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A new exponentiated Weibull distribution's extension: copula, mathematical properties and applications

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Abstract

We introduce a new continuous distribution based on the zero truncated Poisson model which accommodates many important failure rate shapes. Some of its mathematical properties are derived. The density of the new distribution can be expressed as a combination of exponentiated Weibull densities. The method of the maximum likelihood is considered to estimate the model parameters. The importance and flexibility of the proposed distribution are also demonstrated via modeling three data sets.

Keywords: exponentiated Weibull distribution; Poisson distribution; Farlie–Gumbel–Morgenstern copulas; Clayton copula; maximum likelihood.

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1. Introduction

In this paper, we use the zero truncated Poisson (ZTP) distribution to propose a new flexible compound model. Suppose that a system has M subsystems functioning independently at a given time where M has ZTP distribution with parameter $\lambda = 1$. It is the conditional probability distribution of a Poisson-distributed random variable (RV), given that the value of the RV is not zero. The probability mass function (PMF) of M is given by

$$P_{ZTP}^{(\lambda=1)}\,(M=m) = \frac{1}{m!\mathrm{c}_{[1]}} \exp{(-1)}|_{(m=1,2,\ldots)}, \tag{1}$$

where $\mathrm{c}_{[1]} = 1 - \exp{(-1)}$. The expected value ($\mathbf{E}(M|_{\lambda=1})$) and variance ($\mathbf{V}(M|_{\lambda=1})$) based on (1) are given by

$$\mathbf{E}(M|_{\lambda=1}) = \mathrm{c}_{[1]}^{-1} \quad \text{and} \quad \mathbf{V}(M|_{\lambda=1}) = \mathrm{c}_{[1]}^{-1} - \mathrm{c}_{[1]}^{-2}, \tag{2}$$

respectively. Suppose that the failure time of each subsystem has the Burr type X exponentiated Weibull distribution (BXEW), which was recently proposed in [23]. The cumulative distribution function (CDF) of the BXEW is given by

$$\mathbf{H}_{\underline{\mathbf{V}}}(w)|_{\underline{\mathbf{V}}=\theta,\alpha,\beta} = \left[1 - \exp\left(-\left\{[\varrho_\beta\,(w)]^{-\alpha} - 1\right\}^{-2}\right)\right]^{\theta}, \tag{3}$$

where

$$\varrho_\beta\,(w) = 1 - \exp\left(-w^{\beta}\right),$$

and $\theta, \alpha, \beta > 0$ are the shape parameters. Let Y_i denote the failure time of the i^{th} subsystem and let

$$W = \min\{Y_1, Y_2, \cdots, Y_M\}.$$

Then, the conditional CDF of W given M is

$$F\,(w|M) = 1 - \Pr{(W > w|M)} = 1 - \left[1 - \mathbf{H}_{\underline{\mathbf{V}}}(w)\right]^{M}. \tag{4}$$

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Therefore, the unconditional CDF of the quasi Poisson Burr X exponentiated Weibull (QP-BXEW) probability density function (PDF) (for details, see [7]) can be expressed as

$$F_{\underline{\mathbf{V}}}(w) = \frac{1}{c_{[1]}}\left[1-\exp\left(-\left\{1-\exp\left[-\phi_{\alpha,\beta}(w)\right]\right\}^{\theta}\right)\right], \tag{5}$$

where

$$\phi_{\alpha,\beta}(w) = \left[\varrho_{\beta}^{-\alpha}(w)-1\right]^{-2}.$$

The corresponding PDF can be expressed as

$$f_{\underline{\mathbf{V}}}(w) = 2\theta\alpha\beta c_{[1]}^{-1}\frac{w^{\beta-1}\,\varrho_{\beta}^{2\alpha-1}(w)\exp\left[-w^{\beta}-\phi_{\alpha,\beta}(w)\right]}{\left[1-\varrho_{\beta}^{\alpha}(w)\right]^{3}\left\{1-\exp\left[-\phi_{\alpha,\beta}(w)\right]\right\}^{1-\theta}}\exp\left(-\left\{1-\exp\left[-\phi_{\alpha,\beta}(w)\right]\right\}^{\theta}\right). \tag{6}$$

