Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **10** (2024) 11–19
DOI: 10.47443/cm.2024.042

Research Article

Suspension bridge with Kelvin-Voigt damping

Leandro Correia[1,2,*], Carlos Raposo[3], Joilson Ribeiro[2], Luiz Gutemberg[3]

[1]*Department of Exact and Technological Sciences, State University of Southwest Bahia, Vitória da Conquista, Bahia, Brazil*
[2]*Department of Mathematics, Federal University of Bahia, Salvador, Bahia, Brazil*
[3]*Department of Mathematics, Federal University of Pará, Salinópolis, Pará, Brazil*

(Received: 8 July 2024. Received in revised form: 6 August 2024. Accepted: 8 August 2024. Published online: 9 August 2024.)

Abstract

In this article, we study a suspension bridge model with Kelvin-Voigt damping. We use the Lumer-Phillips theorem and semigroup theory to prove the existence of the solution. We obtain exponential stability of the semigroup associated with a suitable energy space.

Keywords: suspension bridge; Timoshenko-Enrenfest beam; partial differential equations; Lummer-Phillips theorem.

2020 Mathematics Subject Classification: 35A01, 35A02, 35B35.

1. Introduction

Several engineering problems can be transformed into mathematical problems using an appropriate approach. For example, we point out the study of bridges, which play a fundamental role in the advancement of human development. Among the various bridge models, suspension bridges (see Figure 1.1) have a prominent place because they have a longer span than other bridge types. The Akashi-Kaikyo Bridge (with a span of 1991 meters) in Japan was thought to be the suspension bridge with the longest span in the world until March 2022, when Turkey opened the 1915 Çanakkale Bridge, which currently has the longest span (of more than 2000 meters) of any suspension bridge.

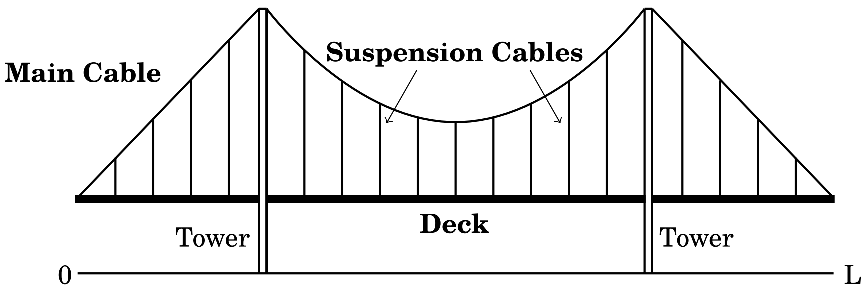

Figure 1.1: Suspension bridge. This figure is taken from the reference [9].

The study of suspension bridges is the subject of numerous research articles. In 1984, Hayashikawa and Watanabe [4] formulated a problem of free vertical vibration of suspension bridges. Mukiawa et al. [7] studied the existence and uniqueness of the solution of a thermal-Timoshenko-beam system with suspenders and Kelvin-Voigt damping type, where the heat was given by Cattaneo's law; also, a similar study for a suspension bridge with laminated beams was done by Raposo in [9]. For a suspension bridge with internal damping, Raposo et al. [10] proved that the solution not only decays exponentially, but it is also analytic.

As in [10], we assume that when compared to the length (span of the bridge), the transversal section dimensions of the deck are negligible, which allows us to use Timoshenko's one-dimensional theory to study a suspension bridge as a beam of length L, see [11–13].

In this work, $\varphi = \varphi(x,t)$ is the displacement of the cross-section at $x \in (0,L)$ and $\psi = \psi(x,t)$ is the rotation angle of the cross-section, where x denotes the distance along the center line of the beam in its equilibrium configuration and t is the time variable. The main cables are modeled by an elastic string $u = u(x,t)$, which is coupled to the deck employing suspension cables.

*Corresponding author (leandro.araujo@uesb.edu.br).

By considering the viscoelastic Kelvin-Voigt damping, we have the following coupled system:

$$u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) - \gamma_1 u_{txx} = 0 \quad \text{in } (0, L) \times (0, \infty), \tag{1}$$

$$\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \lambda(\varphi - u) - \gamma_2 \varphi_{txx} = 0 \quad \text{in } (0, L) \times (0, \infty), \tag{2}$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) - \gamma_3 \psi_{txx} = 0 \quad \text{in } (0, L) \times (0, \infty). \tag{3}$$

