambda_1 := \lambda_{\alpha,\beta;\,1}$ and $\lambda_n := \lambda_{\alpha,\beta;\,n}$.*

**(i).** *If the inequality*

$$\lambda_1 \leq (\alpha+\beta)\sqrt{\frac{(k+2)\delta^2}{n} + \frac{e^2}{n(n-k-2)}},$$

*holds then $G$ is traceable or $G$ is $K_{k,\,k+2}$.*

**(ii).** *If the inequality*

$$\lambda_n \geq (\alpha+\beta)\sqrt{\frac{(n-k-2)\Delta^2}{n} + \frac{e^2}{n(k+2)}},$$

*holds then $G$ is traceable or $G$ is $K_{k,\,k+2}$.*

## 2. Lemmas

This section gives the known results that are used to prove Theorem 1.1 and Theorem 1.2.

**Lemma 2.1** (see [2]). *Let $G$ be a $k$-connected graph of order $n \geq 3$. If $\gamma \leq k$, then $G$ is Hamiltonian.*

**Lemma 2.2** (see [2]). *Let $G$ be a $k$-connected graph of order n. If $\gamma \leq k+1$, then $G$ is traceable.*

**Lemma 2.3** (see [6]). *Let $G$ be a balanced bipartite graph of order $2n$ with bipartition $(A,\,B)$. If $d(x)+d(y) \geq n+1$ for any $x \in A$ and any $y \in B$ with $xy \notin E$, then $G$ is Hamiltonian.*

**Lemma 2.4** (see [4]). *Let $G$ be a 2-connected bipartite graph with bipartition $(A,\,B)$, where $|A| \geq |B|$. If each vertex in $A$ has degree at least $s$ and each vertex in $B$ has degree at least $t$, then $G$ contains a cycle of length at least $2\min(|B|, s+t-1, 2s-2)$.*

The following result is the well-known Rayleigh-Ritz theorem:

**Lemma 2.5** (see the theorem on Page 176 in [3]). *Let $M$ be an $n \times n$ Hermitian matrix with the largest eigenvalue $\lambda_1$ and the smallest eigenvalue $\lambda_n$. Suppose $X$ is any non-zero $n$-dimensional row vector. Then*

$$\lambda_1 \geq \frac{XMX^*}{XX^*} \geq \lambda_n,$$

*where $X^*$ is the transpose conjugate of $X$.*

## 3. Proofs of Theorem 1.1 and Theorem 1.2

**Proof of Theorem 1.1.** Let $G$ be a $k$-connected ($k \geq 2$) graph with $n \geq 3$ vertices and $e$ edges. Suppose $G$ is not Hamiltonian. Then Lemma 2.1 implies that $\gamma \geq k+1$. Also, we have that $n \geq 2\delta+1 \geq 2k+1$, otherwise $\delta \geq k \geq n/2$ and $G$ is Hamiltonian. Let $I_1 := \{u_1, u_2, \ldots, u_\gamma\}$ be a maximum independent set in $G$. Then $I := \{u_1, u_2, \ldots, u_{k+1}\}$ is an independent set in $G$. Thus,

$$\sum_{u \in I} d(u) = |E(I, V-I)| \leq \sum_{v \in V-I} d(v).$$

Since

$$\sum_{u \in I} d(u) + \sum_{v \in V-I} d(v) = 2e,$$

we have that

$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V-I} d(v).$$

Let $V - I = \{v_1, v_2, \ldots, v_{n-(k+1)}\}$. From Cauchy-Schwarz inequality, we have

$$\sum_{r=1}^{n-(k+1)} 1^2 \sum_{r=1}^{n-(k+1)} d^2(v_r) \geq \left(\sum_{r=1}^{n-(k+1)} d(v_r)\right)^2 \geq e^2.$$

Consequently, it holds that

$$\sum_{v\in V-I} d^2(v) \geq \frac{e^2}{n-k-1}.$$

Therefore,

$$M := (k+1)\delta^2 + \frac{e^2}{n-k-1} \leq \sum_{u\in I} d^2(u) + \sum_{v\in V-I} d^2(v) = \sum_{v\in V} d^2(v)$$

with equality if and only if $d(u) = \delta$ for each $u \in I$, $\sum_{v\in V-I} d(v) = e$ (implying $\sum_{u\in I} d(u) = e$ and thereby $G$ is bipartite with partition sets of $I$ and $V - I$), and $\Delta = d(v)$ for each $v \in V - I$.

