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Research Article

Odd facial total-coloring of maximal plane and outerplane graphs

Július Czap*

Department of Applied Mathematics and Business Informatics, Faculty of Economics, Technical University of Košice, Košice, Slovakia

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Abstract

A facial total-coloring of a plane graph G is a coloring of the vertices and edges such that no facially adjacent edges, no adjacent vertices, and no edge and its endvertices are assigned the same color. A facial total-coloring of G is odd if, for every face f and every color c, either no element or an odd number of elements incident with f is colored by c. In this article, it is shown that every outerplane graph with triangular internal faces admits an odd facial total-coloring with at most 12 colors. In the case of maximal outerplane graphs, 9 colors are sufficient for such a coloring. It is also shown that every maximal plane graph has an odd facial total-coloring with 6 or 7 colors. All of the obtained bounds are tight.

Keywords: facial total-coloring; odd facial total-coloring; outerplane graph; plane graph.

2020 Mathematics Subject Classification: 05C10, 05C15.

1. Introduction

A *drawing* of a graph maps each vertex to a point in the plane and each edge to a Jordan arc between its endvertices. A drawing is planar if no two edges intersect each other, except at their endvertices. A *planar graph* is a graph that has a planar drawing. A planar drawing partitions the plane into connected regions, called faces. There are two types of faces. The bounded faces are *internal*, while the unbounded face is the *outer face*. An outerplanar drawing is a planar drawing such that all the vertices are incident to the outer face. An *outerplanar graph* is a graph that admits an outerplanar drawing. A planar (outerplanar) graph is *maximal planar* (*maximal outerplanar*) if it is not possible to add an edge such that the resulting graph is still planar (outerplanar). For convenience, we often use the abbreviation plane (outerplane) graph for a planar (outerplanar) drawing of a planar (outerplanar) graph. A *cactus* is a connected outerplane graph in which any two cycles have at most one vertex in common. A *cactus forest* is an outerplane graph such that each of its components is a cactus.

Two edges of a plane graph G are *facially adjacent* if they are adjacent and consecutive in the cyclic order around their common endvertex. A *facial total-coloring* of a plane graph G is a coloring of the vertices and edges such that no facially adjacent edges, no adjacent vertices, and no edge and its endvertices are assigned the same color. This concept was introduced in 2016, by Fabrici, Jendroľ, and Vrbjarová [8]. Motivated by the above-mentioned concept and by odd facial vertex-coloring and odd facial edge-coloring of plane graphs (see e.g. [5,6]), in 2021, Czap and Šugerek [7] introduced the concept *odd facial total-coloring*. An odd facial total-coloring of a plane graph is a facial total-coloring such that for every face f and every color c, either no element or an odd number of elements incident with f is colored by c (an element incident with f is a vertex or an edge incident with f). Let $\chi''_o(G)$ denote the minimum number of colors required in an odd facial total-coloring of a plane graph G. In [7] it is proven that every tree on at least three vertices admits an odd facial total-coloring with exactly five colors. From this it follows that $\chi''_o(T) \in \{3,5\}$ for any nontrivial tree T, moreover it is easy to check whether $\chi''_o(T) = 3$ or $\chi''_o(T) = 5$. In the case of connected unicyclic plane graphs 10 colors are sufficient for such a coloring [3]. Moreover, infinitely many unicyclic plane graphs need 10 colors for an odd facial total-coloring. Odd facial total-coloring of plane graphs with many edge-disjoint cycles were studied in [4]. It is proven that every cactus forest has an odd facial total-coloring with at most 16 colors. Moreover, this bound is tight.

All of the above mentioned results on odd facial total-coloring concern sparse plane graphs. This article focuses on dense plane graphs. The odd facial total-coloring of maximal plane and outerplane graphs is investigated. It is shown that every maximal plane graph admits an odd facial total-coloring with 6 or 7 colors. In the case of maximal outerplane graphs, 9 colors are sufficient. Moreover, these bounds are tight. It is also shown that every outerplane graph with only triangular internal faces has an odd facial total-coloring with at most 12 colors. This bound is also tight.