The suspender cables are assumed to be linear elastic strings with standard stiffness $\lambda > 0$. The constant $\alpha > 0$ is the elastic modulus of the string (holding the main cable to the deck). The positive coefficients ρ_1 and ρ_2 are the mass density and the moment of mass inertia of the beam, respectively. Moreover, b represents the cross section's rigidity coefficient and k represents the elasticity's shear modulus. Finally, the constants $\gamma_1, \gamma_2, \gamma_3 > 0$ are the coefficients of the damping force. We consider the initial data

$$\begin{cases} u(0,t) = u_0(x), u_t(x,0) = u_1(x), x \in (0,L), \\ \varphi(0,t) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), x \in (0,L), \\ \psi(0,t) = \psi_0(x), \psi_t(x,0) = \psi_1(x), x \in (0,L), \end{cases} \tag{4}$$

and the Dirichlet boundary conditions

$$\begin{cases} u(0,t) = u(L,t) = 0, t > 0, \\ \varphi(0,t) = \varphi(L,t) = 0, t > 0, \\ \psi(0,t) = \psi(L,t) = 0, t > 0. \end{cases} \tag{5}$$

Our functions belong to the functional spaces

$$\begin{cases} u_0(x) \in H_0^1(0,L), \quad u_1(x) \in L^2(0,L), \\ \varphi_0(x) \in H_0^1(0,L), \quad \varphi_1(x) \in L^2(0,L), \\ \psi_0(x) \in H_0^1(0,L), \quad \psi_1(x) \in L^2(0,L). \end{cases} \tag{6}$$

The remaining part of the paper is organized as follows. In Section 2, we give some preliminary results. We introduce the energy functional of the model in Section 3. In Section 4, we present the semigroup setting and establish the well-posedness of the system. Finally, in Section 5, we show the exponential decay of the system (1)–(3).

2. Preliminary results

Theorem 2.1. *Let $\mathcal{A}$ be an unbounded linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is dissipative and 0 belongs to the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$, then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$.*

Proof. See Theorem 1.2.4 on Page 3 in [6]. □

Theorem 2.2 (Lax-Milgram)**.** *Assume that $\mathbb{B}(u,v)$ is a continuous coercive sesquilinear form on a Hilbert space H. Given any $g \in H'$, which is the dual space of H, there exists a unique element $u \in H$ such that $\mathbb{B}(u,v) = \mathbb{L}(v) \ \ \forall \, v \in H$, where $\mathbb{L}(v) = \langle g, v \rangle_{H' \times H}$.*

Proof. See Corollary 5.8 on Page 140 in [1]. □

Lemma 2.1. *Let H be a Hilbert space. Let $B, L : H \to H$ be bounded linear operators such that L has a bounded inverse. If*

$$||B||_{\mathcal{L}(H)} < \frac{1}{||L^{-1}||_{\mathcal{L}(H)}},$$

then $B + L$ is a bounded and invertible linear operator.

Proof. First, we prove that $B + L$ is invertible; that is, $B + L$ is bijective. Let $y \in H$. For $x \in H$, $P(x) = L^{-1}y - L^{-1}Bx$ is a bounded linear operator. On the other hand,

$$\begin{aligned} ||P(z) - P(x)||_{\mathcal{L}(H)} = ||L^{-1}Bz - L^{-1}Bx||_{\mathcal{L}(H)} &\le ||L^{-1}||_{\mathcal{L}(H)} ||Bz - Bx||_{\mathcal{L}(H)} \\ &\le ||L^{-1}||_{\mathcal{L}(H)} |||B||_{\mathcal{L}(H)} ||z - x||_{\mathcal{L}(H)} \\ &\le C||z - x||_{\mathcal{L}(H)}. \end{aligned}$$

Since $C = ||L^{-1}||_{\mathcal{L}(H)}|||B||_{\mathcal{L}(H)}$, we have $0 < C < 1$. By the contraction mapping theorem, there exists a unique point $x \in X$ such that $P(x) = x$. Since $L^{-1}y - L^{-1}Bx = x$, we get $Lx = y - Bx$ and hence x is the unique solution of the problem $(B+L)x = y$. It is clear that $(B+L)x = 0$ has $x = 0$ as the unique solution. We have that $B+L$ is surjective and injective. Finally, as $B+L$ is bounded, by the closed graph theorem, $(B+L)^{-1}$ is also bounded. □

Theorem 2.3 (Gagliardo-Nirenberg)**.** *Let j and m be integers satisfying $0 \le j < m$. Let $1 \le q, r \le \infty$, $p \in \mathbb{R}$, and $\frac{j}{m} \le a \le 1$ such that*

$$\frac{1}{p} - \frac{j}{n} = a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}.$$

(a). *For any $u \in W^{m,r}(\mathbb{R}) \cap L^q(\mathbb{R})$, there is a positive constant C depending only on $n, m, j, q, r,$ and a such that the following inequality holds:*

$$\|D^j u\|_{L^p(\mathbb{R}^n)} \le C\|D^m u\|^a_{L^r(\mathbb{R}^n)}\|u\|^{1-a}_{L^q(\mathbb{R}^n)} \tag{7}$$

with the following exception: if $1 < r < \infty$ and $m - j - \frac{n}{r}$ is a nonnegative integer, then (7) holds only for a satisfying $\frac{j}{m} \le a < 1$.