Some new useful Weibull extensions can be found in [3,4,6,8–11,15–17,25,26,32,35,36,42–45,47–49]. Figure 1 gives PDF and hazard rate function (HRF) plots for the QP-BXEW model. From Figure 1 (left panel), we note that the QP-BXEW model may be suitable for modeling "asymmetric left skewed", "symmetric", "asymmetric right skewed" and "asymmetric unimodal" data sets. Also, from Figure 1 (right panel), we note that the HRF of the QP-BXEW can be "monotonically increasing" ($\theta = 2.5, \alpha = 1, \beta = 0.45$) or "bathtub" ($\theta = 1, \alpha = 0.45, \beta = 1.25$) or "monotonically decreasing" ($\theta = 0.1, \alpha = 0.1, \beta = 0.75$) or "upside down" ($\theta = 2.5, \alpha = 1, \beta = 0.25$) or "J shaped" ($\theta = 3, \alpha = 0.4, \beta = 0.3$).

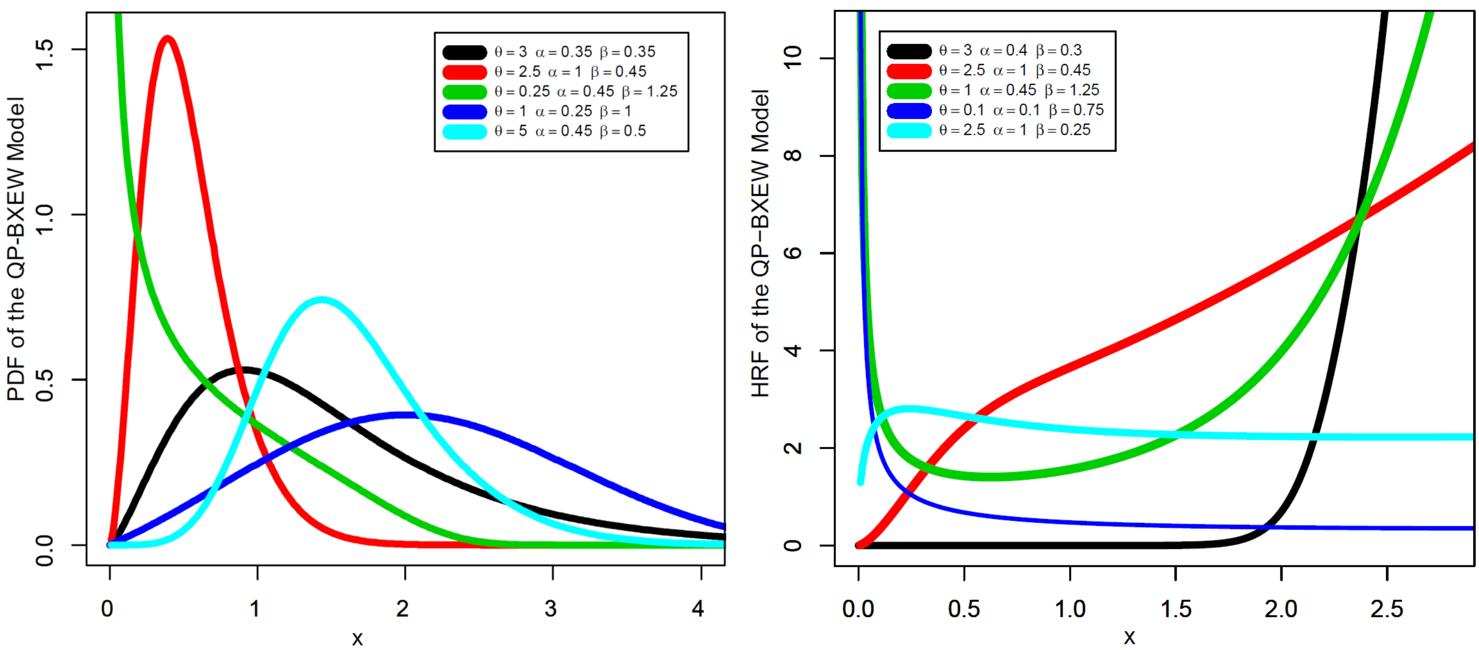

Figure 1: PDF and HRF plots for the QP-BXEW model.

