Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **1** (2020) 67–79
DOI: 10.47443/cm.2020.0021

A survey of antiregular graphs

Akbar Ali[1,2,*]

[1]*Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il, Saudi Arabia*
[2]*Knowledge Unit of Science, University of Management and Technology, Sialkot, Pakistan*

(Received: 7 September 2020. Received in revised form: 26 September 2020. Accepted: 26 September 2020. Published online: 1 October 2020.)

Abstract

The set of all different degrees of the vertices of a graph G is known as the degree set of G. A nontrivial graph of order n whose degree set consists of $n-1$ elements is called an antiregular graph. Antiregular graphs have been studied in literature also under other names, including "quasi-perfect graphs", "maximally nonregular graphs" and "degree antiregular graphs". This paper aims to gather the known results concerning the antiregular graphs.

Keywords: antiregular graph; quasi-perfect graph; maximally nonregular graph; pairlone graph; half-complete graph.

2020 Mathematics Subject Classification: 05C07, 05C31, 05C50.

1. Introduction

The terminology and notation not defined in this paper can be found in some relevant (well–known) books, like [12, 16, 19, 32].

A graph in which all vertices have the same degree is called a regular graph. In contrast, a graph whose all vertices have different degrees was named as the "perfect graph" by Behzad and Chartrand [9]. However, Berge [6, 7] had already used the term "perfect graph" to mean something entirely different – particularly, according to Berge, a graph in which the chromatic number and the clique number have the same value for each of its induced subgraphs is called perfect graph – in the remaining part of this paper, the terminology of perfect graphs suggested by Berge is used. Certainly, there exist at least two regular graphs of order n for every $n \geq 2$. However, on the other hand, there does not exist any nontrivial graph whose all vertices have different degrees – this fact was noticed firstly by Dirac (as mentioned in [17]) and was independently proven by Behzad and Chartrand [9]. The authors of [9] also characterized the graphs of order n containing exactly $n-2$ vertices of different degrees for every $n \geq 2$. Such graphs were referred as the quasi-perfect graphs in [9], maximally nonregular graphs in [62], degree antiregular graphs in [47] and antiregular graphs in [45, 48]. Nowadays, it seems that "antiregular graphs" is a generally accepted term for referring such kind of graphs and thereby we use this name in the remaining part of this paper.

It is known [9] that for every integer $n \geq 2$, there are exactly two nonisomorphic antiregular graphs of order n and that these two graphs are complementary (two nonisomorphic graphs G and G' of same order, satisfying the property $\overline{G} \cong G'$ are called complementary graphs [53], where $\overline{G}$ is the complement of the graph G). For $n \geq 2$, denote by A_n and $\overline{A}_n$ the connected antiregular and disconnected antiregular, respectively, graphs of order n. It needs to be mentioned here that connected antiregular graphs were referred as pairlone graphs in [57] and half-complete graphs in [29]. Antiregular graphs have several interesting properties, and the connected antiregular graphs have found some applications in control theory [38–40]. Also, antiregular graphs are sometimes considered as the graphs opposite to the regular graphs [1, 13, 15, 34].

The main purpose of this paper is to give the known results regarding the antiregular graphs A_n and $\overline{A}_n$. The paper is organized as follows. Basic properties of antiregular graphs are listed in the next section. Section 3 is concerned with some certain polynomials of antiregular graphs. In Section 4, results regarding the graph invariants of antiregular graphs are given. Section 5 is devoted to the results about the spectral study of antiregular graphs.

2. Basic properties

In this section, various basic properties of the antiregular graphs are given. Recall that we have defined antiregular graphs of order at least 2. For completeness, we take A_1 as the unique graph of order 1 then obviously $\overline{A}_1 \cong A_1$.

*Email address: akbarali.maths@gmail.com

Theorem 2.1. [9] *For $n \geq 3$, the antiregular connected graph A_n has exactly one vertex of degree $n-1$ and the antiregular disconnected graph $\overline{A}_n$ has exactly one vertex of degree 0. Also, for $n \geq 2$, the equal degrees are $\lfloor n/2 \rfloor$ in A_n and $\lfloor (n-1)/2 \rfloor$ in $\overline{A}_n$.*

Let D_2 be the unique connected graph of order 2. For every integer $n \geq 3$, denote by D_n the complement of the graph obtained from D_{n-1} by adding an isolated vertex. Nebeský [52] and Munarini [51] independently noted that $A_n \cong D_n$ and $\overline{A}_n \cong \overline{D}_n$, where $\overline{D}_n$ is the complement of D_n. Thus, we may say that the graph A_n (and hence $\overline{A}_n$) can be drawn recursively.

The two vertices of an antiregular graph that have the same degree are called exceptional vertices [52].

Theorem 2.2. [52] *For the connected antiregular graph A_n, the following properties hold.*

(i) *Two vertices u and v of A_n are adjacent if and only if $\deg u + \deg v \geq n$.*

(ii) *Let i be an integer satisfying $1 \leq i \leq \lfloor n/2 \rfloor$ and assume that $u_i, v_i \in V(A_n)$ such that $\deg u_i = i$, $\deg v_i = n - i$, $u_{\lfloor n/2 \rfloor} \neq v_{\lfloor n/2 \rfloor}$ then $u_i v_i \in E(A_n)$ and the set $\{u_1v_1, u_2v_2, \cdots, u_{\lfloor n/2 \rfloor}v_{\lfloor n/2 \rfloor}\}$ is a maximum matching of A_n.*

(iii) *Size of A_n is $\lfloor n/2 \rfloor \lceil n/2 \rceil$.*

(iv) *For $n \geq 3$, if u is an exceptional vertex of A_n then $A_n - u$ is isomorphic to A_{n-1}.*

(v) *For $n \geq 3$, the graph obtained from A_n by identifying its exceptional vertices is isomorphic to A_{n-1}.*

(vi) *If r is a positive integer then A_n contains a subgraph isomorphic to K_r (complete graph of order r) if and only if $r \leq \lfloor (n+1)/2 \rfloor$.*

(vii) *The graph A_n is planar if and only if $n \leq 7$.*

Note that by using Theorem 2.2(i), one can iteratively construct the graphs $A_2, A_3, A_4, \cdots$, starting from A_2 (that is, the path graph of order two). More precisely, for every $n \geq 3$, the graph A_n can be obtained from A_{n-1} by adding a new vertex and connecting it to all vertices with degree greater than $\lfloor (n-1)/2 \rfloor$ and to one of the two vertices with the same degree $\lfloor (n-1)/2 \rfloor$.

A graph G of order n is called tree-complete if for every tree T of order n there is a tree T' spanning the graph G such that T and T' are isomorphic [58].

Theorem 2.3. [58] *The graph A_n is tree-complete.*

Given a class $\mathbb{F}$ of graphs, a graph G is said to be universal for $\mathbb{F}$ if G contains every graph of $\mathbb{F}$ as a subgraph [10]. Some detail about universal graphs can be found in the references [21,22,49,55]. It should be noted here that the definition of the universal graph for trees and that of tree-complete graph are coincident. Thus, the main result (Theorem 2) of the paper [45] is same as Theorem 2.3.