(b). *For any $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$, there are two positive constants C_1 and C_2 such that*

$$\|D^j u\|_{L^p(\Omega)} \le C_1\|D^m u\|^a_{L^r(\Omega)}\|u\|^{1-a}_{L^q(\Omega)} + C_2\|u\|_{L^q(\Omega)},$$

with the same exception as in **(a)***, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. In particular, for any $u \in W^{m,r}_0(\Omega) \cap L^q(\Omega)$, the constant C_2 can be taken as zero.*

Theorem 2.4 (Gearhart-Huang-Prüss)**.** *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on a Hilbert space $\mathcal{H}$. Then, $S(t)$ is exponentially stable if and only if $i\mathbb{R} \subset \rho(\mathcal{A})$ and*

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty,$$

where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$.

Proof. See [2, 3, 5, 8]. □

3. Energy of the system

Multiplying (1) by u_t, we obtain

$$u_t u_{tt} - \alpha u_t u_{xx} - \lambda u_t(\varphi - u) - \gamma_1 u_t u_{txx} = 0.$$

Integrating over $(0, L)$, we arrive at

$$\int_0^L u_t u_{tt} dx - \alpha \int_0^L u_t u_{xx} dx - \lambda \int_0^L u_t(\varphi - u) dx = \gamma_1 \int_0^L u_t u_{txx} dx.$$

Integrating by parts and using (5), we obtain

$$\frac{d}{dt}\frac{1}{2}\int_0^L |u_t|^2 dx + \alpha \int_0^L u_{tx} u_x dx - \lambda \int_0^L u_t(\varphi - u) dx = -\gamma_1 \int_0^L |u_{tx}|^2 dx.$$

Then,

$$\frac{d}{dt}\frac{1}{2}\int_0^L |u_t|^2 dx + \frac{d}{dt}\frac{\alpha}{2}\int_0^L |u_x|^2 dx - \lambda \int_0^L u_t(\varphi - u) dx = -\gamma_1 \int_0^L |u_{tx}|^2 dx. \tag{8}$$

Multiplying (2) by φ_t, we get

$$\rho_1 \varphi_t \varphi_{tt} - k\varphi_t(\varphi_x + \psi)_x + \lambda \varphi_t(\varphi - u) - \gamma_2 \varphi_t \varphi_{txx} = 0.$$

Integrating over $(0, L)$, we obtain

$$\rho_1 \int_0^L \varphi_t \varphi_{tt} dx - k \int_0^L \varphi_t(\varphi_x + \psi)_x dx + \lambda \int_0^L \varphi_t(\varphi - u) dx = \int_0^L \gamma_2 \varphi_t \varphi_{txx} dx.$$

Integrating by parts and using (5), we arrive at

$$\frac{d}{dt}\frac{\rho_1}{2}\int_0^L |\varphi_t|^2 dx + k \int_0^L \varphi_{tx}(\varphi_x + \psi) dx + \lambda \int_0^L \varphi_t(\varphi - u) dx = -\int_0^L \gamma_2 |\varphi_{tx}|^2 dx. \tag{9}$$

Multiplying (3) by ψ_t, we obtain

$$\rho_2\psi_t\psi_{tt} - b\psi_t\psi_{xx} + \lambda\psi_t(\varphi_x + \psi) - \gamma_3\psi_t\psi_{txx} = 0.$$

Integrating over $(0, L)$, we get

$$\rho_2 \int_0^L \psi_t\psi_{tt}dx - b\int_0^L \psi_t\psi_{xx}dx + \lambda\int_0^L \psi_t(\varphi_x + \psi)dx = \int_0^L \gamma_3\psi_t\psi_{txx}dx.$$

Integrating by parts and using (5), we obtain

$$\frac{d}{dt}\frac{\rho_2}{2}\int_0^L |\psi_t|^2dx + b\int_0^L \psi_{tx}\psi_x dx + \lambda\int_0^L \psi_t(\varphi_x + \psi)dx = -\int_0^L \gamma_3|\psi_{tx}|^2dx.$$