2. Copula

In this Section, we derive some new bivariate type QP-BXEW (Biv-QP-BXEW) models using Farlie Gumbel Morgenstern (FGM) copula (see [12–14,21,22,34]), modified FGM copula (see [38]), Clayton copula and Renyi's entropy (see [37]). The Multivariate QP-BXEW (MvQP-BXEW) type is also presented. However, future works may be allocated to study these new models. First, we consider the joint CDF of the FGM family, $\mathbb{H}_{\boldsymbol{\Omega}}(\varsigma,\tau) = \varsigma\tau\left(1+\boldsymbol{\Omega}\varsigma^{*}\tau^{*}\right)|_{\varsigma^{*}=1-\varsigma}$, where $\varsigma = F_1$, $\tau = F_2$, $\boldsymbol{\Omega} \in (-1,1)$ is a dependence parameter and for every $\varsigma,\tau \in (0,1)$, $\mathbb{H}(\varsigma,0) = \mathbb{H}(0,\tau) = 0$ which is "grounded minimum" and $\mathbb{H}(\varsigma,1) = \varsigma$ and $\mathbb{H}(1,\tau) = \tau$ which is "grounded maximum", $\mathbb{H}(\varsigma_1,\tau_1) + \mathbb{H}(\varsigma_2,\tau_2) - \mathbb{H}(\varsigma_1,\tau_2) - \mathbb{H}(\varsigma_2,\tau_1) \geq 0$.

2.1 FGM copula

A copula is continuous in ς and τ; actually, it satisfies the stronger Lipschitz condition, where

$$|\mathbb{H}(\varsigma_2,\tau_2) - \mathbb{H}(\varsigma_1,\tau_1)| \leq |\varsigma_2-\varsigma_1| + |\tau_2-\tau_1|.$$

For $0 \leq \varsigma_1 \leq \varsigma_2 \leq 1$ and $0 \leq \tau_1 \leq \tau_2 \leq 1$, we have

$$\Pr\left(\varsigma_1 \leq \varsigma \leq \varsigma_2, \tau_1 \leq W \leq \tau_2\right) = \mathbb{H}\left(\varsigma_1, \tau_1\right) + \mathbb{H}\left(\varsigma_2, \tau_2\right) - \mathbb{H}\left(\varsigma_1, \tau_2\right) - \mathbb{H}\left(\varsigma_2, \tau_1\right) \geq 0.$$

Then, by setting

$$\varsigma^* = 1 - F_{\underline{\mathbf{V}}_1}(x_1)|_{[\varsigma^*=(1-\varsigma)\in(0,1)]} \quad \text{and} \quad \tau^* = 1 - F_{\underline{\mathbf{V}}_2}(x_2)|_{[\tau^*=(1-\tau)\in(0,1)]},$$

we can easily get the joint CDF of the FGM family. The joint PDF can then derived from

$$c_{\mathbf{\Omega}}(\varsigma, \tau) = 1 + \mathbf{\Omega}\varsigma^{\cdot}\tau^{\cdot}|_{(\varsigma^{\cdot}=1-2\varsigma \text{ and } \tau^{\cdot}=1-2\tau)} \quad \text{or from} \quad f(x_1, x_2) = \mathbb{H}\left(F_1, F_2\right) f_1 f_2.$$

2.2 Modified FGM copula

The modified FGM copula is defined as $\mathbb{H}_{\mathbf{\Omega}}(\varsigma,\tau) = \varsigma\tau\left[1 + \mathbf{\Omega}\psi\left(\varsigma\right)\mathbf{O}\left(\tau\right)\right]|_{\mathbf{\Omega}\in(-1,1)}$ or $\mathbb{H}_{\mathbf{\Omega}}(\varsigma,\tau) = \varsigma\tau + \mathbf{\Omega}\dot{\psi}_{\varsigma}\dot{\mathbf{O}}_{\tau}|_{\mathbf{\Omega}\in(-1,1)}$, where $\dot{\psi}_{\varsigma} = \varsigma\psi\left(\varsigma\right)$ and $\dot{\mathbf{O}}_{\tau} = \tau\mathbf{O}\left(\tau\right)$; $\psi\left(\varsigma\right)$ and $\mathbf{O}\left(\tau\right)$ are two continuous functions on $(0,1)$ with $\psi\left(0\right) = \psi\left(1\right) = \mathbf{O}\left(0\right) = \mathbf{O}\left(1\right) = 0$. Let