*E-mail address: julius.czap@tuke.sk

2. Results

Theorem 2.1. *If G is a maximal plane graph on at least 3 vertices, then $6 \leq \chi''_o(G) \leq 7$. Moreover, the bounds are tight.*

Proof. If G is a maximal plane graph on at least 3 vertices, then every face of G (including the outer face) is bounded by a triangle (see Page 248 in [2]). Let C_3 denote the cycle on 3 vertices. Clearly, no color can appear more than once on C_3 in any odd facial total-coloring, hence $\chi''_o(C_3) = 6$. This implies that $\chi''_o(G) \geq 6$.

Every plane graph admits a proper vertex-coloring with at most 4 colors [1]. Let c be a proper vertex-coloring of G with colors $1, 2, 3, 4$. Now, color the edge uv with color

5 if $\{c(u), c(v)\} = \{1, 2\}$ or $\{c(u), c(v)\} = \{3, 4\}$,
6 if $\{c(u), c(v)\} = \{1, 3\}$ or $\{c(u), c(v)\} = \{2, 4\}$,
7 if $\{c(u), c(v)\} = \{1, 4\}$ or $\{c(u), c(v)\} = \{2, 3\}$.

In such a way, we obtain an odd facial total-coloring which uses 7 colors and hence $\chi''_o(G) \leq 7$.

The upper bound 7 is tight since the triangulation on 4 vertices has no odd facial total-coloring with 6 colors. □

A graph G is called Eulerian if every vertex of G has even degree.

Theorem 2.2. *If G is an Eulerian maximal plane graph, then $\chi''_o(G) = 6$.*

Proof. Every Eulerian maximal plane graph has a proper vertex-coloring with 3 colors (see Page 427 in [2]). Let c be a proper vertex-coloring of G with colors $1, 2, 3$. Now, color the edge uv with color 4 if $\{c(u), c(v)\} = \{1, 2\}$, with color 5 if $\{c(u), c(v)\} = \{1, 3\}$, with color 6 if $\{c(u), c(v)\} = \{2, 3\}$. In such a way we obtain an odd facial total-coloring of G with 6 colors. □

Theorem 2.3. *If G is a maximal outerplane graph, then $\chi''_o(G) \leq 9$. Moreover, this bound is tight.*

Proof. If G is a maximal outerplane graph, then the boundary of its outer face is a Hamiltonian cycle of G and every internal face is bounded by a triangle (see Page 264 in [2]). Let C be the boundary of the outer face. Let $C = v_1 e_1 v_2 e_2 \dots v_n e_n v_1$, where $v_1, v_2, \dots, v_n$ are the vertices, $e_1, e_2, \dots, e_n$ are the edges, $e_i = v_i v_{i+1}$ for $i = 1, 2, \dots, n-1$ and $e_n = v_n v_1$. We distinguish two cases according to the length of C.

First, assume that the length of C is odd and assume that G has at least 7 vertices. In the following, we define an edge-coloring c' of C, we color the edges in order $e_1, e_2, \dots, e_n$.

If $n = 6k - 1, k \geq 2$, then we use four times the pattern $1, 2$ and then $2k - 3$ times the pattern $1, 2, 3$.

If $n = 6k + 1, k \geq 1$, then we use two times the pattern $1, 2$ and then $2k - 1$ times the pattern $1, 2, 3$.

If $n = 6k + 3, k \geq 1$, then we use $2k + 1$ times the pattern $1, 2, 3$.

Every maximal outerplane graph admits a proper vertex-coloring with 3 colors (see Page 448 in [2]). Let c be a proper vertex-coloring of G with colors $4, 5, 6$. If there is an even number of vertices of color i, then we recolor one of them with color $i + 3$.

The colorings c' and c define a partial total-coloring of G with at most 9 colors. Observe that there is no uncolored element on the outer face, moreover, if a color appears on the outer face, then an odd number of elements are colored with it. This partial total-coloring of G can be extended to an odd facial total-coloring which uses at most 9 colors. Consider an internal face f. Since G is maximal, f is incident with 3 vertices and 3 edges. The 3 vertices have mutually distinct colors, because c is a proper vertex-coloring. Clearly, no uncolored edge of f is incident with the outer face. Consequently, every uncolored edge is incident with two triangles. This implies that there are at most 8 forbidden colors for every uncolored edge, hence there is at least 1 admissible color for every such edge. So, we can color the uncolored edges using a greedy coloring.

