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Research Article

Maximal elliptic Sombor index of bicyclic graphs

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Abstract

Let G be a simple connected graph with edge set $E(G)$. The concept of the elliptic Sombor index (ESO) was recently introduced by Gutman, Furtula, and Oz in mathematical chemistry. It is a vertex-degree-based topological index and is defined as $ESO(G) = \sum_{v_iv_j \in E(G)} (d_G(v_i) + d_G(v_j))\sqrt{d_G(v_i)^2 + d_G(v_j)^2}$, where $d_G(v_i)$ is the degree of the vertex v_i in G. In this paper, the bicyclic graph of a given order with the maximal elliptic Sombor index is determined.

Keywords: elliptic Sombor index; extremal value; bicyclic graph.

2020 Mathematics Subject Classification: 05C09, 05C35.

1. Introduction

Let G be a finite, undirected, and simple graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. The cardinalities of $V(G)$ and $E(G)$ are called the order and size of G, respectively. The degree of a vertex v in the graph G, denoted by $d_G(v)$, is the number of neighbors of v in G. The c-cyclic graphs are connected graphs with order n and size $n-1+c$; specifically, for $c = 0, 1, 2$, the corresponding c-cyclic graphs are called trees, unicyclic graphs, and bicyclic graphs, respectively.

From a geometric perspective, Gutman [1] introduced the following Sombor index and studied its basic properties:

$$SO(G) = \sum_{v_iv_j \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2}.$$

The extremal properties of the Sombor index has been the subject of many publications, as demonstrated by the numerous results reported in reference [3].

In 2024, Gutman, Furtula, and Oz [2] introduced a new vertex-degree-based topological index, called the ellipse Sombor index, using a novel geometric method. The ellipse Sombor index is defined as

$$ESO(G) = \sum_{v_iv_j \in E(G)} (d_G(v_i) + d_G(v_j))\sqrt{d_G(v_i)^2 + d_G(v_j)^2}.$$

In [2], several basic mathematical properties of this new index were established, and an extremal problem about this index for trees was studied. Recently, the maximal value of the elliptic Sombor index of trees with a given diameter or matching number or number of pendent vertices was determined in [5], and the corresponding extremal graphs were characterized there. Moreover, the ordering relations in benzenoid systems with respect to the ellipse Sombor index were given in [4].

Let $\mathcal{B}_n$ be the set of bicyclic graphs of order n. In this paper, we study an extremal problem for ESO over $\mathcal{B}_n$, and obtain the following result:

Theorem 1.1. *Let $n \geq 7$ and $G \in \mathcal{B}_n$. Then*

$$ESO(G) \leq n(n-4)\sqrt{n^2-2n+2} + 2(n+1)\sqrt{n^2-2n+5} + (n+2)\sqrt{n^2-2n+10} + 10\sqrt{13}$$

with equality if and only if

$$G \cong B_n(n-4, 0, 0),$$

where the graph $B_n(n-4,0,0)$ is shown in Figure 1.1.

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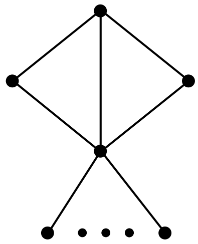

Figure 1.1: The graph $B_n(n-4,0,0)$.

2. Proof of Theorem 1.1

Let $K_4 - e$ denote the graph obtained from the complete graph K_4 by removing an edge. Let $B_n(r,s,t)$ be the bicyclic graph obtained from $K_4 - e$ by attaching r pendent vertices to the vertex $u_1 \in V(K_4 - e)$ of degree 3, s pendent vertices to the other vertex $u_3 \in V(K_4 - e)$ of degree 3, and t pendent vertices to the vertex $u_4 \in V(K_4 - e)$ of degree 2, where $r,s,t \geq 0$ and $r+s+t = n-4$. Let H_5 denote the graph of order 5 that is obtained by joining two copies of the cycle graph C_3 with a common vertex. Let $B'_n(p,q)$ be the bicyclic graph obtained from the graph H_5 by adding p pendent vertices to the vertex $u \in V(H_5)$ of degree 4 and q pendent vertices to another vertex $v \in V(H_5)$, where $p \geq q \geq 1$ and $p+q = n-5$. The graphs $B_n(r,s,t)$ and $B'_n(p,q)$ are shown in Figure 2.1.