The neighborhood graph of a vertex u in a graph G is the subgraph induced by all those vertices of G that are adjacent to u. A graph whose all vertices have nonisomorphic neighborhood graphs is called C-graph [28]. Since two vertices of different degrees in a graph have nonisomorphic neighborhood graphs, one may think that there is a good relation between the C-graphs and antiregular graphs – however, from the next result it can be seen that this is not the case.

Theorem 2.4. [28] *If G is an antiregular graph then G is not a C-graph.*

A graph whose every induced subgraph contains either an isolated vertex or an universal vertex (that is, a vertex adjacent to all other vertices) is called threshold graph [27]. To the best of author's knowledge, threshold graphs were firstly introduced by Chvátal and Hammer [23,24] as well as Henderson and Zalcstein [37] independently. In the next theorem, it is stated that antiregular graphs are threshold graphs, and hence antiregular graphs enjoy all the properties of threshold graphs – here, it should be mentioned that the theory of threshold graphs is well-established and that threshold graphs have numerous applications in computer science and psychology [44].

Theorem 2.5. [47]

(i) *Antiregular graphs are threshold graphs (this fact was also proved independently by Levit and Mandrescu [43]).*

(ii) *The graph A_n is perfect and its line graph is Hamiltonian.*

For a given binary string $b = b_1b_2\cdots b_n$ with $b_1 = 0$, let G_1 be a trivial graph with $V(G_1) = \{v_1\}$ and then recursively define for $j = 2, 3, \cdots, n$, a graph G_j obtained from G_{j-1} by adding a new vertex v_j and making v_j a universal vertex if $b_j = 1$, or leaving v_j as an isolated vertex if $b_j = 0$. After the nth step, the resulting graph $G = G(b)$ is a threshold graph; G is clearly connected if and only if $b_n = 1$.

If G is a connected threshold graph with binary string $b = b_1b_2\cdots b_n$ then we write the string b as $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\cdots 0^{s_k}1^{s_k}$ where $n = \sum_{i=1}^{k}(s_i + t_i)$, 0^{s_i} is short-hand for $s_i \geq 1$ consecutive zeros and 1^{t_i} is short-hand for $t_i \geq 1$ consecutive ones.

Theorem 2.3 states that that every tree of order n is isomorphic to a subgraph of the connected antiregular graph A_n. Aguilar *et al.* [2] showed that a similar property of the connected antiregular graphs holds also for the class of threshold graphs.

Theorem 2.6. [2] *Every connected threshold graph of order $n \geq 2$ is isomorphic to an induced subgraph of the connected antiregular graph A_{2n-2}. In fact, let G be a connected threshold graph of order n and with binary string $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\cdots 0^{s_k}1^{s_k}$ where $n = \sum_{i=1}^{k}(s_i + t_i)$. Let $n' = 2(n-k)$ if $s_1 = 1$ and let $n' = 2(n-k) - 1$ if $s_1 \geq 2$. Then G is an induced subgraph of $A_{n'}$. Moreover, $A_{n'}$ is the smallest antiregular graph containing G as an induced subgraph.*

The next result gives information about the (largest) connected antiregular graph contained in a threshold graph.

Theorem 2.7. [2] *Let G be a connected threshold graph with binary string $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\cdots 0^{s_k}1^{s_k}$. Let $r = 2k$ if $s_1 = 1$ and let $r = 2k + 1$ if $s_1 \geq 2$. Then the connected antiregular graph A_r is an induced subgraph of G. In either case, A_r is the largest antiregular graph contained in G as an induced subgraph.*

By a clique in a graph G, we mean a subgraph of G that is complete, and by a clique set in G, we mean the vertex set of a complete induced subgraph of G. A graph whose vertex set can be partitioned into the disjoint union of an independent set and a clique set (either of which may be empty) is called a split graph [30]

Theorem 2.8. [14] *Antiregular graphs are split graphs.*

For a given graph F, the F-degree of a vertex v in a graph G is the number of those subgraphs of G isomorphic to F that contain v [18]. A concept related to the F-degree of a vertex was also discussed by Kocay [42].

Theorem 2.9. [57] *For every $n \geq 4$, the antiregular graph A_n contains exactly $n-2$ vertices having different P_3-degrees, where P_3 is the path graph of order 3.*

Let $N_G(v)$ be the set of all those vertices of G that are adjacent to v and let $N_G[v] = N_G(v) \cup \{v\}$. A vertex v of a graph G is a simplicial vertex if the graph induced by the vertices of $N_G[v]$ is a complete subgraph graph of G. A graph G in which every vertex is either a simplicial vertex or is adjacent to a simplicial vertex is called simplicial graph [20]. A graph whose order is equal to the sum of its matching number and independence number is called König-Egerváry graph [26, 59].

Theorem 2.10. [43] *Every antiregular graph is a simplicial graph as well as a König-Egerváry graph.*

For a nontrivial graph G of order n, a function $f : V(G) \to \{0, 1\}$ is said to be an antiregular dominating function of G if $\left|\left\{ \sum_{v\in N_G[u]} f(u) \ : \ u \in V(G)\right\}\right| = n-1$ and $\sum_{v\in N_G[u]} f(u) \geq 1$ for every vertex $u \in V(G)$.

Theorem 2.11. [35] *Every antiregular graph has an antiregular dominating function.*

A connected graph in which every pair of adjacent vertices has different degrees is called totally segregated graph [41]. Totally segregated graphs are studied also under the name of neighborly irregular graphs [11].

Theorem 2.12. [56] *For $n \geq 3$, the connected antiregular graph A_n is totally segregated if and only if n is odd.*

A connected graph G with the maximum degree Δ is said to be maximally irregular graph if the degree set (that is, the set of all different vertex degrees) of G contains exactly Δ elements [50].

Theorem 2.13. [56] *Every connected antiregular graph is also a maximally irregular graph.*

Further results related to connected antiregular graphs can found in [29].

3. Some certain polynomials

This section is devoted to outline the known results concerning some particular polynomials, different from characteristic polynomials, of the antiregular graphs. Firstly, we present a result related to the chromatic polynomial.

Theorem 3.1. [58] *The chromatic polynomial of the connected antiregular graph A_n is*

$$x\left(x-\frac{n}{2}\right)^{\lfloor n/2\rfloor-\lfloor (n-1)/2\rfloor}\prod_{i=1}^{\lfloor (n-1)/2\rfloor}(x-i)^2.$$

The matching polynomial of a graph G of order n is defined by

$$M(G;\lambda)=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^{n-k}m_k(G)\lambda^{n-2k},$$

where $m_k(G)$ is the number of all matchings of G made of k independent edges. Munarini [51] established several results involving matching polynomial of the connected antiregular graph A_n.

Theorem 3.2. [51] *Let $M_n(\lambda)=M(A_n;\lambda)$.*

(i) *The matching polynomials of the connected antiregular graphs satisfy the following recurrence relation*

$$M_{n+2}(\lambda)=(\lambda^2-1)M_n(\lambda)-\lambda M_n'(\lambda),$$

where $M_n'(\lambda)$ denotes the derivative of $M_n(\lambda)$ with respect to λ.