From Cauchy-Schwarz inequality again, we have

$$\sum_{r=1}^{k+1} 1^2 \sum_{r=1}^{k+1} d^2(u_r) \leq \left(\sum_{r=1}^{k+1} d(u_r)\right)^2 \leq e^2.$$

Thus,

$$\sum_{u\in I} d^2(u) \leq \frac{e^2}{k+1}.$$

Therefore,

$$N := \frac{e^2}{k+1} + (n-k-1)\Delta^2 \geq \sum_{u\in I} d^2(u) + \sum_{v\in V-I} d^2(v) = \sum_{v\in V} d^2(v)$$

with equality if and only if $d(v) = \Delta$ for each $v \in V - I$, $\sum_{u\in I} d(u) = e$ (implying $\sum_{v\in V-I} d(v) = e$ and thereby $G$ is bipartite with partition sets of $I$ and $V - I$), and $\delta = d(u)$ for each $u \in I$.

For any real row vector $X = (x_1, x_2, \dots, x_n)$, we have

$$XM(G;\alpha,\beta)X^T = (\alpha - \beta)\sum_{i=1}^{n} x_i^2 + \beta \sum_{uv\in E} (d(u) + d(v))^2 \geq 0,$$

where $X^T$ is the transpose of $X$. Thus, $M(G;\alpha,\beta)$ is positive semidefinite and therefore,

$$\lambda_1 = \lambda_{\alpha,\beta;\,1} \geq \lambda_{\alpha,\beta;\,2} \geq \cdots \geq \lambda_{\alpha,\beta;\,n} = \lambda_n \geq 0.$$

Hence $\lambda_1^2 = \lambda_{\alpha,\beta;\,1}^2 \geq \lambda_{\alpha,\beta;\,2}^2 \geq \cdots \geq \lambda_{\alpha,\beta;\,n}^2 = \lambda_n^2 \geq 0$ are the eigenvalues of $M^2(G;\alpha,\beta)$.

Since $M^2(G;\alpha,\beta) = \alpha^2 D^2 + \alpha\beta DA + \alpha\beta AD + \beta^2 A^2$, the sum of all the entries in the $uth$ row of $M^2(G;\alpha,\beta)$ is equal to the sum of all the entries in the $uth$ rows of $\alpha^2 D^2$, $\alpha\beta DA$, $\alpha\beta AD$, and $\beta^2 A^2$, where $u$ is any vertex in $G$. Notice that the sums of all the entries of the $uth$ rows of $D^2$, $DA$, $AD$, and $A^2$ are equal to $d^2(u)$, $d^2(u)$, $\sum_{v\in N(u)} d(v)$, and $\sum_{v\in N(u)} d(v)$, respectively (see Page 805 in [5]). Hence, the sum of all the entries in the $uth$ row, denoted as $RS(u)$, in $M^2(G;\alpha,\beta)$ is

$$\alpha(\alpha+\beta)d^2(u) + \beta(\alpha+\beta)\sum_{v\in N(u)} d(v).$$

Let $Y = (1, 1, \dots, 1)$ be an $n$-dimensional row vector. Applying Lemma 2.5 to $M^2(G;\alpha,\beta)$, we have

$$\lambda_1^2 \geq \frac{YM(G;\alpha,\beta)Y^*}{YY^*} \geq \lambda_n^2.$$

Notice that

$$\begin{aligned} YM(G;\alpha,\beta)Y^* &= \sum_{u\in V} RS(u) \\ &= \alpha(\alpha+\beta)\sum_{u\in V} d^2(u) + \beta(\alpha+\beta)\sum_{u\in V}\sum_{v\in N(u)} d(v) \\ &= \alpha(\alpha+\beta)\sum_{u\in V} d^2(u) + \beta(\alpha+\beta)\sum_{u\in V} d^2(u) \\ &= (\alpha+\beta)^2 \sum_{u\in V} d^2(u). \end{aligned}$$

Hence, the following chain of inequalities holds:

$$\lambda_1^2 \geq (\alpha+\beta)^2 \frac{\sum_{u\in V} d^2(u)}{n} \geq \lambda_n^2.$$

**(i).** From the given condition, we have

$$\begin{aligned}(\alpha+\beta)^2\left(\frac{(k+1)\delta^2}{n}+\frac{e^2}{n(n-k-1)}\right) &\geq \lambda_1^2\\ &\geq (\alpha+\beta)^2\frac{\sum_{u\in V} d^2(u)}{n}\\ &\geq (\alpha+\beta)^2\frac{M}{n}\\ &= (\alpha+\beta)^2\left(\frac{(k+1)\delta^2}{n}+\frac{e^2}{n(n-k-1)}\right).\end{aligned}$$