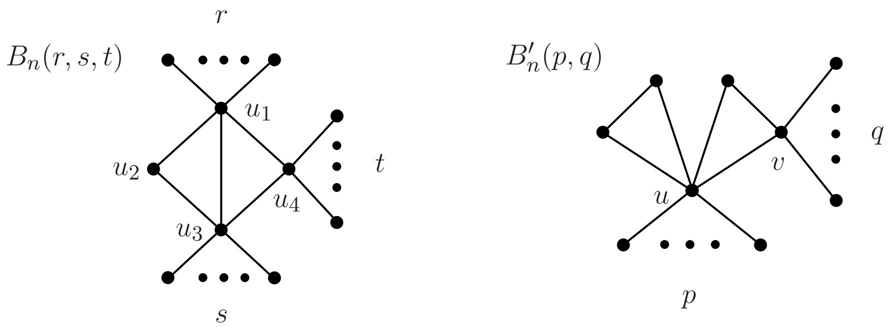

Figure 2.1: The bicyclic graphs $B_n(r,s,t)$ and $B'_n(p,q)$.

Lemma 2.1 (see [5]). *Let* $f(x,y) = (x+y)\sqrt{x^2+y^2}$ *for* $x,y \geq 1$ *and*

$$g(x,y) = (x+1+y)\sqrt{(x+1)^2+y^2} - (x+y)\sqrt{x^2+y^2}$$

for $x,y \geq 3$. *Then both the functions* $f(x,y)$ *and* $g(x,y)$ *are increasing functions in* x *and in* y.

Lemma 2.2. *For any integer* $n \geq 7$ *and* $r,s,t \geq 1$ *with* $r+s+t = n-4$, *the inequality*

$$ESO(B_n(r,s,t)) \leq ESO(B_n(n-4,0,0))$$

holds, where the equality holds if and only if $B_n(r,s,t) \cong B_n(n-4,0,0)$.

Proof. Without loss of generality, we can assume that $r \geq t \geq 1$. By Lemma 2.1, we have

$$\begin{aligned}
ESO(B_n(r+1,s,t-1)) - ESO(B_n(r,s,t)) &= r[f(r+4,1) - f(r+3,1)] - (t-1)[f(t+2,1) - f(t+1,1)] \\
&\quad + [f(r+4,1) - f(t+2,1)] + [f(r+4,t+1) - f(r+3,t+2)] \\
&\quad + [f(r+4,s+3) - f(r+3,s+3)] - [f(t+2,s+3) - f(t+1,s+3)] \\
&\quad + [f(r+4,2) - f(r+3,2)] \\
&> [rg(r+3,1) - (t-1)g(t+1,1)] + [f(r+4,t+1) - f(r+3,t+2)] \\
&\quad + [g(r+3,s+3) - g(t+1,s+3)] + g(r+3,2). \qquad (1)
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&[rg(r+3,1)-(t-1)g(t+1,1)]+[f(r+4,t+1)-f(r+3,t+2)]\\
&+[g(r+3,s+3)-g(t+1,s+3)]+g(r+3,2)\\
&=[rg(r+3,1)-(t-1)g(t+1,1)]+[g(r+3,s+3)-g(t+1,s+3)]\\
&+(r+t+5)[\sqrt{r^2+t^2+8r+2t+17}-\sqrt{r^2+t^2+6r+4t+13}]+g(r+3,2)>0. \qquad (2)
\end{aligned}
$$

From (1) and (2), we have

$$ESO(B_n(r+1,s,t-1))>ESO(B_n(r,s,t)).$$

Similarly, we prove that

$$ESO(B_n(r,s,0))<ESO(B_n(r+1,s-1,0))$$

for $r\geq s\geq 1$.

By applying the above transformations repeatedly, we obtain

$$ESO(B_n(r,s,t))\leq ESO(B_n(n-4,0,0))$$

with equality if and only if $B_n(r,s,t)\cong B_n(n-4,0,0)$. □

Lemma 2.3. *For any integer $n\geq 7$ and $p\geq q\geq 1$ with $p+q=n-5$, the following inequality holds:*

$$ESO(B'_n(p,q))<ESO(B'_n(p+1,q-1))<ESO(B_n(n-4,0,0)).$$

Proof. By Lemma 2.1, we have

$$
\begin{aligned}
ESO(B'_n(p+1,q-1))-ESO(B'_n(p,q))&=p[f(p+5,1)-f(p+4,1)]-(q-1)[f(q+2,1)-f(q+1,1)]\\
&+[f(p+5,1)-f(q+2,1)]+[f(p+5,q+1)-f(p+4,q+2)]\\
&+3[f(p+5,2)-f(p+4,2)]-[f(q+2,2)-f(q+1,2)]\\
&>[pg(p+4,1)-(q-1)g(q+1,1)]+[f(p+5,q+1)-f(p+4,q+2)]\\
&+[3g(p+4,2)-g(q+1,2)]\\
&=[pg(p+4,1)-(q-1)g(q+1,1)]+[3g(p+4,2)-g(q+1,2)]\\
&+(p+q+6)[\sqrt{p^2+q^2+10p+2q+26}-\sqrt{p^2+q^2+8p+4q+20}]\\
&>0.
\end{aligned}
$$