That is,

$$\frac{d}{dt}\frac{\rho_2}{2}\int_0^L |\psi_t|^2dx + \frac{d}{dt}\frac{b}{2}\int_0^L |\psi_x|^2dx + \lambda\int_0^L \psi_t(\varphi_x + \psi)dx = -\int_0^L \gamma_3|\psi_{tx}|^2dx. \tag{10}$$

Summing (8), (9), and (10), we obtain

$$\frac{d}{dt}\frac{1}{2}\int_0^L \left[|u_t|^2 + \alpha|u_x|^2 + b|\psi_x|^2 + \rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + \lambda|\varphi - u|^2 + k|\varphi_x + \psi|^2\right] dx$$

$$= -\gamma_1\int_0^L |u_{tx}|^2dx - \gamma_2\int_0^L |\varphi_{tx}|^2dx - \gamma_3\int_0^L |\psi_{tx}|^2dx.$$

We define the energy of the system (1)–(3) by

$$E(t) = \frac{1}{2}\int_0^L \left[|u_t|^2 + \alpha|u_x|^2 + b|\psi_x|^2 + \rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + \lambda|\varphi - u|^2 + k|\varphi_x + \psi|^2\right] dx. \tag{11}$$

This shows the dissipative nature of the energy functional $E(t)$.

4. Existence and uniqueness of the solution

In this section, we use the results of the semigroup theory to obtain an existence theorem of the system (1)–(3). Taking $u_t = v, \varphi_t = w$ and $\psi_t = z$, we get a vector function given as $U = (u, v, \varphi, w, \psi, z)^T$,

$$U_t = \begin{pmatrix} v \\ \alpha u_{xx} + \lambda(\varphi - u) - \gamma_1 v_{xx} \\ w \\ k(\varphi_x + \psi)_x - \lambda(\varphi - u) - \gamma_2 w_{xx} \\ z \\ b\psi_{xx} - k(\varphi_x + \psi) - \gamma_3 z_{xx} \end{pmatrix} := \mathcal{A}U.$$

We can write (1)–(3) as a first-order evolution Cauchy problem:

$$\begin{cases} U_t - \mathcal{A}U = 0, \\ U(0) = U_0, \end{cases} \tag{12}$$

where $U \in \mathcal{H} = \{H_0^1(0, L) \times L^2(0, L)\}^3$. The domain of the unbounded linear operator $\mathcal{A}$ is

$$D(\mathcal{A}) = \{H_0^1(0, L) \cap H^2(0, L) \times L^2(0, L)\}^3$$

and $\mathcal{H}$ is a Hilbert space with the inner product

$$\langle U, \tilde{U}\rangle_{\mathcal{H}} = \int_0^L v\tilde{v}dx + \alpha\int_0^L u_x\tilde{u}_x dx + \rho_1\int_0^L w\tilde{w}dx + \rho_2\int_0^L z\tilde{z}dx + \lambda\int_0^L (\varphi - u)(\tilde{\varphi} - \tilde{u})dx + k\int_0^L (\varphi_x + \psi)(\tilde{\varphi_x} + \tilde{\psi})dx$$

We define a norm in $\mathcal{H}$ by

$$||U||_{\mathcal{H}}^2 = \langle U, U\rangle.$$

From Sobolev's space theory, we get that $D(\mathcal{A})$ is dense on $\mathcal{H}$. We want to show that $\mathcal{A}$ is a generator of a C_0-semigroup of contractions $S(t) = e^{At}, t \geq 0$, on the Hilbert Space $\mathcal{H}$. For this, we need the following theorem:

Theorem 4.1. *Let $\mathcal{A}$ be an unbounded linear operator such that the domain $D(A)$ is dense in a Hilbert space H. If $\mathcal{A}$ is dissipative and 0 belongs to the resolvent set $\rho(A)$ of $\mathcal{A}$, then $\mathcal{A}$ generates a C_0-semigroup of contractions on H.*

Proof. See Theorem 1.2.4 on Page 3 in [6]. □

Lemma 4.1. *The operator $\mathcal{A}$ is dissipative.*

Proof. For every $U = (u, v, \varphi, w, \psi, z) \in D(\mathcal{A})$, we have

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\gamma_1 \int_0^L |v_x|^2 dx - \gamma_2 \int_0^L |w_x|^2 dx - \gamma_3 \int_0^L |z_x|^2 dx \leq 0. \tag{13}$$