Thus, each of the above inequalities becomes an equality. Therefore, $d(u)=\delta$ for each $u\in I$, $\sum_{v\in V-I} d(v)=e$ (implying $\sum_{u\in I} d(u)=e$ and thereby $G$ is bipartite with partition sets of $I$ and $V-I$), and $\Delta=d(v)$ for each $v\in V-I$. Hence,

$$(k+1)\delta = |E(I, V-I)| = \Delta(n-k-1) \geq \delta(n-k-1).$$

Therefore, $2k+2\geq n\geq 2k+1$. If $n=2k+2$, then $\delta=\Delta$. Lemma 2.3 implies $G$ is Hamiltonian, a contradiction. If $n=2k+1$, then $G$ is $K_{k,\,k+1}$. This completes the proof of Theorem 1.1(i).

**(ii).** From the given condition, we have

$$\begin{aligned}(\alpha+\beta)^2\left(\frac{(n-k-1)\Delta^2}{n}+\frac{e^2}{n(k+1)}\right) &\leq \lambda_n^2\\ &\leq (\alpha+\beta)^2\frac{\sum_{u\in V} d^2(u)}{n}\\ &\leq (\alpha+\beta)^2\frac{N}{n}\\ &= (\alpha+\beta)^2\left(\frac{(n-k-1)\Delta^2}{n}+\frac{e^2}{n(k+1)}\right).\end{aligned}$$

Thus, each of the above inequalities becomes an equality. Therefore, $d(v)=\Delta$ for each $v\in V-I$, $\sum_{u\in I} d(u)=e$ (implying $\sum_{v\in V-I} d(v)=e$ and thereby $G$ is bipartite with partition sets of $I$ and $V-I$), and $\delta=d(u)$ for each $u\in I$. Hence,

$$(k+1)\delta = |E(I, V-I)| = \Delta(n-k-1) \geq \delta(n-k-1).$$

Therefore, $2k+2\geq n\geq 2k+1$. If $n=2k+2$, then $\delta=\Delta$. Lemma 2.3 implies that $G$ is Hamiltonian, which is a contradiction. If $n=2k+1$, then $G$ is $K_{k,\,k+1}$. This completes the proof of Theorem 1.1(ii). □

Although the proof of Theorem 1.2 is similar to the proof of Theorem 1.1, we present here a proof of Theorem 1.2 for the sake of completeness.

**Proof of Theorem 1.2.** Let $G$ be a $k$-connected ($k\geq 1$) graph with $n\geq 9$ vertices and $e$ edges. Suppose that $G$ is not traceable. Then, Lemma 2.2 implies that $\gamma\geq k+2$. Also, we have that $n\geq 2\delta+2\geq 2k+2$, otherwise $\delta\geq k\geq (n-1)/2$ and $G$ is traceable. Using the ideas in the proof of Theorem 1.1, we have an independent set $I$ of size $k+2$ in $G$ such that

$$M_1 := (k+2)\delta^2+\frac{e^2}{n-k-2} \leq \sum_{u\in I} d^2(u) + \sum_{v\in V-I} d^2(v) = \sum_{v\in V} d^2(v)$$

with equality if and only if $d(u)=\delta$ for each $u\in I$, $\sum_{v\in V-I} d(v)=e$ (implying $\sum_{u\in I} d(u)=e$ and thereby $G$ is bipartite with partition sets of $I$ and $V-I$), and $\Delta=d(v)$ for each $v\in V-I$, and

$$N_1 := \frac{e^2}{k+2}+(n-k-2)\Delta^2 \geq \sum_{u\in I} d^2(u) + \sum_{v\in V-I} d^2(v) = \sum_{v\in V} d^2(v)$$

with equality if and only if $d(v)=\Delta$ for each $v\in V-I$, $\sum_{u\in I} d(u)=e$ (implying $\sum_{v\in V-I} d(v)=e$ and thereby $G$ is bipartite with partition sets of $I$ and $V-I$), and $\delta=d(u)$ for each $u\in I$.