Thus,

$$ESO(B'_n(p+1,q-1))>ESO(B'_n(p,q)).$$

By applying the above transformation repeatedly, we have

$$ESO(B'_n(p,q))\leq ESO(B'_n(n-5,0))$$

with equality if and only if $B'_n(p,q)\cong B'_n(n-5,0)$. By direct calculations, we have

$$ESO(B'_n(n-5,0))=(n-5)n\sqrt{(n-1)^2+1}+4(n+1)\sqrt{(n-1)^2+4}+16\sqrt{2}$$

and

$$ESO(B_n(n-4,0,0))=n(n-4)\sqrt{(n-1)^2+1}+2(n+1)\sqrt{(n-1)^2+4}+(n+2)\sqrt{(n-1)^2+9}+10\sqrt{13}.$$

Therefore, we have

$$ESO(B'_n(n-5,0))<ESO(B_n(n-4,0,0)).$$

This completes the proof. □

Proof of Theorem 1.1. Let Δ be the maximum degree of G. Let v_iv_j be any edge in G such that $d_G(v_i) \geq d_G(v_j)$. We discuss three cases.

Case 1. $d_G(v_i) + d_G(v_j) \leq n$ for every edge $v_iv_j \in E(G)$. Let $h(x) = \sqrt{x^2 + (n-x)^2}$. Then $h(x) \leq \max\{h(n-3), h(3)\}$ for $3 \leq x \leq n-3$. If $3 \leq \Delta \leq n-3$, then we have

$$\begin{aligned}(d_G(v_i) + d_G(v_j))\sqrt{d_G(v_i)^2 + d_G(v_j)^2} &\leq n\sqrt{d_G(v_i)^2 + (n - d_G(v_i))^2} \\ &= n \cdot h(d_G(v_i)) \\ &\leq n \cdot \max\{h(n-3), h(3)\} \\ &= n\sqrt{(n-3)^2 + 9}.\end{aligned}$$

Thus,

$$\begin{aligned}ESO(G) &= \sum_{v_iv_j \in E(G)} (d_G(v_i) + d_G(v_j))\sqrt{d_G(v_i)^2 + d_G(v_j)^2} \\ &\leq n(n+1)\sqrt{(n-3)^2 + 9} \\ &< ESO(B_n(n-4, 0, 0)).\end{aligned}$$

If $\Delta = n-2$, then $G \cong G_1$ or $G \cong G_2$ (see Figure 2.2). By direct calculations, we have

$$\begin{aligned}ESO(G_1) &= (n-6)(n-1)\sqrt{(n-2)^2 + 1} + 4n\sqrt{(n-2)^2 + 4} + 24\sqrt{2} \\ &< ESO(B_n(n-4, 0, 0)) \quad \text{and}\end{aligned}$$

$$\begin{aligned}ESO(G_2) &= (n-7)(n-1)\sqrt{(n-2)^2 + 1} + 5n\sqrt{(n-2)^2 + 4} + 16\sqrt{2} + 3\sqrt{5} \\ &< ESO(B_n(n-4, 0, 0)).\end{aligned}$$

Figure 2.2: The graphs G_1 and G_2.

Case 2. $d_G(v_i) + d_G(v_j) = n+1$ for an edge $v_iv_j \in E(G)$. Then G is one of $G_3, G_4, \ldots, G_8$ shown in Figure 2.3. We can assume that $r \geq r_3 + 1, s \geq s_3$ and $r \geq r_4 + 1, s \geq s_4$. By Lemma 2.1, we have

$$\begin{aligned}ESO(G_7) &= r_3 f(r_3+3, 1) + 2f(r_3+3, 2) + f(r_3+3, s_3+3) + s_3 f(s_3+3, 1) + 2f(s_3+3, 2) + f(2, 2) \\ &< rf(r+3, 1) + 2f(r+3, 2) + f(r+3, s+3) + sf(s+3, 1) + 2f(s+3, 2) \\ &= B_n(r, s, 0)\end{aligned}$$

and

$$\begin{aligned}ESO(G_8) &= (r_4 - 1)f(r_4+3, 1) + 3f(r_4+3, 2) + f(2, 1) + s_4 f(s_4+3, 1) + 2f(s_4+3, 2) + f(r_4+3, s_4+3) \\ &< rf(r+3, 1) + 2f(r+3, 2) + f(r+3, s+3) + sf(s+3, 1) + 2f(s+3, 2) \\ &= B_n(r, s, 0).\end{aligned}$$