$$c_1 = \inf\left\{\dot{\psi}_{\varsigma} : \frac{\partial}{\partial\varsigma}\dot{\psi}_{\varsigma}|_{\sigma_1}\right\} < 0,\ c_2 = \sup\left\{\dot{\psi}_{\varsigma} : \frac{\partial}{\partial\varsigma}\dot{\psi}_{\varsigma}|_{\sigma_1}\right\} < 0,\ d_1 = \inf\left\{\dot{\mathbf{O}}_{\tau} : \frac{\partial}{\partial\tau}\dot{\mathbf{O}}_{\tau}|_{\sigma_2}\right\} > 0 \text{ and } d_2 = \sup\left\{\dot{\mathbf{O}}_{\tau} : \frac{\partial}{\partial\tau}\dot{\mathbf{O}}_{\tau}|_{\sigma_2}\right\} > 0.$$

Then, $1 \leq \min\left(c_1c_2, d_1d_2\right) < \infty$, where

$$\varsigma\frac{\partial}{\partial\varsigma}\psi\left(\varsigma\right) = \frac{\partial}{\partial\varsigma}\dot{\psi}_{\varsigma} - \psi\left(\varsigma\right), \quad \sigma_1 = \left\{\varsigma : \varsigma \in (0,1)\,|_{\frac{\partial}{\partial\varsigma}\dot{\psi}_{\varsigma}\ \text{exists}}\right\} \quad \text{and} \quad \sigma_2 = \left\{\tau : \tau \in (0,1)\,|_{\frac{\partial}{\partial\tau}\dot{\mathbf{O}}_{\tau}\ \text{exists}}\right\}.$$

2.2.1 Type-I Biv-QP-BXEW-FGM model

Consider the functional form for both $\psi\left(\varsigma\right)$ and $\mathbf{O}\left(\tau\right)$. Then, the Biv-QP-BXEW-FGM (Type-I) can be derived from $\mathbb{H}_{\mathbf{\Omega}}(\varsigma,\tau) = \varsigma\tau + \mathbf{\Omega}\dot{\psi}_{\varsigma}\dot{\mathbf{O}}_{\tau}|_{\mathbf{\Omega}\in(-1,1)}$ where $\dot{\psi}_{\varsigma} = \varsigma\left[1 - F_{\underline{\mathbf{V}}_1}(\varsigma)\right]$ and $\dot{\mathbf{O}}_{\tau} = \tau\left[1 - F_{\underline{\mathbf{V}}_2}(\tau)\right]$.

2.2.2 Type-II Biv-QP-BXEW-FGM model

Let $\psi\left(\varsigma\right)$ and $\mathbf{O}\left(\tau\right)$ be two functional form satisfying all the conditions stated earlier where $\psi\left(\varsigma\right)^*|_{(\mathbf{\Omega}_1>0)} = \varsigma^{\mathbf{\Omega}_1}\left(1-\varsigma\right)^{1-\mathbf{\Omega}_1}$ and $\mathbf{O}\left(\tau\right)^*|_{(\mathbf{\Omega}_2>0)} = \tau^{\mathbf{\Omega}_2}\left(1-\tau\right)^{1-\mathbf{\Omega}_2}$. Then, the corresponding Biv-QP-BXEW-FGM (Type-**II**) can be derived from

$$\mathbb{H}_{\mathbf{\Omega},\mathbf{\Omega}_1,\mathbf{\Omega}_2}(\varsigma,\tau) = \varsigma\tau\left[1 + \mathbf{\Omega}\psi\left(\varsigma\right)^*\ \mathbf{O}\left(\tau\right)^*\right].$$