(ii) *The matching polynomial $M_n(\lambda)$ has n real roots and for $n\geq 3$ all these roots are contained in the interval*

$$\Big(-2\sqrt{n-2},\ 2\sqrt{n-2}\Big).$$

(iii) *The matching polynomials of the connected antiregular graphs form a Sturm sequence.*

(iv) *The exponential generating series for $M_{2n}(\lambda)$ and $M_{2n+1}(\lambda)$ are*

$$\sum_{n\geq 0}M_{2n}(\lambda)\frac{t^n}{n!}=e^{-t}e^{-\frac{\lambda^2}{2}(e^{-2t}-1)}$$

and

$$\sum_{n\geq 0}M_{2n+1}(\lambda)\frac{t^n}{n!}=\lambda e^{-2t}e^{-\frac{\lambda^2}{2}(e^{-2t}-1)}.$$

Also, the following identities hold

$$M_{2n}(\lambda)=\frac{1}{\lambda}\sum_{k=0}^{n}\binom{n}{k}M_{2k+1}(\lambda)\quad\text{and}\quad M_{2n+1}(\lambda)=\lambda\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}M_{2k}(\lambda).$$

(v) *The matching polynomial $M_{2n}(\lambda)$ has $2n$ simple real roots, from which n are negative and n are positive. While, $M_{2n+1}(\lambda)$ has $2n+1$ simple real roots, from which n are negative, one is zero and remaining n are positive. Moreover, if $r_{-n},\cdots,r_{-1},r_1,\cdots,r_n$ are the roots of $M_{2n}(\lambda)$ and $s_{-n-1},\cdots,s_{-1},s_1,\cdots,s_{n+1}$ are the roots of $M_{2n+2}(\lambda)$ such that $r_{-n}<\cdots<r_{-1}<r_1<\cdots<r_n$ and $s_{-n-1}<\cdots<s_{-1}<s_1<\cdots<s_{n+1}$ then*

$$s_{-n-1}<r_{-n}<s_{-n}<\cdots<r_{-1}<s_{-1}<0<s_1<r_1<\cdots<s_n<r_n<s_{n+1}.$$

Similarly, if $r_{-n},\cdots,r_{-1},0,r_1,\cdots,r_n$ are the roots of $M_{2n+1}(\lambda)$ and $s_{-n-1},\cdots,s_{-1},0,s_1,\cdots,s_{n+1}$ are the roots of $M_{2n+3}(\lambda)$ such that $r_{-n}<\cdots<r_{-1}<0<r_1<\cdots<r_n$ and $s_{-n-1}<\cdots<s_{-1}<0<s_1<\cdots<s_{n+1}$ then

$$s_{-n-1}<r_{-n}<s_{-n}<\cdots<r_{-1}<s_{-1}<0<s_1<r_1<\cdots<s_n<r_n<s_{n+1}.$$

Denote by $I(G;\lambda)$ the independence polynomial of a graph G.

Theorem 3.3. [43] *The independence polynomial of A_n is*

$$I(A_n;\lambda)=\begin{cases}(\lambda+1)^{k-1}(\lambda+2)-1 & \text{if } n=2k-1,\\ 2(\lambda+1)^k-1 & \text{if } n=2k.\end{cases}$$

Also, the independence polynomial of $\overline{A}_n$ is

$$I(A_n;\lambda)=\begin{cases}2(\lambda+1)^k-\lambda-1 & \text{if } n=2k-1,\\ (\lambda+1)^k(\lambda+2)-\lambda-1 & \text{if } n=2k.\end{cases}$$

A finite sequence $(a_0, a_1, a_2, \cdots, a_n)$ of real numbers satisfying the inequality $a_i^2 \geq a_{i-1}a_{i+1}$ for every $i \in \{1, 2, \cdots, n-1\}$ is called log-concave. A polynomial whose coefficients form a log-concave sequence is called log-concave polynomial. The next two corollaries are the direct consequences of Theorem 3.3.

Corollary 3.1. [43]

(i) *The independence polynomial of every antiregular graph is log-concave.*

(ii) *In the family of threshold graphs, there is no non-antiregular graph whose independence polynomial is equal to the independence polynomial of an antiregular graph. In other words, within the family of threshold graphs, every antiregular graph is uniquely determined by its independence polynomial.*

Corollary 3.2. [43]

(i) *The independence polynomial $I(A_{2k}; \lambda)$ has only one real root, that is $-1 + \frac{1}{\sqrt[k]{2}}$, for every odd k and exactly two real roots, namely $-1 \pm \frac{1}{\sqrt[k]{2}}$, for each even k.*

(ii) *The independence polynomial $I(A_{2k-1}; \lambda)$ has only one real root, that belongs to the open interval $(-1, 0)$, for every odd k and exactly two real roots, one belongs to $(-1, 0)$ and the other belongs to $(-3, -2)$, for each even k.*

4. Graph invariants

The aim of this section is to gather known results regarding different graph invariants of the antiregular graphs. Firstly, we note from Theorem 2.2(ii) that the matching number of the connected antiregular graph A_n is $\lfloor n/2 \rfloor$ – this value was independently calculated by Merris [45].

Corollary 4.1. [45, 52] *The matching number of A_n is $\lfloor n/2 \rfloor$.*

Nebeský [52] gave the chromatic number of the connected antiregular graph A_n and the same was calculated independently by Merris [45] and Salehi [57].

Proposition 4.1. [45, 52, 57] *The chromatic number of A_n is $\lceil (n+1)/2 \rceil = \lfloor n/2 \rfloor + 1$.*

Sedláček [58] determined several graph invariants of connected antiregular graphs.

Theorem 4.1. [58] *For the connected antiregular graph A_n, the following properties hold.*

(i) *The number of all trees spanning the graph A_n is*

$$\frac{(n-1)!}{\lceil n/2 \rceil}.$$

(ii) *The edge chromatic number of A_n is $n-1$ and the total chromatic number of A_n is*

$$\begin{cases} 3 & \text{if } n = 2, \\ n & \text{if } n \geq 3. \end{cases}$$

(iii) *The independence number of A_n is $\lfloor (n+1)/2 \rfloor$, the number of all maximal independent sets of A_n is*

$$\begin{cases} 1 & \text{if } n \text{ is odd}, \\ 2 & \text{otherwise}, \end{cases}$$

the matching number of A_n is $\lfloor n/2 \rfloor$ and the number of all maximal independent edge sets of A_n is

$$\begin{cases} 2^{(n-1)/2} & \text{if } n \text{ is odd}, \\ 1 & \text{otherwise}. \end{cases}$$

The formula for the number of all trees spanning the graph A_n given in Theorem 4.1(i) was also derived in [51], but using an alternative way.

Theorem 4.2. [47] *The algebraic connectivity of A_n is 1 for $n \geq 3$.*

The inverse of the matrix $I_n + D(G) - A(G)$, appeared within the field of chemical graph theory [33], is known as the doubly stochastic matrix [46] where I_n is the identity matrix of order n, $D(G)$ is the diagonal matrix of a graph G of order n and $A(G)$ is the adjacency matrix of G. (It needs to be mentioned here that the terminology of doubly stochastic matrix is also used in [36] in connection with the strongly connected digraphs.) Merris [47] conjectured that the minimal value of entries of the doubly stochastic matrix of the graph A_n is $\frac{1}{2(n+1)}$ for $n \geq 3$ and this conjecture was proved by Berman and Zhang [8].