Therefore, $\mathcal{A}$ is dissipative. □

Lemma 4.2. *The number 0 belongs to $\rho(\mathcal{A})$.*

Proof. Let $F = (f^1, f^2, f^3, f^4, f^5, f^6)^T \in \mathcal{H}$ and consider the resolvent equation

$$-AU = F. \tag{14}$$

In terms of U and F, we get

$$-v = f^1 \text{ in } H_0^1(0, L), \tag{15}$$

$$-\alpha u_{xx} - \lambda(\varphi - u) + \gamma_1 v_{xx} = f^2 \text{ in } L^2(0, L), \tag{16}$$

$$-w = f^3 \text{ in } H_0^1(0, L), \tag{17}$$

$$-k(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma_2 w_{xx} = f^4 \text{ in } L^2(0, L), \tag{18}$$

$$-z = f^5 \text{ in } H_0^1(0, L), \tag{19}$$

$$-b\psi_{xx} + \lambda(\varphi_x + \psi) + \gamma_3 z_{xx} = f^6 \text{ em } L^2(0, L). \tag{20}$$

Using (15)–(20), we obtain

$$-\alpha u_{xx} - \lambda(\varphi - u) = \gamma_1 f^1_{xx} + f^2 := g_1 \in L^2(0, L), \tag{21}$$

$$-k(\varphi_x + \psi)_x + \lambda(\varphi - u) = \gamma_2 f^3_{xx} + f^4 := g_2 \in L^2(0, L), \tag{22}$$

$$-b\psi_{xx} + \lambda(\varphi_x + \psi) = \gamma_3 f^5_{xx} + f^6 := g_3 \in L^2(0, L). \tag{23}$$

Multiplying (21) by $\tilde{u} \in H_0^1(0, L)$, (22) by $\tilde{\varphi} \in H_0^1(0, L)$, and (23) by $\tilde{\psi} \in H_0^1(0, L)$. Then, integrating by parts, we obtain

$$-\alpha \int_0^L \tilde{u}_x u_x dx - \int_0^L \lambda\tilde{u}(\varphi - u)dx = \int_0^L \tilde{u} g_1 dx \in L^2(0, L) \tag{24}$$

$$-k \int_0^L \tilde{\varphi}_x(\varphi_x + \psi)dx + \lambda \int_0^L \tilde{\varphi}(\varphi - u)dx = \int_0^L \tilde{\varphi} g_2 dx \in L^2(0, L) \tag{25}$$

$$-b \int_0^L \tilde{\psi}_x \psi_x dx + \lambda \int_0^L \tilde{\psi}(\varphi_x + \psi)dx = \int_0^L \tilde{\psi} g_3 dx \in L^2(0, L) \tag{26}$$

Summing (24), (25), and (26), we obtain a variational problem

$$B((u, \varphi, \psi); (\tilde{u}, \tilde{\varphi}, \tilde{\psi})) = L((\tilde{u}, \tilde{\varphi}, \tilde{\psi})), \tag{27}$$

where $B : [H_0^1(0, L) \times H_0^1(0, L) \times H_0^1(0, L)]^2 \to \mathbb{C}$ with

$$B((u, \varphi, \psi); (\tilde{u}, \tilde{\varphi}, \tilde{\psi})) = \alpha \int_0^L u_x \tilde{u}_x dx + \lambda \int_0^L (\varphi - u)(\tilde{\varphi} - \tilde{u})dx + k \int_0^L (\varphi_x + \psi)(\tilde{\varphi}_x + \tilde{\psi}) + b \int_0^L \psi_x \tilde{\psi}_x dx,$$

and $L : [H_0^1(0, L) \times H_0^1(0, L) \times H_0^1(0, L)] \to \mathbb{C}$ with

$$L((\tilde{u}, \tilde{\varphi}, \tilde{\psi})) = \int_0^L \tilde{u} g_1 dx + \int_0^L \tilde{\varphi} g_2 dx + \int_0^L \tilde{\psi} g_3 dx.$$

Now, we define the norm in $H_0^1(0, L) \times H_0^1(0, L) \times H_0^1(0, L)$ as $||(u, \varphi, \psi)||^2 = B((u, \varphi, \psi); (u, \varphi, \psi))$. Note that B is bilinear, continuous, and coercive in $H_0^1(0, L) \times H_0^1(0, L) \times H_0^1(0, L)$. Hence, L is a continuous linear form.

Now, by the Lax-Milgram theorem, there exists a unique solution $(u,\varphi,\psi) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$ of (27) for every $(\tilde{u},\tilde{\varphi},\tilde{\psi}) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$. By the theory of elliptic equations (see Chapter 1 in [6]), (21), (22), and (23) yield $u,\varphi,\psi \in H^2(0,L)$; that is, $u,\varphi,\rho \in H_0^1 \cap H^2(0,L)$. On the other hand, from (15), (17), and (19), it follows that $v,w,z \in H_0^1(0,L)$. Thus, $U \in D(\mathcal{A})$ and $0 \in \rho(\mathcal{A})$. □

Theorem 4.2. *The operator $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = e^{At},\ t \geq 0$, in the Hilbert $\mathcal{H}$.*

Proof. By Lemmas 4.1 and 4.2, $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$. Also, $\mathcal{A}$ is dense. Now, the result holds by Theorem 4.1. □

5. Asymptotic behavior

To prove the main result of this section, we need some lemmas first.