2.2.3 Type-III Biv-QP-BXEW-FGM model

Let $\widetilde{\psi^*\left(\varsigma\right)} = \varsigma\left[\log\left(1+\varsigma^*\right)\right]$ and $\widetilde{\mathbf{O}^*\left(\tau\right)} = \tau\left[\log\left(1+\tau^*\right)\right]$ for all $\psi\left(\varsigma\right)$ and $\mathbf{O}\left(\tau\right)$, which satisfy all the conditions stated earlier. In this case, one can also derive a closed form expression for the associated CDF of the Biv-QP-BXEW-FGM (Type-**III**) from $\mathbb{H}_{\mathbf{\Omega}}(\varsigma,\tau) = \varsigma\tau\left(1 + \mathbf{\Omega}\widetilde{\psi^*\left(\varsigma\right)}\ \widetilde{\mathbf{O}^*\left(\tau\right)}\right)$.

2.2.4 Type-IV Biv-QP-BXEW-FGM model

The CDF of the Biv-QP-BXEW-FGM (Type-**IV**) model can be derived from $\mathbb{H}(\varsigma,\tau) = \varsigma F^{-1}_{\underline{\mathbf{V}}_1}(\tau) + \tau F^{-1}_{\underline{\mathbf{V}}_1}(\varsigma) - F^{-1}_{\underline{\mathbf{V}}_1}(\varsigma)F^{-1}_{\underline{\mathbf{V}}_2}(\tau)$ where $F^{-1}_{\underline{\mathbf{V}}_1}(\varsigma)$ and $F^{-1}_{\underline{\mathbf{V}}_2}(\tau)$ can be derived using (13).

2.3 Clayton copula

The Clayton copula can be considered as $\mathbb{H}(\tau_1,\tau_2) = \left[(1/\tau_1)^{\mathbf{\Omega}} + (1/\tau_2)^{\mathbf{\Omega}} - 1\right]^{-\mathbf{\Omega}^{-1}}|_{\mathbf{\Omega}\in(0,\infty)}$. Setting $\tau_1 = F_{\underline{\mathbf{V}}_1}(t)$ and $\tau_2 = F_{\underline{\mathbf{V}}_1}(x)$. Then, the Biv-QP-BXEW type can be derived from $\mathbb{H}(\tau_1,\tau_2) = \mathbb{H}(F_{\mathbf{O}_1}(t), F_{\mathbf{O}_2}(x))$. Similarly, the MvQP-BXEW (m-dimensional extension) from the above can be derived from $\mathbb{H}(\tau_{\hbar}) = \left(\sum_{\hbar=1}^{m}\tau_{\hbar}^{-\mathbf{\Omega}} + 1 - m\right)^{-\mathbf{\Omega}^{-1}}$.

2.4 Renyi's entropy copula

Using a theorem of Pougaza and Djafari [37], $\mathbb{H}(\varsigma,\tau) = x_2\varsigma + x_1\tau - x_1x_2$. Then, the associated Biv-QP-BXEW is $\mathbb{H}(\varsigma,\tau) = \mathbb{H}(F_{\underline{\mathbf{V}}_1}(x_1), F_{\underline{\mathbf{V}}_2}(x_2))$.

3. Mathematical properties

3.1 Expansions

Using the power series

$$\exp\left(-\frac{\varsigma_1}{\varsigma_2}\right)=\sum_{k_3=0}^{\infty}\frac{1}{k_3!}\left(-\frac{\varsigma_1}{\varsigma_2}\right)^{k_3},$$

the PDF in (6) can be written as

$$f_{\underline{\mathbf{V}}}(w)=\sum_{k_3=0}^{\infty}\frac{2\theta\alpha\beta c_{[1]}^{-1}w^{\beta-1}\,(-1)^{k_3}\left[1-\exp\left(-\left\{[\varrho_\beta(w)]^{-\alpha}-1\right\}^{-2}\right)\right]^{\theta(1+k_3)-1}}{[\varrho_\beta(w)]^{-2\alpha+1}\,k_3!\left[1-\varrho_\beta^{\alpha}(w)\right]^3\exp\left(w^\beta+\left\{[\varrho_\beta(w)]^{-\alpha}-1\right\}^{-2}\right)}. \tag{7}$$