Theorem 4.3. [8] *The minimal value of entries of the doubly stochastic matrix of the graph A_n is $\frac{1}{2(n+1)}$ for $n \geq 3$.*

A matching whose edges contains all the vertices of the graph is called perfect matching and a matching whose edges contains all the vertices, except one, of the graph is known as quasi-perfect matching. Denote by M_n the total number of matchings of A_n.

Theorem 4.4. [51]

(i) *The graph A_{2n} has exactly one perfect matching and A_{2n+1} has 2^n quasi-perfect matchings.*

(ii) *It holds that*

$$M_{2n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} M_{2k+1} \quad \textit{and} \quad M_{2n+1} = \sum_{k=0}^{n} \binom{n}{k} M_{2k}\,.$$

(iii) *The exponential generating series for the numbers M_{2n} and M_{2n+1} are*

$$\sum_{n\geq 0} M_{2n}\frac{t^n}{n!} = e^{(e^t-2t-1)/2} \quad \textit{and} \quad \sum_{n\geq 0} M_{2n+1}\frac{t^n}{n!} = e^{(e^t-4t-1)/2}\,.$$

Also, the ordinary generating series for the numbers M_{2n} and M_{2n+1} are

$$\sum_{n\geq 0} M_{2n}t^n = \sum_{n\geq 0} \frac{t^k}{(1-t)(1-3t)\cdots(1-(2k+1)t)}$$

and

$$\sum_{n\geq 0} M_{2n+1}t^n = \sum_{n\geq 0} \frac{t^k}{(1-2t)(1-4t)\cdots(1-(2k+2)t)}\,.$$

Theorem 4.5. [61] *Let t_n and $\bar{t}_n$ be the number of triangles in A_n and $\overline{A}_n$, respectively. Also, let m_n be the number of edges of A_n. If $n \geq 4$ then*

$$t_n = t_{n-2} + m_{n-2} = \begin{cases} \dfrac{n(n-1)(n-2)}{24} & \textit{if } n \textit{ is even,} \\ \dfrac{(n+1)(n-1)(n-3)}{24} & \textit{otherwise,} \end{cases}$$

and

$$\bar{t}_n = t_{n-1} = \begin{cases} \dfrac{n(n-2)(n-4)}{24} & \textit{if } n \textit{ is even,} \\ \dfrac{(n-1)(n-2)(n-3)}{24} & \textit{otherwise.} \end{cases}$$

The number of all independent sets of a graph G is called the Fibonacci number of G [54]. The next result is a direct consequence of Theorem 3.3.

Corollary 4.2. [43]

(i) *The Fibonacci number of A_n is*

$$\begin{cases} 2^{(n+2)/2} - 1 & \textit{if } n \textit{ is even,} \\ 3 \times 2^{(n-1)/2} - 1 & \textit{otherwise.} \end{cases}$$

Also, the Fibonacci number of $\overline{A}_n$ is

$$\begin{cases} 3 \times 2^{n/2} - 2 & \textit{if } n \textit{ is even,} \\ 2^{(n+3)/2} - 2 & \textit{otherwise.} \end{cases}$$

(ii) *For the connected antiregular graph A_n, the number of independent sets of odd size is greater by one than the number of independent sets of even size, while for the disconnected antiregular graph $\overline{A}_n$, these two numbers are equal.*

An irregularity measure (IM) of a connected graph G is a non-negative graph invariant satisfying the property: $IM(G) = 0$ if and only if G is regular. The total irregularity of a graph G, denoted by $irr_t(G)$, is one of the much studied irregularity measures, and it is defined as

$$irr_t(G) = \sum_{\{u,v\}\subseteq V(G)} |\deg u - \deg v|.$$

Antiregular graphs belong to the class of the graphs that attain the maximum value of irr_t among all the connected graphs of a fixed order $n \geq 3$. The problem of devising the irregularity measure(s), for which only antiregular graphs have the maximum value among all the connected graphs of a fixed order $n \geq 3$, was posed in [56]. This problem was solved in the recent paper [4] by introducing the following two irregularity measures

$$IRA(G) = \frac{n(n-1)}{2} \cdot \frac{1}{N_0(G)} - 1 \quad \text{and} \quad IRB(G) = 1 - \frac{2}{n(n-1)} \cdot N_0(G)\,,$$

where

$$N_0(G) = \sum_{i=1}^{\Delta} \frac{n_i(n_i-1)}{2}\,,$$

and n_i is the number of vertices of degree i in G.

5. Spectral theory

In this section, known results about the spectral study of the antiregular graphs are outlined. The first such result, related to the eigenvalues of the adjacency matrix and Laplacian matrix of a connected antiregular graph, is due to Merris [47].

Theorem 5.1. [47]

(i) *If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of the adjacency matrix of the connected antiregular graph A_n, then either $\lambda_s = 0 = \lambda_{n-s+1}$, or they have opposite signs, where $1 \leq s \leq n$.*

(ii) *The Laplacian eigenvalues of A_n consist of all but one of the integers $0, 1, 2, \cdots, n$ and the "missing eigenvalue" is $\lfloor (n+1)/2 \rfloor$.*

Denote by $\varphi_n(\lambda)$ the characteristic polynomial of the connected antiregular graph A_n. Munarini [51] derived several results involving the characteristic polynomial $\varphi_n(\lambda)$ of A_n.

Theorem 5.2. [51]

(i) *The characteristic polynomials of the connected antiregular graphs satisfy the second order non-linear recurrence relation*

$$\varphi_{n+2}(\lambda) = (-1)^n \varphi_{n+1}(-\lambda - 1) + (\lambda^2 + \lambda)\varphi_n(\lambda)$$

as well as the fourth order linear recurrence relation

$$\varphi_{n+4}(\lambda) = (2\lambda^2 + 2\lambda - 1)\varphi_{n+2}(\lambda) + (\lambda^2 + \lambda)^2 \varphi_n(\lambda).$$

(ii) *The characteristic polynomials of the connected antiregular graphs satisfy the following explicit expressions*

$$\begin{aligned}
\varphi_{2n}(\lambda) &= \sum_{k=0}^{n} \binom{n+k}{2k} (-1)^k \left(\frac{2k}{n+k} \cdot \lambda + 1 \right) (\lambda^2 + \lambda)^{n-k} \\
&= (\lambda + 1) \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-1)^k \frac{n}{n-k} (n\lambda - n + k)\, H_{nk}(\lambda) \\
&= \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (-1)^k \frac{2(2k+1)\lambda^2 - 2(n-4k+1)\lambda - n - 1}{2k+1} h_{nk}(\lambda)
\end{aligned}$$

and

$$\varphi_{2n+1}(\lambda) = \sum_{k=0}^{n} \binom{n+k}{2k} (-1)^k \left(\lambda - \frac{n-k}{2k+1} \right) (\lambda^2 + \lambda)^{n-k}$$

$$= \lambda \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-1)^k \frac{n}{n-k} \left(n\lambda^2 + 2k\lambda - 2n + 3k\right) H_{nk}(\lambda)$$
$$= \frac{\lambda}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (-1)^k \frac{2(2k+1)\lambda^2 - 2(n-4k-1)\lambda - 3n + 4k - 1}{2k+1} h_{nk}(\lambda),$$

where

$$H_{nk}(\lambda) = (2\lambda^2 + 2\lambda - 1)^{n-2k-1}(\lambda^2 + \lambda)^{2k}$$

and

$$h_{nk}(\lambda) = (2\lambda^2 + 2\lambda - 1)^{n-2k-1}(4\lambda^2 + 4\lambda - 1)^k.$$