Lemma 5.1. $i\mathbb{R} \subset \rho(\mathcal{A})$.

Proof. Suppose that $i\mathbb{R} \subset \rho(\mathcal{A})$ is false. Then, there exists θ and a sequence $\beta^n \to \theta, |\beta^n| < |\theta|$, such that

$$||(i\beta^n - A)^{-1}||_{L(\mathcal{H})} \to \infty$$

and for all $M > 0$, there exists $n_0 \in \mathbb{N}$ such that $n > n_0$ with

$$||(i\beta^n - A)^{-1}||_{L(\mathcal{H})} > M.$$

Therefore, there exists $0 \neq y^n \in \mathcal{H}$ such that

$$\frac{||(i\beta^n - A)^{-1}y^n||_{\mathcal{H}}}{||y^n||_{\mathcal{H}}} > M.$$

Writing $g^n = (i\beta^n - A)^{-1}y^n$, we obtain

$$\frac{||g^n||_{\mathcal{H}}}{||(i\beta^n - A)g^n||_{\mathcal{H}}} > M$$

or

$$\frac{||(i\beta^n - A)g^n||_{\mathcal{H}}}{||g^n||_{\mathcal{H}}} < \frac{1}{M}.$$

Thus,

$$||(i\beta^n - A)U^n||_{\mathcal{H}} < \frac{1}{M},$$

where $n > n_0$ and $U^n = \dfrac{g^n}{||g^n||} \in D(\mathcal{A})$ with $||U^n|| = 1$. Consequently, we conclude that

$$||(i\beta^n - A)U^n||_{\mathcal{H}} \to 0, \tag{28}$$

that is,

$$i\beta^n u^n - v^n \to 0 \text{ in } H_0^1(0,L), \tag{29}$$

$$i\beta^n v^n - \alpha u^n_{xx} + \lambda(\varphi^n - u^n) + \gamma_1 v^n_{xx} \to 0 \text{ in } L^2(0,L), \tag{30}$$

$$i\beta^n \varphi^n - w^n \to 0 \text{ in } H_0^1(0,L), \tag{31}$$

$$i\beta^n w^n - k(\varphi^n_x + \psi^n)_x + \lambda(\varphi^n - u^n) + \gamma_2 w^n_{xx} \to 0 \text{ in } L^2(0,L), \tag{32}$$

$$i\beta^n \psi^n - z^n \to 0 \text{ in } H_0^1(0,L), \tag{33}$$

$$i\beta^n z^n - b\psi^n_{xx} + k(\varphi^n + \psi^n) - \gamma_2 z^n_{xx} \to 0 \text{ in } L^2(0,L). \tag{34}$$

Now, observe that

$$\langle (i\beta^n - A)U^n, U^n\rangle_{\mathcal{H}} = i\beta^n ||U^n||_{\mathcal{H}} - \langle AU^n, U^n\rangle_{\mathcal{H}}.$$

Taking the real part, we obtain

$$\Re\langle (i\beta^n - A)U^n, U^n\rangle_{\mathcal{H}} = \gamma_1 \int_0^L |v^n_x|^2 dx + \gamma_2 \int_0^L |w^n_x|^2 dx + \gamma_3 \int_0^L |z^n_x|^2 dx.$$