Now, assume that the length of C is even and assume that G has at least 7 vertices. If the length of C is $4k + 2, k \geq 2$, then we color the edges of C with colors 1 and 2 such that adjacent edges receive different colors. Then we find a proper vertex-coloring of G with colors $3, 4, 5$. If there is an even number of vertices of color i, then we recolor one of them with color $i + 3$. In such a way we obtain a partial total-coloring of G which can be extended to an odd facial total-coloring as in the previous case. Now, assume that the length of C is $4k, k \geq 2$. First, we color the edges of C in order $e_1, e_2, \dots, e_n$. We use two times the pattern $1, 2, 3$ and then $2k - 3$ times the pattern $1, 2$. Clearly, exactly two edges have color 3, and both colors 1 and 2 are used an odd number of times. Now, we find a proper vertex-coloring of G with colors $4, 5, 6$. If at most three colors are used an even number of times on C, then we choose one element for each such color and recolor them with new colors $7, 8, 9$. In such a way we obtain an odd total-coloring of C. This coloring can be extended to an odd facial total-coloring as above. Next, assume that the colors $3, 4, 5, 6$ appear an even number of times on C. There are exactly two internal faces of G, say f_1, f_2, which are incident with an edge of color 3. Let v be a vertex of G which is incident with neither f_1 nor f_2. Such a vertex exists since G has at least 7 vertices. Without loss of generality, assume that v has color 4. Now, we recolor v with color 3. After that we choose one vertex of color 5 and recolor it with 7, then choose a vertex of color 6 and recolor it with color 8. We obtain a total-coloring of C such that every used color appears an odd number of times on C. Finally, we extend this coloring to an odd facial total-coloring of G using a greedy coloring.

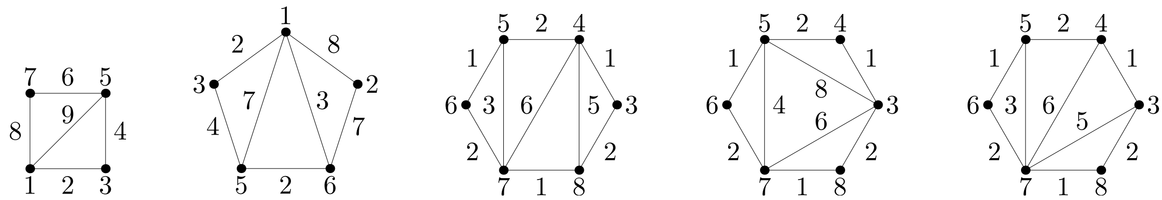

Figure 2.1: Odd facial total-colorings of small maximal outerplane graphs.

Figure 2.1 gives all maximal outerplane graphs that have at least 9 elements and fewer than 7 vertices. Each of them admits an odd facial total-coloring with at most 9 colors. Hence, $\chi''_o(G) \le 9$ for every maximal outerplane graph G. □

In what follows, we investigate outerplane graphs that have only triangular internal faces and are not maximal.

Theorem 2.4. *If G is an outerplane graph that has only triangular internal faces, then $\chi''_o(G) \le 12$. Moreover, this bound is tight.*

Proof. First, we prove that every connected outerplane graph which has only triangular internal faces admits a facial total-coloring with at most 6 colors, such that for every internal face f and every color c, either no element or an odd number of elements incident with f is colored by c. Suppose to the contrary, that H is a counterexample with the minimal number of vertices. Clearly, it has at least 4 vertices. Every outerplanar graph contains a vertex of degree 2 or less (see Page 264 in [2]). First, assume that H contains a vertex v of degree 1. Let u be the neighbor of v and let e_1, e_2 be the edges facially adjacent to uv in H (if the vertex u has degree 2, then $e_1 = e_2$). Let $H - v$ be the graph obtained from H by removing the vertex v. It has fewer vertices than H, each of its internal faces is triangular, therefore $H - v$ has a facial total-coloring φ with at most 6 colors, such that for every internal face f and every color c, either no element or an odd number of elements incident with f is colored by c. Without loss of generality, we can assume that $\varphi(e_1) = 1, \varphi(u) = 2, \varphi(u_2) = 3$. We can extend this coloring of $H - v$ to a required one of H easily. It is sufficient to color uv, v with colors $4, 5$ (see Figure 2.2 for illustration).