(iii) *The characteristic polynomials of the connected antiregular graphs can be expressed in terms of the Chebyshev polynomials as given below:*

$$\varphi_{2n}(\lambda) = \frac{2\lambda(\lambda+2)(\lambda^2+\lambda)^n}{2\lambda^2+2\lambda-1}(T_n(f(\lambda))) - \frac{(2\lambda+1)(\lambda^2+\lambda)^n}{2\lambda^2+2\lambda-1}(U_n(f(\lambda)))$$

and

$$\varphi_{2n+1}(\lambda) = \frac{2\lambda(\lambda+1)^2(\lambda^2+\lambda)^n}{2\lambda^2+2\lambda-1}(T_n(f(\lambda))) - \frac{\lambda(2\lambda+3)(\lambda^2+\lambda)^n}{2\lambda^2+2\lambda-1}(U_n(f(\lambda)))$$

where U_n and T_n are the Chebyshev polynomial functions of the first and second kind depending on $f(\lambda)$, and

$$f(\lambda) = \frac{2\lambda^2+2\lambda-1}{2\lambda(\lambda+1)}.$$

Let $p(x)$ be a polynomial of degree n with real roots $r_1, r_2, \cdots, r_n$ and let $q(x)$ be a polynomial of degree $n+1$ with real roots $q_1, q_2, \cdots, q_{n+1}$, such that $r_1 \le r_2 \le \cdots \le r_n$ and $q_1 \le q_2 \le \cdots \le q_{n+1}$. These polynomial have the interlacing property when $q_1 \le r_1 \le q_2 \le r_2 \le \cdots \le q_n \le r_n \le q_{n+1}$. A sequence $\{p_n(x)\}_n$ of polynomials is a Sturm sequence when every $p_n(x)$ is a real polynomial of degree n with n real distinct roots, and $p_n(x)$ and $p_{n+1}(x)$ have the interlacing property for every positive integer n.

Theorem 5.3. [51] *The characteristic polynomials of the connected antiregular graphs form a Sturm sequence.*

As usual, let $\mathbf{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ be the vector of length n with the entry equal to 1 in position i. The energy of a graph G is denoted by $\mathcal{E}(G)$ and is defined as the sum of the absolute values of all eigenvalues of G.

Theorem 5.4. [51]

(i) *For every $n \ge 1$, $\lambda = -1$ is a simple eigenvalue of A_{2n} and the associated eigenspace is $\langle(-\mathbf{e}_n, \mathbf{e}_1)\rangle$, and $\lambda = 0$ is an eigenvalue of A_{2n+1} and its associated eigenspace is $\langle(\mathbf{e}_n, -1, \mathbf{0})\rangle$.*

(ii) *If λ_n is the maximum eigenvalue of A_n then it holds that*

$$\frac{n}{2}\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor \le \lambda_n \le \frac{n\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor + 1}.$$

(iii) *The antiregular graph A_{2n} has $2n$ simple real eigenvalues: n negative and n positive. The antiregular graph A_{2n+1} has $2n+1$ simple real eigenvalues: n negative, one zero and n positive.*

(iv) *The energy of A_n, that is $\mathcal{E}(A_n)$ satisfies*

$$2\sqrt{\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor} \le \mathcal{E}(A_n) \le 2\left\lfloor \frac{n}{2} \right\rfloor \sqrt{\left\lceil \frac{n}{2} \right\rceil}.$$

(v) *The determinant of the adjacency matrix of A_n is*

$$\begin{cases} (-1)^{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Here, we mention that the determinant of the adjacency matrix of A_n, when n is even, was also found in [5]. The next result gives a characterization for the eigenvalues of the connected antiregular graphs.

Theorem 5.5. [3] *Let*

$$\theta = \arccos\left(\frac{-2\lambda^2 - 2\lambda + 1}{2\lambda(\lambda+1)}\right).$$

For $n = 2k$, λ is an eigenvalue of the connected antiregular graph A_n if and only if

$$\lambda = \frac{\sin(k\theta)}{\sin(k\theta) + \sin((k-1)\theta)}.$$

For $n = 2k+1$, $\lambda \neq 0$ is an eigenvalue of the connected antiregular graph A_n if and only if

$$\frac{2-\lambda^2}{\lambda(\lambda+1)} = \frac{\sin((k-1)\theta)}{\sin(k\theta)}.$$

The numbers -1 and 0 are usually considered as the trivial eigenvalues of the threshold graphs (and hence of the antiregular graphs). Aguilar *et al.* [3] showed that connected antiregular graphs have no non-trivial eigenvalues in the interval $[(-1-\sqrt{2})/2, (-1+\sqrt{2})/2]$ and Ghorbani [31] generalize this result by proving that threshold graphs have no non-trivial eigenvalues in the aforementioned interval.

Theorem 5.6. [3, 31] *No non-trivial eigenvalue of the antiregular graphs belongs to the interval*

$$\left[\frac{-1-\sqrt{2}}{2}, \frac{-1+\sqrt{2}}{2}\right].$$

Let $\lambda_1^+, \lambda_2^+, \cdots, \lambda_k^+$ and

$$\begin{cases} \lambda_1^-, \lambda_2^-, \cdots, \lambda_{k-1}^- & \text{if } n = 2k, \\ \lambda_1^-, \lambda_2^-, \cdots, \lambda_k^- & \text{if } n = 2k+1, \end{cases}$$

be the positive and negative (excluding -1 when $n = 2k$), respectively, eigenvalues of the connected antiregular graph A_n such that $\lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+$ and

$$\begin{cases} \lambda_1^- < \lambda_2^- < \cdots < \lambda_{k-1}^- < -1 & \text{if } n = 2k, \\ \lambda_1^- < \lambda_2^- < \cdots < \lambda_k^- & \text{if } n = 2k+1. \end{cases}$$

Theorem 5.7. [3]

(i) *Let r be a fixed positive number less than 1 and ϵ be an arbitrary positive number. Then, for sufficiently large k, the inequality*

$$|\lambda_i^+ + \lambda_i^- + 1| < \epsilon$$

holds for all $i \in \{1, 2, \cdots, k-1\}$ such that $\frac{2i}{2k-1} \leq r$ if $n = 2k$ and $\frac{i}{k} \leq r$ if $n = 2k+1$ is odd.