As U^n is bounded and $(i\beta^n - A)U^n \to 0$, we have

$$v_x^n \to 0 \text{ in } L^2(0,L), \tag{35}$$

$$w_x^n \to 0 \text{ in } L^2(0,L), \tag{36}$$

$$z_x^n \to 0 \text{ in } L^2(0,L). \tag{37}$$

Using the Poincaré inequality, we have

$$v^n \to 0 \text{ in } L^2(0,L), \tag{38}$$

$$w^n \to 0 \text{ in } L^2(0,L), \tag{39}$$

$$z^n \to 0 \text{ in } L^2(0,L). \tag{40}$$

Using (38) in (29), (39) in (31), and (40) in (33), we obtain

$$u^n \to 0 \text{ in } L^2(0,L), \tag{41}$$

$$\varphi^n \to 0 \text{ in } L^2(0,L), \tag{42}$$

$$\psi^n \to 0 \text{ in } L^2(0,L). \tag{43}$$

However, we need that

$$u^n \to 0 \text{ in } H_0^1(0,L), \tag{44}$$

$$\varphi^n \to 0 \text{ in } H_0^1(0,L), \tag{45}$$

$$\psi^n \to 0 \text{ in } H_0^1(0,L). \tag{46}$$

For this, using (38), (41), and (42) in (30), we obtain

$$-\alpha u_{xx}^n + \gamma_1 v_{xx}^n \to 0 \text{ in } L^2(0,L). \tag{47}$$

Integrating from 0 to x, we arrive at

$$-\alpha(u_x^n - u_x^n(0)) + \gamma_1(v_x^n - v_x^n(0)) \to 0 \text{ in } L^2(0,L). \tag{48}$$

But, by (35), we have

$$-\alpha(u_x^n - u_x^n(0)) \to 0 \text{ in } L^2(0,L). \tag{49}$$

As α is constant, it follows from the Gagliardo-Nirenberg inequality that

$$u_x^n \to 0 \text{ in } L^2(0,L). \tag{50}$$

Similarly, we obtain that $\varphi_x^n, \psi_x^n \to 0$ in $L^2(0,L)$. Therefore, we get that $u^n, \varphi^n, \psi^n \to 0$ in $H_0^1(0,L)$. This implies that $\|U^n\| \to 0$, which is a contradiction to $\|U^n\| = 1$. □

Lemma 5.2. $\overline{\lim\limits_{|\beta|\to\infty}} \|(i\beta - A)^{-1}\| < \infty$.

Proof. Again, proceeding by contradiction, we have (26). However, we need to be more careful because $\beta^n \to \infty$. Dividing by β^n, we have

$$iu^n - \frac{1}{\beta^n} v^n \to 0 \text{ in } H_0^1(0,L), \tag{51}$$

$$iv^n - \frac{1}{\beta^n}[\alpha u_{xx}^n + \lambda(\varphi^n - u^n) + \gamma_1 v_{xx}^n] \to 0 \text{ in } L^2(0,L), \tag{52}$$

$$i\varphi^n - \frac{1}{\beta^n} w^n \to 0 \text{ in } H_0^1(0,L), \tag{53}$$

$$i\beta^n w^n + \frac{1}{\beta^n}[-k(\varphi_x^n + \psi^n)_x + \lambda(\varphi^n - u^n) + \gamma_2 w_{xx}^n] \to 0 \text{ in } L^2(0,L), \tag{54}$$

$$i\psi^n - \frac{1}{\beta^n} z^n \to 0 \text{ in } H_0^1(0,L), \tag{55}$$

$$i\beta^n z^n + \frac{1}{\beta^n}[-b\psi_{xx}^n + k(\varphi^n + \psi^n) - \gamma_2 z_{xx}^n] \to 0 \text{ in } L^2(0,L). \tag{56}$$

Now, note that

$$\Re\left\langle\left(i-\frac{1}{\beta^n}A\right)U^n,U^n\right\rangle_{\mathcal{H}}=\frac{\gamma_1}{\beta^n}\int_0^L|v_x^n|^2dx+\frac{\gamma_2}{\beta^n}\int_0^L|w_x^n|^2dx+\frac{\gamma_3}{\beta^n}\int_0^L|z_x^n|^2dx.$$

As U^n is bounded and $\left(i-\frac{1}{\beta^n}A\right)U^n\to 0$, we have that

$$\frac{v_x^n}{\beta^n}\to 0 \text{ in } L^2(0,L), \tag{57}$$

$$\frac{w_x^n}{\beta^n}\to 0 \text{ in } L^2(0,L), \tag{58}$$

$$\frac{z_x^n}{\beta^n}\to 0 \text{ in } L^2(0,L). \tag{59}$$

Using the Poincaré inequality, we arrive at

$$\frac{v^n}{\beta^n}\to 0 \text{ in } L^2(0,L), \tag{60}$$

$$\frac{w^n}{\beta^n}\to 0 \text{ in } L^2(0,L), \tag{61}$$

$$\frac{z^n}{\beta^n}\to 0 \text{ in } L^2(0,L). \tag{62}$$

Using (60) in (51), (61) in (53), and (62) in (55), we obtain

$$u^n\to 0 \text{ in } L^2(0,L), \tag{63}$$

$$\varphi^n\to 0 \text{ in } L^2(0,L), \tag{64}$$

$$\psi^n\to 0 \text{ in } L^2(0,L). \tag{65}$$

Now, multiplying (52) by v, we have

$$i\|v\|-\frac{1}{\beta^n}[-\alpha\langle u_{xx}^n,v\rangle+\lambda\langle(\varphi^n-u^n),v\rangle+\gamma_1\langle v_{xx}^n,v\rangle]\to 0 \text{ in } L^2(0,L). \tag{66}$$

