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Research Article

A new characterization of slices in warped products

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Abstract

A new characterization of slices in certain warped products among starshaped hypersurfaces is given. Namely, if there exists a positive constant c such that the higher order mean curvatures H_s and H_{s-1} satisfy $H_s^{\frac{1}{s}} \leqslant c \leqslant H_{s-1}^{\frac{1}{s-1}}$ then the hypersurface is a slice.

Keywords: warped products; geodesic spheres; higher order mean curvatures.

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1. Introduction

The question of the characterization of round spheres in space forms has a long history. One of the most famous results on this topic is the well-known Alexandrov theorem [1], which ensures that a closed embedded hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ with constant mean curvature must be a round sphere. This result has been extended to scalar curvature and then higher order mean curvatures by Ros [9, 10]. Later, Montiel and Ros [8] proved that the Alexandrov theorem for the mean curvature as well as for higher order mean curvatures is also true for hypersurfaces of hyperbolic spaces and half-spheres. There has been a wide literature on the topic under discussion and many other characterizations have been obtained for space forms with different types of conditions. One can refer to [11] and references therein for instance.

Recently, Brendle [3] has obtained an Alexandrov-type theorem for hypersurfaces into certain warped products (geodesic sphere are thus replaced by totally umbilical hypersurfaces or slices); he and Eichmair [4] has extended this study for higher order mean curvatures. After that, many other characterizations of slices in warped products have been obtained (see [6, 7, 11, 13] for instance).

The aim of this article is to give a new characterization of slices into warped products. This characterization is not of Alexandrov type but lies on some inequalities between two consecutive higher order mean curvatures. The geometric setting is the following. Let $n \geqslant 2$ be an integer and (M^n, g_M) be a compact Riemannian manifold of dimension n satisfying

$$\mathrm{Ric}_M \geqslant (n-1)kg,$$

for some constant k. Moreover, let $t_0 > 0$ and let $h : [0, t_0) \longrightarrow \mathbb{R}$ be a positive function satisfying the following four conditions

$$h'(0) = 0 \text{ and } h''(0) > 0, \tag{H1}$$

$$h'(t) > 0 \text{ for all } t \in (0, t_0), \tag{H2}$$

$$\text{the function } r \longmapsto 2\frac{h''(t)}{h(t)} - (n-1)\frac{k - h'(t)^2}{h(t)^2} \text{ is non-decreasing on } (0, t_0), \tag{H3}$$

$$\frac{h''(t)}{h(t)} + \frac{k - h'(t)^2}{h(t)^2} > 0 \text{ for all } t \in (0, t_0). \tag{H4}$$

Consider the warped product P defined by $P = [0, t_0) \times M$ endowed with the metric $g_P = dt^2 \oplus h(t)g_M$. This wide class of Riemannian manifolds includes the three real space forms $\mathbb{M}^{n+1}(\delta)$; namely, the Euclidean space $\mathbb{R}^{n+1}$ if $\delta = 0$, the half-sphere $\mathbb{S}_+^{n+1}$ if $\delta = 1$, and the hyperbolic space $\mathbb{H}^{n+1}$ if $\delta = -1$.

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The main result of the present article is given as follows:

Theorem 1.1. *Let $n \geqslant 2$ and $s \in \{2, \cdots, n\}$ be two integers. Let Σ be a closed and oriented hypersurface embedded into the warped product (P^{n+1}, g_P) satisfying the four conditions (H1) – (H4). If Σ is star-shaped and s-convex, and if there exists a constant c such that*

$$H_s^{\frac{1}{s}} \leqslant c \leqslant H_{s-1}^{\frac{1}{s-1}}$$

then Σ is a slice $\{t_1\} \times M$.