(ii) *Let $\lambda_1^+(k)$ be the smallest positive eigenvalue of the connected antiregular graph A_n and denote by $\lambda_1^-(k)$ the negative eigenvalue of A_n closest to the trivial eigenvalue, where $n = 2k$ if n is even and $n = 2k+1$ if n is odd. Then, the sequence $\{\lambda_1^+(k)\}_{k=1}^{\infty}$ is strictly decreasing and converges to $(-1+\sqrt{2})/2$, while the sequence $\{\lambda_1^-(k)\}_{k=1}^{\infty}$ is strictly increasing and converges to $(-1-\sqrt{2})/2$.*

(iii) *If $\sigma(n)$ is the set of all eigenvalues of A_n then closure of the set $\bigcup_{n\geq 1} \sigma(n)$ is*

$$(-\infty, (-1-\sqrt{2})/2] \cup \{-1, 0\} \cup [(-1+1\sqrt{2})/2, \infty).$$

Note that the function F defined by

$$F(\theta) = \frac{\sin(k\theta)}{\sin(k\theta) + \sin((k-1)\theta)} \tag{1}$$

has vertical asymptotes at

$$\gamma_j = \frac{2j\pi}{2k-1}, \tag{2}$$

$j = 1, 2, \cdots, k-1$, in the interval $(0, \pi)$. Also, the equation

$$\theta = \arccos\left(\frac{-2\lambda^2 - 2\lambda + 1}{2\lambda(\lambda+1)}\right)$$

has the following two solutions

$$\lambda = f_1(\theta) = \frac{-(\cos\theta + 1) + \sqrt{(\cos\theta+1)(\cos\theta+3)}}{2(\cos\theta+1)} \tag{3}$$

$$\lambda = f_2(\theta) = \frac{-(\cos\theta + 1) - \sqrt{(\cos\theta+1)(\cos\theta+3)}}{2(\cos\theta+1)}. \tag{4}$$

Theorem 5.8. [3] *With the notations defined in Equations (1), (2), (3) and (4), the following statements hold.*

(i) *The functions F and f_1 intersect exactly k times in the interval $(0,\pi)$. If $\theta_1^+, \theta_2^+, \cdots, \theta_k^+$ are the intersection points satisfying $\theta_1^+ < \theta_2^+ < \cdots < \theta_k^+$ then the positive eigenvalues of A_{2k} are the outputs of f_1 at these intersection points and it holds that*
$$f_1(\theta_1^+) < f_1(\theta_2^+) < \cdots < f_1(\theta_k^+)$$
and
$$f_1(\gamma_{j-1}) < f_1(\theta_j^+) < f_1(\gamma_j) \quad \text{for } j = 1,2,\cdots,k-1.$$

(ii) *The functions F and f_2 intersect exactly $(k-1)$ times in the interval $(0,\pi)$. If $\theta_1^-, \theta_2^-, \cdots, \theta_{k-1}^-$ are the intersection points satisfying $\theta_1^- < \theta_2^- < \cdots < \theta_{k-1}^-$ then the negative eigenvalues of A_{2k} are the outputs of f_2 at these intersection points, together with -1, and it holds that*
$$f_2(\theta_{k-1}^-) < \cdots < f_2(\theta_2^-) < f_2(\theta_1^-) < -1$$
and
$$f_2(\gamma_j) < f_2(\theta_j^-) < f_2(\gamma_{j-1}) \quad \text{for } j = 1,2,\cdots,k-1.$$

(iii) *If $n = 2k$ then the inequality*
$$|\lambda_j^+ + \lambda_j^- + 1| < \frac{2\pi f_1'(\gamma_j)}{2k-1}$$
holds for all $j \in \{1,2,\cdots,k-1\}$. In particular, if $r \in (0,1)$ is fixed and ϵ is a given positive number, and if k satisfies $\frac{2\pi f_1'(\gamma_i j)}{2k-1} < \epsilon$ then the inequality
$$|\lambda_j^+ + \lambda_j^- + 1| < \epsilon$$
holds for all $j \in \{1,2,\cdots,\frac{(2k-1)r}{2}\}$.

(iv) *If $n = 2k$ then the inequalities*
$$|\lambda_j^+ - f_1(\gamma_j)| < \frac{2\pi f_1'(\gamma_j)}{2k-1}$$
and
$$|\lambda_j^- - f_1(\gamma_j)| < \frac{2\pi f_1'(\gamma_j)}{2k-1}$$
hold for all $j \in \{1,2,\cdots,k-1\}$.

(v) *Let $\lambda_{\max} > 0$ and $\lambda_{\min} < 0$ be the largest and smallest eigenvalues, respectively, of the connected antiregular graph A_n where n is even. Then,*
$$F(\pi) = \frac{n}{2} < \lambda_{\max} \quad \text{and} \quad f_2\left(\frac{2(n/2-1)\pi}{n-1}\right) < \lambda_{\min}.$$

Note that n is even in all the results presented in Theorem 5.8. These results hold almost verbatim for the case $n = 2k+1$; the only change is that the ratio $\frac{2\pi}{2k-1}$ is now $\frac{\pi}{k}$, see [3] for details.

Define the matrices
$$U = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$
For stating the next result, we assume that $n = 2k$ and define $k-1$ matrices as follows. Let H_0 be the $n \times n$ matrix whose principal submatrix indexed by the rows and the columns 1, 2, 3 equals U and with its remaining entries equal to zero. For $i = 1,2,\cdots,k-2$, let H_i be the $n \times n$ matrix whose principal submatrix indexed by the rows and the columns $2i+1$, $2i+2$, $2i+3$ equals V and with its remaining entries equal to zero. Let H_{k-1} be the $n \times n$ matrix whose principal submatrix indexed by the rows and the columns $2k-1$, $2k$ equals W and with its remaining entries equal to zero. With this notation we have the following result.

Theorem 5.9. [5] *For $n = 2k$, the inverse of the adjacency matrix of the connected antiregular graph A_n is*
$$H_0 + H_1 + \cdots + H_{k-1}.$$

Theorem 5.9 gives expression for the inverse of the adjacency matrix of the connected antiregular graph A_n involving sums of certain matrices. The next result gives a closed-form expression for the aforementioned inverse matrix.

Theorem 5.10. [3] *For $n = 2k$, the inverse of the adjacency matrix of the connected antiregular graph A_n is*

$$\begin{bmatrix} S & T \\ T & O \end{bmatrix},$$

where O is the $k \times k$ null matrix and, S and T are $k \times k$ matrices defined as

$$S = \begin{bmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} & & & -1 & 1 \\ & & \iddots & \iddots & \\ & \iddots & \iddots & & \\ -1 & \iddots & & & \\ 1 & & & & \end{bmatrix}.$$

Let $\mu^+(A_n) = \lambda_1^+$ and

$$\mu^-(A_n) = \begin{cases} \lambda_{k-1}^- & \text{if } n = 2k, \\ \lambda_k^- & \text{if } n = 2k+1. \end{cases}$$

Proposition 5.1. [2] *If $n \geq 3$ is odd, then $\mu^-(A_n) \geq \mu^-(A_{n+1})$ and $\mu^+(A_n) \geq \mu^+(A_{n+1})$. Also, if $n \geq 4$ is even, then $\mu^-(A_n) \leq \mu^-(A_{n+1})$ and $\mu^+(A_n) \leq \mu^+(A_{n+1})$.*

In the remaining part of this paper, for a graph G of order n, we denote by $\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)$ all the eigenvalues of G satisfying $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$. Also, we denote by $m_{\lambda_i}(G)$ the algebraic multiplicity of an eigenvalue $\lambda_i(G)$. The inertia of G is the triple $i(G) = (i_-(G), i_0(G), i_+(G))$ where $i_-(G)$ is the number of negative, $i_+(G)$ is the number of positive, and $i_0(G)$ is the number of zero eigenvalues of G.