Taking the real part, we obtain

$$\frac{1}{\beta^n}\alpha\langle u_{xx}^n,v\rangle-\frac{1}{\beta^n}\lambda\langle(\varphi^n-u^n),v\rangle-\frac{1}{\beta^n}\gamma_1\langle v_{xx}^n,v\rangle]\to 0 \text{ in } L^2(0,L).$$

Integrating by parts, we have

$$\frac{\alpha}{\beta^n}\langle u_{xx}^n,v\rangle\to 0 \text{ in } L^2(0,L). \tag{67}$$

Applying (67) in (66), we arrive at $v^n\to 0$ in $L^2(0,L)$. In this way, by (52), we have

$$\frac{1}{\beta^n}[\alpha u_{xx}^n+\lambda(\varphi^n-u^n)+\gamma_1 v_{xx}^n]\to 0 \text{ in } L^2(0,L).$$

Multiplying by β^n, we obtain

$$\alpha u_{xx}^n+\lambda(\varphi^n-u^n)+\gamma_1 v_{xx}^n\to 0 \text{ in } L^2(0,L).$$

Using (63), (64), and the Gagliardo-Nirenberg inequality, we conclude that $u_x^n\to 0$ in $L^2(0,L)$. Similarly, we obtain that $\varphi_x^n,\psi_x^n\to 0$ in $L^2(0,L)$. Therefore, we get that $u^n,\varphi^n,\psi^n\to 0$ in $H_0^1(0,L)$ and hence $\|U^n\|\to 0$, which is a contradiction with $\|U^n\|=1$. □

Theorem 5.1. *The C_0-semigroup of contractions $S(t)=e^{\mathcal{A}t},t\geq 0$, generated by $\mathcal{A}$ is exponentially stable.*

Proof. In view of Theorem 2.4, the result follows from Lemmas 5.1 and 5.2. □

6. Conclusion

In this article, the Timoshenko-Enrenfest theory has been used to study a suspension bridge system with Kelvin-Voigt damping as a beam. With the techniques of semigroup theory, it has been shown that this system of equations has a solution. In fact, a suitable Hilbert space has been used, where a semigroup has been built, to prove that its energy is dissipative, and the Lummer-Phillips theorem has been applied to obtain the solution of the system. It has been proved that the mentioned semigroup has an exponential decay. Investigations of a suspension bridge system with other types of damping are open for future work.

Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable suggestions, which greatly improved the exposition of the paper. Also, C. A. Raposo would like to thank CNPq (Grant No. 307447/2023-5) for the financial support.

References

[1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[2] K. J. Engel, R. Nagel, *One-Parameter Semigroup for Linear Evolution Equation*, Springer, New York, 2000.

[3] L. Gearhart, Spectral theory for contraction semigroups on Hilbert space, *Trans. Amer. Math. Soc.* **236** (1978) 385–394.

[4] T. Hayashikawa, N. Watanabe, Vertical vibration in Timoshenko beam suspension bridges, *J. Eng. Mech.* **110** (1984) 341–356.

[5] F. L. Huang, Characteristic condition for exponential stability of linear dynamical systems Hilbert spaces, *Ann. Diff. Equ.* **1** (1985) 43–56.

[6] Z. Liu, S. Zheng, *Semigroups Associated with Dissipative Systems*, Chapman & Hall, London, 1999.

[7] S. E. Mukiawa, Y. Khan, H. A. Sulaimani, M. E. Omaba, C. D. Enyi, Thermal Timoshenko beam system with suspenders and Kelvin-Voigt damping, *Front. Appl. Math. Stat.* **9** (2023) #1153071.

[8] J. Prüss, On the spectrum of C_0-semigroups, *Trans. Amer. Math. Soc.* **284** (1984) 847–857.

[9] C. Raposo, Suspension bridge model with laminated beam, *Math. Morav.* **27** (2023) 77–90.

[10] C. Raposo, L. Correia, J. Ribeiro, A. Cunha, Suspension bridge with internal damping, *Acta Mech.* **235** (2024) 203–214.

[11] S. P. Timoshenko, On the correction for shear of the differential equation for transverse vibration of prismatic bars, *Lond. Edinb. Dubl. Phil. Mag. J. Sci.* **41** (1921) 744–746.

[12] S. P. Timoshenko, Theory of suspension bridges, *J. Franklin Inst.* **235** (1943) 213–238.

[13] S. P. Timoshenko, Theory of suspension bridges, *J. Franklin Inst.* **235** (1943) 327–349.