Remark that, as we will see in the next section, by the s-convexity assumption, we always have $H_s^{\frac{1}{s}} \leqslant H_{s-1}^{\frac{1}{s-1}}$. The condition here is to assume that some constant c can be inserted between the two quantities $H_s^{\frac{1}{s}}$ and $H_{s-1}^{\frac{1}{s-1}}$, which, by the theorem, forces the hypersurface to be a slice. Note also that from the s-convexity assumption, if such a constant exists, then it is positive. Finally, in the case where (P^{n+1}, g_P) is one of the real space forms $\mathbb{M}^{n+1}(\delta)$, we recover the result of Stong [12].

2. Preliminaries

We consider the warped product (P, g_p) of the form given in the introduction and satisfying the four conditions (H1), (H2), (H3), and (H4). Let Σ be a closed, embedded, connected, and oriented hypersurface of (P^{n+1}, g_P). We consider the function $f = h'$ and the vector field $X = h\frac{\partial}{\partial t}$. We also assume that Σ is starshaped; that is, the function $\langle X, \nu \rangle$ has a fixed sign on Σ, where ν is the outward normal unit vector field. In [3], Brendle used the fact that X is a conformal vector field, that is $\mathcal{L}_X g = 2fg$, to obtain the following Hsiung-Minkowski formula for hypersurfaces of these warped products:

$$\int_\Sigma H\langle X, \nu \rangle dv_g = \int_\Sigma f dv_g, \tag{1}$$

where g is the induced metric on Σ and H is the mean curvature of Σ. More generally, the higher order mean curvatures are extrinsic quantities defined from the second fundamental form and generalizing the notion of mean curvature. Up to a normalization the mean curvature H is the trace of the second fundamental form B:

$$H = \frac{1}{n}\mathrm{tr}\,(B). \tag{2}$$

In other words, the mean curvature is

$$H = \frac{1}{n}S_1(\kappa_1, \dots, \kappa_n), \tag{3}$$

where S_1 is the first elementary symmetric polynomial and $\kappa_1, \dots, \kappa_n$ are the principal curvatures. Higher order mean curvatures are defined in a similar way for $r \in \{1, \dots, n\}$ by

$$H_r = \frac{1}{\binom{n}{r}} S_r(\kappa_1, \cdots, \kappa_n), \tag{4}$$

where S_r is the r-th elementary symmetric polynomial; that is, for any n-tuple $(x_1, \cdots, x_n)$,

$$S_r(x_1, \dots, x_n) = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} x_{i_1} \cdots x_{i_r}. \tag{5}$$

Note that H_1 is nothing else but the mean curvature H.

We recall some classical inequalities between the higher order mean curvatures which are well-known. If $H_s > 0$, then $H_r > 0$ for any $r \in \{1, \cdots, s-1\}$ and

$$H_s^{\frac{1}{s}} \leqslant H_{s-1}^{\frac{1}{s-1}} \leqslant \cdots \leqslant H_2^{\frac{1}{2}} \leqslant H. \tag{6}$$

These classical inequalities can be found in [7] (for instance), where Li, Wei, and Xiong extended to warped products the proof of Barbosa and Colares [2] for space forms.

Finally, we recall the following general Hsiung-Minkowski-type formula in warped products. Namely, if conditions (H1)-(H4) are satisfied and if Σ is starshaped, then

$$\int_\Sigma f H_{s-1} dv_g \leqslant \int_\Sigma H_s \langle X, \nu \rangle dv_g. \tag{7}$$

This inequality has been first proved by Brendle-Eichmair [4] if Σ is convex but it turned out that the convexity assumption is not necessary and can be replaced by the s-convexity; that is, H_s is positive everywhere, which is a weaker assumption than convexity which means that all the principal curvatures are positive.

Note that (1) and (7) are the generalizations in this class of warped products of the so-called Hsiung-Minkowski formulas [5] for space forms (see [7] for instance). A crucial difference is that in space forms, they are equalities, which is not the case here except for identity (1).

Now, we have all the ingredients to prove Theorem 1.1.