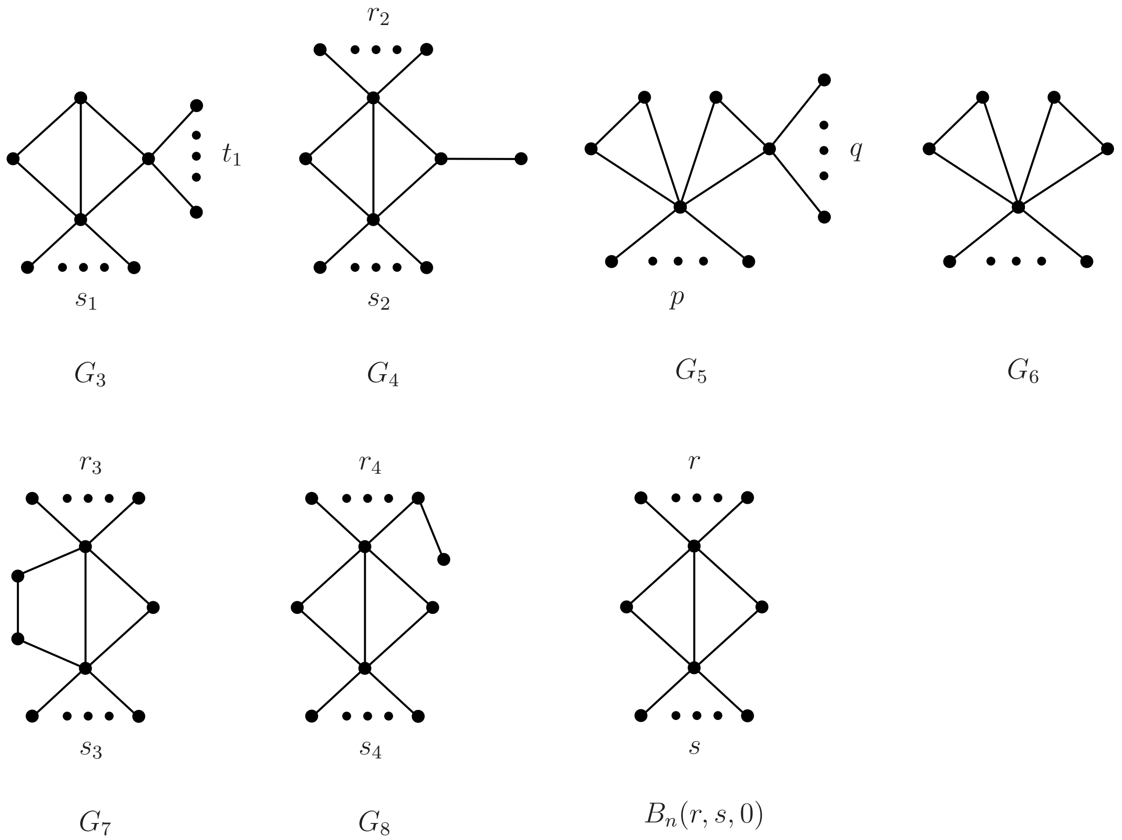

Figure 2.3: The graphs $G_3 - G_8$ and $B_n(r,s,0)$, where $r \geq r_3+1, s \geq s_3$ and $r \geq r_4+1, s \geq s_4$.

By Lemma 2.2, we have

$$ESO(G_7) < ESO(B(r,s,0)) < ESO(B_n(n-4,0,0)),$$

$$ESO(G_8) < ESO(B(r,s,0)) < ESO(B_n(n-4,0,0)),$$

$$ESO(G_3) < ESO(B_n(n-4,0,0)) \quad \text{and} \quad ESO(G_4) < ESO(B_n(n-4,0,0)).$$

Also, by Lemma 2.3, we have

$$ESO(G_5) < ESO(B_n(n-4,0,0)) \quad \text{and} \quad ESO(G_6) < ESO(B_n(n-4,0,0)).$$

Case 3. $d_G(v_i) + d_G(v_j) = n+2$ for an edge $v_iv_j \in E(G)$. Then $G \cong B_n(r,s,0)$ (see Figure 2.3). By Lemma 2.2, we have

$$ESO(B_n(r,s,0)) < ESO(B_n(n-4,0,0)).$$

Combining the conclusion made in the above three cases, we have the desired result (that is, Theorem 1.1). □

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