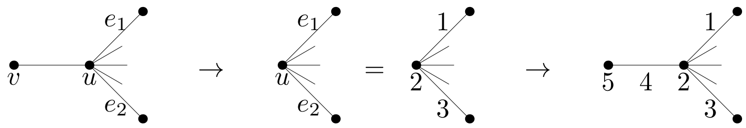

Figure 2.2: A vertex of degree 1 is a reducible configuration.

Now, assume that the minimum degree of H is 2. Let v be a vertex of degree 2 and let v_1, v_2 be its neighbors.

First, assume that v is incident with no triangular internal face (i.e. the edges incident with v are bridges). Since H is outerplanar, the vertices v_1 and v_2 are not adjacent in H. Let $H - v + v_1v_2$ be the graph obtained from H by removing the vertex v and adding the edge v_1v_2. This graph has fewer vertices than H so it has a required coloring φ. Without loss of generality, we can assume that $\varphi(v_1) = 1, \varphi(v_1v_2) = 2, \varphi(v_2) = 3$. Let ψ be a total-coloring defined in the following way: $\psi(u) = \varphi(u)$ for every vertex $u \in V(H) - v$, $\psi(e) = \varphi(e)$ for every edge $e \in E(H) - \{vv_1, vv_2\}$, $\psi(vv_1) = \varphi(v_1v_2)$, $\psi(v) = 4$ and $\psi(vv_2) \in \{1, 5, 6\} - \{\varphi(e_1), \varphi(e_2)\}$, where e_1, e_2 are the edges facially adjacent to vv_2 different from vv_1. It is easy to see that ψ is a required coloring of H (see Figure 2.3 for illustration).

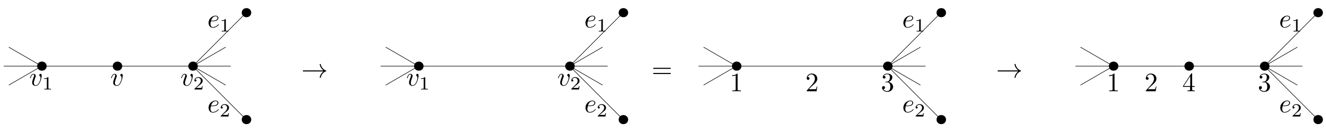

Figure 2.3: A vertex of degree 2 incident with two bridges is a reducible configuration.

Now, assume that v is incident with a triangular internal face. We distinguish some cases according to the degrees of its neighbors. At least one of the vertices v_1, v_2 has degree at least 3, since H has at least 4 vertices.

Case 1: Without loss of generality assume that $\deg(v_1) = 2$ and $\deg(v_2) \geq 3$.

Case 1.1: First, assume that $\deg(v_2) = 3$. Let v_3 be the third neighbor of v_2. Let $H - \{v, v_1\}$ be the graph obtained from H by removing the vertices v and v_1. It has fewer vertices than H so it admits a required coloring φ. Without loss of generality, we can assume that $\varphi(v_2) = 1, \varphi(v_2v_3) = 2, \varphi(v_3) = 3$. We extend this coloring to a required one by setting $\varphi(v) = 2, \varphi(v_1) = 3, \varphi(vv_1) = 4, \varphi(vv_2) = 5$, and $\varphi(v_1v_2) = 6$ (see Figure 2.4 for illustration).