Theorem 5.11. [2] *Let G be a connected threshold graph of order n and with binary string $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\cdots 0^{s_k}1^{s_k}$ where $n = \sum_{i=1}^{k}(s_i + t_i)$, and let $s = \sum_{i=1}^{k} s_i$, $t = \sum_{i=1}^{k} t_i$.*

(i) *If $s_1 \geq 2$ then*

$$\lambda_i(G) \leq \lambda_i(A_{2k+1}) < -1, \quad \text{for } i = 1, 2, \cdots, k$$

and

$$0 < \lambda_{k+1+i}(A_{2k+1}) \leq \lambda_{n-k+i}(G), \quad \text{for } i = 1, 2, \cdots, k.$$

Consequently, $m_{-1}(G) = t-k$ and $m_0(G) = s-k$, and G has k non-trivial negative and k non-trivial positive eigenvalues (by trivial eigenvalues of G, we mean 0 and -1).

(ii) *If $s_1 = 1$ then*

$$\lambda_i(G) \leq \lambda_i(A_{2k}) < -1, \quad \text{for } i = 1, 2, \cdots, k-1$$

and

$$0 < \lambda_{k+i}(A_{2k}) \leq \lambda_{n-k+i}(G), \quad \text{for } i = 1, 2, \cdots, k.$$

Consequently, $m_{-1}(G) = t-k+1$ and $m_0(G) = s-k$, and G has $(k-1)$ non-trivial negative and k non-trivial positive eigenvalues.

In either case, G has inertia $i(G) = (t, s-k, k)$.

The next corollary is a direct consequence of Theorem 5.11.

Corollary 5.1. *Let G be a connected threshold graph of order n and with binary string $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\cdots 0^{s_k}1^{s_k}$.*

(i) *If $s_1 \geq 2$ then G does not contain non-trivial eigenvalues in the interval*

$$\left[\mu^-(A_{2k+1}), \mu^+(A_{2k+1})\right].$$

(ii) *If $s_1 = 1$ then G does not contain non-trivial eigenvalues in the interval*

$$\left[\mu^-(A_{2k}), \mu^+(A_{2k})\right].$$

Aguilar *et al.* [3] posed the following conjecture.

Conjecture 5.1. [3] *The connected antiregular graph A_n has the smallest positive eigenvalue and has the largest non-trivial negative eigenvalue among all threshold graphs of order n.*

In [2], the authors reported some partial results towards Conjecture 5.1 – more precisely, they prove that this conjecture is true for all threshold graphs of order n except for $n-2$ critical cases, and these cases are considered in a recent preprint [60]. Thus, combining the results obtained in [2, 60], we can say that Conjecture 5.1 holds.

Theorem 5.12. [2] *If $n \geq 2$ is even,*

(i) *then $\mu^+(A_n) \leq \mu^+(G)$ for every threshold graph G of order n;*

(ii) *then $\mu^-(A_n) \geq \mu^-(G)$ for every threshold graph G of order n with binary string $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\cdots 0^{s_k}1^{s_k}$ where $s_1 = 1$;*

(iii) *then $\mu^-(A_n) \geq \mu^-(G)$ for every threshold graph G of order n with binary string $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\cdots 0^{s_k}1^{s_k}$ where $s_1 \geq 2$ and $n > 2k+2$.*

Theorem 5.13. [2] *If $n \geq 3$ is odd,*

(i) *then $\mu^-(A_n) \geq \mu^-(G)$ for every threshold graph G of order n;*

(ii) *then $\mu^+(A_n) \geq \mu^+(G)$ for every threshold graph G of order n with binary string $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\cdots 0^{s_k}1^{s_k}$ where $s_1 \geq 2$;*

(iii) *then $\mu^+(A_n) \geq \mu^+(G)$ for every threshold graph G of order n with binary string $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\cdots 0^{s_k}1^{s_k}$ where $s_1 = 1$ and $n > 2k+1$.*

We recall that the Laplacian polynomial (or admittance polynomial) of a graph G is the characteristic polynomial of the Laplacian matrix of G. Denote by $L_n(\lambda)$ the Laplacian polynomial of the connected antiregular graph A_n.

Theorem 5.14. [51]

(i) *The Laplacian polynomial of the connected antiregular graph A_n is*

$$\frac{\lambda(\lambda-1)\cdots(\lambda-n)}{\lambda-\lceil n/2\rceil}.$$

(ii) *The Kel'mans polynomial (see (1.17) in [25]) of A_n is*

$$\frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}{\lambda+\lceil n/2\rceil}.$$

Acknowledgments

This work is supported by the Deanship of Scientific Research, University of Ha'il, Ha'il, Saudi Arabia, through Project# RG-20-006. The author would like to thank professors Gary Chartrand and Ping Zhang for their correspondence dealing with the topic of this paper.

References

[1] H. Abdo, S. Brandt, D. Dimitrov, The total irregularity of a graph, *Discrete Math. Theor. Comput. Sci.* **16** (2014) 201–206.
[2] C. O. Aguilar, M. Ficarra, N. Schurman, B. Sullivan, The role of the anti-regular graph in the spectral analysis of threshold graphs, *Linear Algebra Appl.* **588** (2020) 210–223.
[3] C. O. Aguilar, J. Lee, E. Piato, B. J. Schweitzer, Spectral characterizations of anti-regular graphs, *Linear Algebra Appl.* **557** (2018) 84–104.
[4] A. Ali, T. Réti, Two irregularity measures possessing high discriminatory ability, *Contrib. Math.* **1** (2020) 27–34.
[5] R. B. Bapat, On the adjacency matrix of a threshold graph, *Linear Algebra Appl.* **439** (2013) 3008–3015.
[6] C. Berge, Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* **10** (1961) 114–115.
[7] C. Berge, Les problémes de colorations en theeorie les graphes, *Publ. Inst. Statist. Univ. Paris* **9** (1960) 123–160.
[8] A. Berman, X. Zhang, A note on degree antiregular graphs, *Linear Multilinear Algebra* **47** (2000) 307–311.
[9] M. Bezhad, G. Chartrand, No graph is perfect, *Amer. Math. Monthly* **74** (1967) 962–963.
[10] S. N. Bhatt, F. R. K. Chung, F. T. Leighton, A. L. Rosenberg, Universal graphs for bounded-degree trees and planar graphs, *SIAM J. Discrete Math.* **2** (1989) 145–155.
[11] S. G. Bhragsam, S. K. Ayyaswamy, Neighbourly irregular graphs, *Indian J. Pure Appl. Math.* **53** (2004) 389–399.
[12] N. Biggs, *Algebraic Graph Theory*, 2nd Edition, Cambridge University Press, Cambridge, 1974.
[13] P. O. Boaventura-Netto, Graph irregularity: discussion, graph extensions and new proposals, *Rev. Mat. Teor. Apl.* **22** (2015) 293–310.
[14] P. O. Boaventura-Netto, P-antiregular graphs, *Electron. Notes Discrete Math.* **22** (2005) 41–48.
[15] P. Boaventura-Netto, L. de Lima, G. Caporossi, Exhaustive and metaheuristic exploration of two new structural irregularity measures, *MATCH Commun. Math. Comput. Chem.* **82** (2019) 561–580.
[16] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, London, 2008.