3. Proof of Theorem 1.1

First of all, from assumptions (H1) and (H2), $f = h'$ is a nonnegative function and

$$\int_\Sigma f dv_g > 0 \quad \text{(in fact } f \text{ can vanish at most in one point, corresponding to } t = 0 \text{ in the product).}$$

Also, as mentioned in the preliminaries section, since we assume that H_s is a positive function, so H is also positive. Moreover, since Σ is supposed to be starshaped, the support function $\langle X, \nu\rangle$ has a fixed sign and so $H\langle X, \nu\rangle$ also has the same fixed sign. Hence, from Brendle identity (1), we have

$$\int_\Sigma H\langle X, \nu\rangle dv_g = \int_\Sigma f dv_g > 0,$$

and so necessarily, $\langle X, \nu\rangle$ has to be positive.

From the assumption $H_s^{\frac{1}{s}} \leqslant c$ and the fact that H_s is positive everywhere, we deduce that the constant c is positive. Also, from the assumption $c \leqslant H_{s-1}^{\frac{1}{s-1}}$, we have

$$\int_\Sigma f H_{s-1} dv_g \geqslant c^{s-1} \int_\Sigma f dv_g, \tag{8}$$

which gives the following inequality by using Brendle identity (1):

$$\int_\Sigma f H_{s-1} dv_g \geqslant c^{s-1} \int_\Sigma H\langle X, \nu\rangle dv_g. \tag{9}$$

On the other hand, from (7), we have

$$\int_\Sigma f H_{s-1} dv_g \leqslant \int_\Sigma H_s\langle X, \nu\rangle dv_g. \tag{10}$$

Using the other part of the assumption, $H_s^{\frac{1}{s}} \leqslant c$, and the fact that $\langle X, \nu\rangle$ is positive everywhere on Σ, we obtain

$$\int_\Sigma f H_{s-1} dv_g \leqslant \int_\Sigma c^s\langle X, \nu\rangle dv_g = c^{s-1} \int_\Sigma c\langle X, \nu\rangle dv_g. \tag{11}$$

From (6) and the assumption $c \leqslant H_{s-1}^{\frac{1}{s-1}}$, we obtain immediately that

$$H \geqslant H_{s-1}^{\frac{1}{s-1}} \geqslant c. \tag{12}$$

Hence, (11), (12), and the positivity of $\langle X, \nu\rangle$ and c imply the next inequality

$$\int_\Sigma f H_{s-1} dv_g \leqslant c^{s-1} \int_\Sigma H\langle X, \nu\rangle dv_g. \tag{13}$$

We use the Brendle identity (1) again to obtain the following inequality from (13):

$$\int_\Sigma f H_{s-1} dv_g \leqslant c^{s-1} \int_\Sigma f dv_g. \tag{14}$$

Thus, (8) and (14) together yield to

$$\int_\Sigma f H_{s-1} dv_g = c^{s-1} \int_\Sigma f dv_g$$

and all inequalities from (8) to (14) are in fact equalities. In particular, we get

$$\int_\Sigma f \left(H_{s-1} - c^{s-1}\right) dv_g = 0.$$

Since f is a positive function and with the assumption $H_{s-1} \geqslant c^{s-1}$, we deduce that

$$H_{s-1} = c^{s-1}.$$

On the other hand, from the fact that (11) and (13) are equalities, it follows that

$$c^{s-1}\int_\Sigma c\langle X,\nu\rangle dv_g = c^{s-1}\int_\Sigma H\langle X,\nu\rangle dv_g,$$

or equivalently (since c is a positive constant), it holds that

$$\int_\Sigma (c-H)\,\langle X,\nu\rangle dv_g = 0. \tag{15}$$

From (15), the fact that $\langle X,\nu\rangle$ is positive, and $H \geqslant c$, it follows that $H = c$. Finally, we have

$$H_{s-1}^{\frac{1}{s-1}} = c = H$$

such that the equality occurs in (6) and so Σ is totally umbilical.

We finish the proof by using the following argument due to Brendle [3]: if conditions (H1) – (H4) are satisfied, Σ is starshaped and totally umbilical, then Σ is a slice. □

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