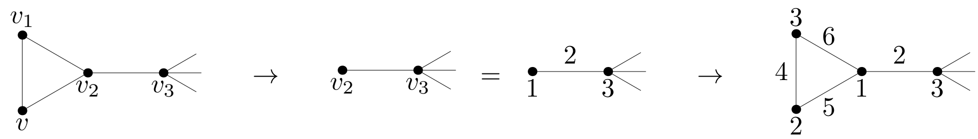

Figure 2.4: A triangle incident with two vertices of degree 2 and a vertex of degree 3 is a reducible configuration.

Case 1.2: Now, assume that $\deg(v_2) \geq 4$. Let e_1 be the edge facially adjacent to v_1v_2 different from vv_1, vv_2. Let e_2 be the edge facially adjacent to vv_2 different from vv_1, v_1v_2. The graph $H - \{v, v_1\}$ has fewer vertices than H so it admits a required coloring φ. Without loss of generality, we can assume that $\varphi(e_1) = 1, \varphi(v_2) = 2, \varphi(e_2) = 3$. We extend this coloring to a required one by setting $\varphi(v) = 1, \varphi(v_1) = 3, \varphi(vv_1) = 4, \varphi(vv_2) = 5$, and $\varphi(v_1v_2) = 6$ (see Figure 2.5 for illustration).

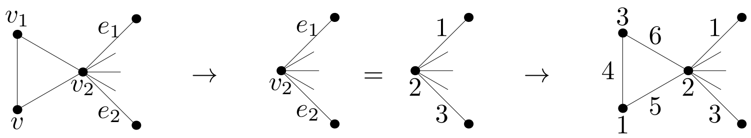

Figure 2.5: A triangle incident with two vertices of degree 2 and a vertex of degree at least 4 is a reducible configuration.

Case 2: Finally, assume that $\deg(v_1) \geq 3$ and $\deg(v_2) \geq 3$. Let e_1 be the edge facially adjacent to vv_1 different from vv_2, v_1v_2 and let e_2 be the edge facially adjacent to vv_2 different from vv_1, v_1v_2. The graph $H - v$ has fewer vertices than H so it admits a required coloring φ. Without loss of generality, we can assume that $\varphi(v_1) = 1, \varphi(v_1v_2) = 2, \varphi(v_2) = 3$. We extend this coloring of $H - v$ to a required one of H in the following way. First, we color the edge vv_1 with a color different from $1, 2, 3, \varphi(e_1)$. Then we color the edge vv_2. There are at most five forbidden colors for vv_2, namely the colors $1, 2, 3, \varphi(e_2)$ and the color of the edge vv_1. This implies that there is an admissible color for vv_2. Finally, we color the vertex v with a color that does not appear on the edges vv_1, vv_2, v_1v_2 and on the vertices v_1, v_2.

Hence, the minimal counterexample does not exist, which means that every connected outerplane graph which has only triangular internal faces admits a facial total-coloring with at most 6 colors, such that for every internal face f and every color c, either no element or an odd number of elements incident with f is colored by c. Now, we take such a coloring of each component of G. If some color c appears an odd number of times on the outer face of G, then we choose one element of color c and recolor it with a new color. Since we recolor at most 6 elements, the new coloring uses at most 12 colors. Clearly, the obtained coloring is an odd facial total-coloring.

The bound 12 is tight since the outerplane graph consisting of two cycles on three vertices needs 12 colors in every odd facial total-coloring. □

We finish the paper with two open problems.

Problem 2.1. *Is there a connected outerplane graph G with only triangular internal faces such that $\chi''_o(G) = 12$?*

If the bound 12 from Theorem 2.4 can be improved for connected outerplane graphs with triangular internal faces, then only by one. The connected graphs G and H depicted in Figure 2.6 need 11 colors in every odd facial total-coloring.

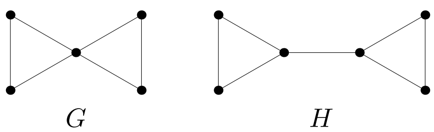

Figure 2.6: Graphs with $\chi''_o(G) = \chi''_o(H) = 11$.

Problem 2.2. *Characterize all graphs attaining the bounds given in Theorem 2.1, Theorem 2.3, and Theorem 2.4.*

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