[17] G. Chartrand, Highly Irregular, In: R. Gera, S. Hedetniemi, C. Larson (Eds.), *Graph Theory: Favorite Conjectures and Open Problems – 1*, Springer, Berlin, 2016, pp. 1–16.
[18] G. Chartrand, K. Holbert, O. Oellermann, and H. Swart, F-degrees in graphs, *Ars Combin.* **24** (1987) 133–148.
[19] G. Chartrand, L. Lesniak, P. Zhang, *Graphs & Digraphs*, Sixth Edition, CRC Press, Boca Raton, 2016.
[20] G. A. Cheston, E. O. Hare, S. T. Hedetniemi, R. C. Laskar, Simplicial graphs, *Congr. Numer.* **67** (1988) 105–113.
[21] F. R. K. Chung, R. L. Graham, On universal graphs, *Ann. New York Acad. Sci.* **319** (1979) 136–140.
[22] F. R. K. Chung, R. L. Graham, On universal graphs for spanning trees, *J. London Math. Soc.* **27** (1983) 203–211.
[23] V. Chvátal, P. L. Hammer, *Aggregation of Inequalities in Integer Programming*, Tech. Rep. STAN-CS-75-518, Stanford University, 1975.
[24] V. Chvátal, P.L. Hammer, Aggregation of inequalities in integer programming, In: P. L. Hammer, E. L. Johnson, B. H. Korte, G. L. Nemhauser (Eds.), *Studies in Integer Programming*, North-Holland, Amsterdam, 1977, pp. 145–162.
[25] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs: Theory and Applications*, Third Edition, Johann Ambrosius Barth, Heidelberg, 1995.
[26] R. W. Deming, Independence numbers of graphs – an extension of the König-Egerváry theorem, *Discrete Math.* **27** (1979) 23–33.
[27] P. Diaconis, S. Holmes, S. Janson, Threshold graph limits and random threshold graphs, *Internet Math.* **5** (2008) 267–320.
[28] A. Diánez, A. Márquez, On Sedlacek's C-graphs, *Bull. Math. de la Soc. Sci. Math. de Roumanie* **37** (1993) 15–17.
[29] P. C. Fishburn, Packing graphs with odd and even trees, *J. Graph Theory* **7** (1983) 369–383.
[30] S. Földes, P. L. Hammer, *Split Graphs*, Proceedings of the 8th South-Eastern Conference on Combinatorics: Graph Theory and Computing, 1977, pp. 311–315.
[31] E. Ghorbani, Eigenvalue-free interval for threshold graphs, *Linear Algebra Appl.* **583** (2019) 300–305.
[32] C. Godsil, G. Royle, *Algebraic Graph Theory*, Springer, New York, 2001.
[33] V. E. Goiender, V. V. Droglav, A. B. Rosenblit, Graph potentials method and its application for chemical information processing, *J. Chem. Inf. Comput. Sci.* **21** (1981) 196–204.
[34] I. Gutman, Irregularity of molecular graphs, *Kragujevac J. Sci.* **38** (2016) 71–81.
[35] J. Hallas, M. Talanda-Fisher, P. Zhang, Dominating functions in graphs - regularity versus irregularity, *Int. J. Comput. Math. Comput. Syst. Theory* **5** (2020) 98–111.
[36] D. J. Hartfiel, J. W. Spellman, A role for doubly stochastic matrices in graph theory, *Proc. Amer. Math. Soc.* **36** (1972) 389–394.
[37] P. B. Henderson, Y. Zalcstein, A graph-theoretic characterization of the PV class of synchronizing primitives, *SIAM J. Comput.* **6** (1977) 88–108.
[38] S. Hsu, Constructing a controllable graph under edge constraints, *Syst. Control Lett.* **107** (2017) 110–116.
[39] S. Hsu, Laplacian controllability of interconnected graphs, *IEEE Trans. Control. Netw. Syst.* **7** (2020) 797–806.
[40] S. Hsu, P. Yang, Laplacian controllable graphs based on connecting two antiregular graphs, *IET Control Theory Appl.* **12** (2018) 2213–2220.
[41] D. E. Jackson, R. Entringer, Totally segregated graphs, *Congr. Numer.* **55** (1986) 159–165.
[42] W. L. Kocay, On reconstructing degree sequence, *Util. Math.* **17** (1980) 151–162.
[43] V. E. Levit, E. Mandrescu, On the independence polynomial of an antiregular graph, *Carpathian J. Math.* **28** (2012) 279–288.
[44] N. V. R. Mahadev, U. N. Peled (Eds.), *Threshold Graphs and Related Topics*, North-Holland, Amsterdam, 1995.
[45] R. Merris, Antiregular graphs are universal for trees, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **14** (2003) 1–3.
[46] R. Merris, Doubly stochastic graph matrices, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* 8 (1997) 64–71.
[47] R. Merris, Doubly stochastic graph matrices II, *Linear Multilinear Algebra* **45** (1998) 275–285.
[48] R. Merris, *Graph Theory*, Wiley, New York, 2000.
[49] W. Moon, On minimal n-universal graphs, *Glasgow Math. Soc.* **7** (1965) 32–33.
[50] S. Mukwembi, On maximally irregular graphs, *Bull. Malays. Math. Sci. Soc.* **36** (2013) 717–721.
[51] E. Munarini, Characteristics, admittance and matching polynomials of an antiregular graph, *Appl. Anal. Discrete Math.* **3** (2009) 157–176.
[52] L. Nebeský, On connected graphs containing exactly two points of the same degree, *Ćasopis Pést. Mat.* **98** (1973) 305–306.
[53] E. A. Nordhaus, J. W. Gaddum, On complementary graphs, *Amer. Math. Monthly* **63** (1956) 175–177.
[54] H. Prodinger, R. F. Tichy, Fibonacci numbers of graphs, *Fibonacci Quart.* **20** (1982) 16–21.
[55] R. Rado, Universal graphs and universal functions, *Acta Arith.* **9** (1964) 331–340.
[56] T. Réti, A. Ali, *On the Comparative Study of Nonregular Networks*, 2019 IEEE 23rd International Conference on Intelligent Engineering Systems (INES), Gödöllő, Hungary, 2019, pp. 289–293.
[57] E. Salehi, On P_3-degree of graphs, *J. Combin. Math. Combin. Comput.* **62** (2007) 45–51.
[58] J. Sedláček, On perfect and quasiperfect graphs, *Ćasopis Pést. Mat.* **100** (1975) 135–141.
[59] F. Sterboul, A characterization of the graphs in which the transversal number equals the matching number, *J. Combin. Theory Ser. B* **27** (1979) 228–229.
[60] F. Tura, A conjecture of eigenvalues of threshold graphs, *ArXiv*: 2006.03136 [math.CO], (2020).
[61] B. Wen, F. Wei, F. Li, W. Liu y. Zhu, *A Note on Antiregular Graphs*, 2010 International Conference on Computer Application and System Modeling (ICCASM 2010), Taiyuan, 2010, pp. 420–421.
[62] A. A. Zykov, *Fundamentals of Graph Theory*, (Translated from the Russian and edited by L. Boron, C. Christenson and B. Smith), BCS Associates, Moscow, 1990.