Consider the following power series

$$\left(1-\frac{\varsigma_1}{\varsigma_2}\right)^{\varsigma_3-1}=\sum_{k_4=0}^{\infty}\frac{\Gamma(\varsigma_3)}{k_4!\,\Gamma(\varsigma_3-k_4)}\left(-\frac{\varsigma_1}{\varsigma_2}\right)^{k_4}\Big|_{\left(\left|\frac{\varsigma_1}{\varsigma_2}\right|<1\text{ and }\varsigma_3>0\right)}. \tag{8}$$

By using (8) in (7), we have

$$f_{\underline{\mathbf{V}}}(w)=\theta\alpha\beta c_{[1]}^{-1}w^{\beta-1}\sum_{k_3,k_4=0}^{\infty}\frac{\lambda^{1+k_3}(-1)^{k_3+k_4}\,\{1-[\varrho_\beta(w)]^{\alpha}\}^{-3}\,\Gamma(\theta(1+k_3))\,2\,[\varrho_\beta(w)]^{2\alpha-1}}{k_4!\,\Gamma(\theta(1+k_3)-k_4)\exp\left(w^\beta\right)\underbrace{\exp\left[(1+k_4)\,\phi_{\alpha,\beta}(w)\right]}_{A(w)}}. \tag{9}$$

Applying the power series again to $A(w)$, the PDF in (9) then becomes

$$f_{\underline{\mathbf{V}}}(w)=\sum_{k_3,k_4,k_1=0}^{\infty}\frac{2\theta\alpha\beta c_{[1]}^{-1}(-1)^{k_3+k_4+k_1}\,(1+k_4)^{k_1}\,\Gamma(\theta(1+k_3))\,w^{\beta-1}\left[\varrho_\beta^{\alpha}(w)\right]^{2\alpha(k_1+1)-1}}{k_4!\,k_1!\Gamma(\theta(1+k_3)-k_4)\exp\left(w^\beta\right)\underbrace{\left[1-\varrho_\beta^{\alpha}(w)\right]^{3+2k_1}}_{B(w)}}. \tag{10}$$

Consider the following series expansion

$$\left(1-\frac{\varsigma_1}{\varsigma_2}\right)^{-\varsigma_3}=\sum_{k_2=0}^{\infty}\frac{\Gamma(\varsigma_3+k_2)}{k_2!\Gamma(\varsigma_3)}\left(\frac{\varsigma_1}{\varsigma_2}\right)^{k_2}\Big|_{\left(\left|\frac{\varsigma_1}{\varsigma_2}\right|<1,\ \varsigma_3>0\right)}. \tag{11}$$

Using (11) in (10) for the term $B(w)$, Equation (10) becomes

$$f_{\underline{\mathbf{V}}}(w)=\sum_{k_1,k_2=0}^{\infty}\mathbf{C}_{k_1,k_2}\,g_{\alpha^\star,\beta}(w)|_{(\alpha^\star=\alpha(2k_1+k_2+2))}, \tag{12}$$

where

$$\mathbf{C}_{k_1,k_2}=\frac{2\theta c_{[1]}^{-1}(-1)^{k_1}\,\Gamma(3+2k_1+k_2)}{k_1!k_2!\Gamma(2k_1+3)(2k_1+k_2+2)}\sum_{k_3,k_4=0}^{\infty}\frac{(-1)^{k_3+k_4}\,\Gamma(\theta(1+k_3))(1+k_4)^{k_1}}{k_4!\,\Gamma(\theta(1+k_3)-k_4)},$$

and $g_{\alpha^\star,\beta}(w)=\alpha^\star\beta w^{\beta-1}\exp\left(-w^\beta\right)\varrho_\beta^{\alpha^\star-1}(w)$ which is the PDF of the EW model with parameters $\alpha^\star$ and β. Similarly, the CDF of the QP-BXEW model is given as

$$F_{\underline{\mathbf{V}}}(w)=\sum_{k_1,k_2=0}^{\infty}\mathbf{C}_{k_1,k_2}\,G_{\alpha^\star,\beta}(w)|_{(\alpha^\star=\alpha(2k_1+k_2+2))}, \tag{13}$$

where $G_{\alpha^\star,\beta}(w)=\varrho_\beta^{\alpha^\star}(w)$ is the CDF of the EW model with parameters $\alpha^\star$ and β.