Using the ideas in the proof of Theorem 1.1 again, we have the following chain of inequalities:

$$\lambda_1^2 \geq (\alpha+\beta)^2 \, \frac{\sum_{u\in V} d^2(u)}{n} \geq \lambda_n^2.$$

**(i).** By the given condition, we have

$$\begin{aligned}(\alpha+\beta)^2\left(\frac{(k+2)\delta^2}{n} + \frac{e^2}{n(n-k-2)}\right) &\geq \lambda_1^2 \\ &\geq (\alpha+\beta)^2 \frac{\sum_{u\in V} d^2(u)}{n} \\ &\geq (\alpha+\beta)^2 \frac{M_1}{n} \\ &= (\alpha+\beta)^2\left(\frac{(k+2)\delta^2}{n} + \frac{e^2}{n(n-k-2)}\right).\end{aligned}$$

Thus, each of the above inequalities becomes an equality. Therefore, $d(u) = \delta$ for each $u \in I$, $\sum_{v\in V-I} d(v) = e$ (implying $\sum_{u\in I} d(u) = e$ and thereby $G$ is bipartite with partition sets of $I$ and $V - I$), and $\Delta = d(v)$ for each $v \in V - I$. Hence,

$$(k+2)\delta = |E(I, V-I)| = \Delta(n-k-2) \geq \delta(n-k-2).$$

Thus, $2k+4 \geq n \geq 2k+2$. Consequently, we have $n = 2k+4$, $n = 2k+3$, or $n = 2k+2$. If $n = 2k+4 \geq 9$, then $\delta = \Delta$ and $k \geq 3$. Lemma 2.3 implies that $G$ is Hamiltonian and thereby $G$ is traceable, which is a contradiction. If $n = 2k+3 \geq 9$, then $k \geq 3$. Lemma 2.4 implies that $G$ has a cycle of length at least $(n-1)$. Hence, $G$ is traceable, which is again a contradiction. If $n = 2k+2$, then $G$ is $K_{k,\,k+2}$. This completes the proof of Theorem 1.2(i).

**(ii).** From the given condition, we have

$$\begin{aligned}(\alpha+\beta)^2\left(\frac{(n-k-2)\Delta^2}{n} + \frac{e^2}{n(k+2)}\right) &\leq \lambda_n^2 \\ &\leq (\alpha+\beta)^2 \, \frac{\sum_{u\in V} d^2(u)}{n} \\ &\leq (\alpha+\beta)^2 \frac{N_1}{n} \\ &= (\alpha+\beta)^2\left(\frac{(n-k-2)\Delta^2}{n} + \frac{e^2}{n(k+2)}\right).\end{aligned}$$

Thus, each of the above inequalities becomes an equality. Therefore, $d(v) = \Delta$ for each $v \in V - I$, $\sum_{u\in I} d(u) = e$ (implying $\sum_{v\in V-I} d(v) = e$ and thereby $G$ is bipartite with partition sets of $I$ and $V - I$), and $\delta = d(u)$ for each $u \in I$. Hence,

$$(k+2)\delta = |E(I, V-I)| = \Delta(n-k-2) \geq \delta(n-k-2).$$

Thus $2k+4 \geq n \geq 2k+2$. Therefore, we have $n = 2k+4$, $n = 2k+3$, or $n = 2k+2$. If $n = 2k+4 \geq 9$, then $\delta = \Delta$ and $k \geq 3$. Lemma 2.3 implies that $G$ is Hamiltonian and thereby $G$ is traceable, which is a contradiction. If $n = 2k+3 \geq 9$, then $k \geq 3$. Lemma 2.4 implies that $G$ has a cycle of length at least $(n-1)$. Hence $G$ is traceable, which is again a contradiction. If $n = 2k+2$, then $G$ is $K_{k,\,k+2}$. This completes the proof of Theorem 1.2(ii). □

From the proof of Theorem 1.1, the next result follows.

**Corollary 3.1.** *Let $G$ be a graph with $n$ vertices and $e \geq 1$ edges. Suppose that $\alpha \geq \beta > 0$ and let $I$ be any independence set of $G$ with $|I| = \gamma$. Set $\lambda_1 := \lambda_{\alpha,\beta;\,1}$ and $\lambda_n := \lambda_{\alpha,\beta;\,n}$. Then*

$$\lambda_1 \geq (\alpha+\beta)\sqrt{\frac{\gamma\delta^2}{n} + \frac{e^2}{n(n-\gamma)}} \quad \textit{and} \quad \lambda_n \leq (\alpha+\beta)\sqrt{\frac{(n-\gamma)\Delta^2}{n} + \frac{e^2}{n\gamma}}.$$

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