3.2 Quantile function (QF)

The QF of W, where $W\sim$QP-BXEW($\underline{\mathbf{V}}$), is obtained by inverting (5) as

$$Q(u)=\left(-\ln\left\{1-\left[1+\left(-\ln\left\{1-\left[-\ln\left(1-uc_{[1]}\right)\right]^{\frac{1}{\theta}}\right\}\right)^{\frac{1}{2}}\right]^{\frac{1}{\alpha}}\right\}\right)^{\frac{1}{\beta}}\Big|_{(0\le u\le 1)}.$$

Simulating the QP-BXEW RV is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the RV $W=Q(U)$ follows (6).

3.3 Ordinary moments

The n^{th} ordinary moment of W, say μ'_n, can be obtained from (12) as

$$\mu'_n = \mathbf{E}\left(W^n\right) = \Gamma\left(1+\frac{n}{\beta}\right) \sum_{k_1,k_2,k_3=0}^{\infty} \mathbf{C}_{k_1,k_2,k_3}^{(n,\alpha^\star)} |_{(n>-\beta)}, \quad (14)$$

where

$$\mathbf{C}_{k_1,k_2,k_3}^{(n,\alpha^\star)} = \mathbf{C}_{k_1,k_2} \; \mathbf{C}_{k_3}^{(n,\alpha^\star)} \quad \text{and} \quad \mathbf{C}_{\varsigma_1}^{\{n,\varsigma_2\}} = \varsigma_2 \left(-1\right)^{\varsigma_1} \left(1+\varsigma_1\right)^{-\left(1+\frac{n}{\beta}\right)} \binom{\varsigma_2 - 1}{\varsigma_1}.$$

Setting $n = 1, 2, 3$ and 4 in (14), we get the mean of W (μ'), $\mathbf{E}\left(W^2\right)$, $\mathbf{E}\left(W^3\right)$ and $\mathbf{E}\left(W^4\right)$ as

$$\mu' = \mathbf{E}\left(W\right) = \Gamma\left(1+\frac{1}{\beta}\right) \sum_{k_1,k_2,k_3=0}^{\infty} \mathbf{C}_{k_1,k_2,k_3}^{(1,\alpha^\star)} |_{(1>-\beta)}\,, \quad \mu'_2 = \mathbf{E}\left(W^2\right) = \Gamma\left(1+\frac{2}{\beta}\right) \sum_{k_1,k_2,k_3=0}^{\infty} \mathbf{C}_{k_1,k_2,k_3}^{(2,\alpha^\star)} |_{(2>-\beta)}\,,$$

$$\mu'_3 = \mathbf{E}\left(W^3\right) = \Gamma\left(1+\frac{3}{\beta}\right) \sum_{k_1,k_2,k_3=0}^{\infty} \mathbf{C}_{k_1,k_2,k_3}^{(3,\alpha^\star)} |_{(3>-\beta)} \quad \text{and} \quad \mu'_4 = \mathbf{E}\left(W^4\right) = \Gamma\left(1+\frac{4}{\beta}\right) \sum_{k_1,k_2,k_3=0}^{\infty} \mathbf{C}_{k_1,k_2,k_3}^{(4,\alpha^\star)} |_{(4>-\beta)}\,.$$

The last four moments can be used to get the variance V(W), skewness (β_1) and kurtosis (β_2).

Table 1: Numerical results for μ'_1, V(W), $\beta_1$and β_2 for the QP-BXEW model.

θ	α	β	μ'_1	V(W)	β_1	β_2
0.00001	1.5	1.5	2.7581×10^{-5}	1.51583×10^{-5}	192.3671	42324.15
0.0001			0.00027513	0.000150788	60.87517	4249.966
0.001			0.00275128	0.001501061	19.18972	423.7988
0.01			0.02746264	0.01432537	5.879609	41.41344
0.1			0.245922	0.08515707	1.340875	3.933828
0.5			0.677138	0.07414028	−0.0430646	**2.333759**
1			0.850623	0.04403703	−0.2644226	2.799186
5			1.103883	0.01125588	−0.0295711	3.066298
20			1.222271	0.00469636	0.2763029	3.182548
50			1.277130	0.00302655	0.4250359	3.342197
150			1.329053	0.00197290	0.5587528	3.551701
500			1.374743	0.00134595	0.6668214	3.746486
1000			1.397313	0.00111286	0.7150307	3.882475
10	0.1	0.5	4.469×10^{-7}	4.8521×10^{-9}	155.85240	24290.96
	0.5		0.2269298	0.0087136	1.0833400	4.880516
	1		0.9000780	0.0576436	0.7061349	3.815006
	5		5.5885400	0.5313196	0.3874134	3.292511
	10		7.1586870	12.495730	**−1.458867**	3.298355
	15		0.4151138	3.8239210	4.5046250	21.32049
	20		0.0021061	0.0205282	68.056120	4635.271
1.5	1.5	0.00001	0.0001469	0.0007344	**245.9918**	**68077.65**
		0.0001	0.0014688	0.0073419	77.77563	6806.865
		0.001	0.0146826	0.0731854	24.55138	679.803
		0.01	0.1459747	0.7069398	7.634374	67.27253
		0.1	1.1179720	3.7691470	2.389238	8.465432
		0.5	0.8956323	0.2158466	0.7142713	3.408528
		1	0.9126955	0.0626191	0.0041641	2.774495
		1.5	0.9325487	0.0307822	−0.2728509	2.986136

3.4 Incomplete moments

The n^{th} incomplete moment of W is defined by $\mathbf{I}_n(\varsigma_2) = \int_{-\infty}^{\varsigma_2} w^n f(w)dw$. Then, we can write

$$\mathbf{I}_n(t) = \Gamma_{\left[\left(\frac{1}{t}\right)^{-\beta}\right]}\left(1+\frac{n}{\beta}\right) \sum_{k_1,k_2,k_3=0}^{\infty} \mathbf{C}_{k_1,k_2,k_3}^{(n,\alpha^\star)} |_{(n>-\beta)},$$

where

$$\Gamma_{[\varsigma_2]}\left(\varsigma_1\right) = \gamma\left(\varsigma_1, \varsigma_2\right) = \int_0^{\varsigma_2} t^{\varsigma_1 - 1} \exp\left(-t\right) dt = \sum_{m=0}^{\infty} \frac{\left(-1\right)^m \varsigma_2^{\varsigma_1+m}}{m!\left(\varsigma_1+m\right)},$$

3.5 The moment generating function (MGF)

The MGF of W, say $M_W(t) = \mathbf{E}(\exp(tW))$, is obtained from (12) as

$$M_W(t) = \Gamma\left(1+\frac{n}{\beta}\right) \sum_{k_1,k_2,k_3,n=0}^{\infty} \mathbf{C}_{k_1,k_2,k_3,n}^{(n,\alpha^\star)}|_{(n>-\beta)}, \quad \text{where} \quad \mathbf{C}_{k_1,k_2,k_3,n}^{(n,\alpha^\star)} = \frac{t^n}{n!}\mathbf{C}_{k_1,k_2,k_3}^{(n,\alpha^\star)}.$$

4. Numerical analysis for μ_1', V(W), β_1 and β_2

The effects of θ, α and β of the standard EW model on the μ_1', V(W), β_1 and β_2 are calculated and listed in Table 1. From Table 1, we note that all parameters effect the μ_1', V(W), β_1 and β_2. The skewness of the QP-BXEW model can rang from -1.458867 to 245.9918. The kurtosis of the QP-BXEW model can range from 2.333759 to 68077.65.

5. Parameter estimation

Let $w_1, \ldots, w_n$ be a random sample (RS) from QP-BXEW model with parameters θ, α and β. Let $\underline{\mathbf{V}}$ be the 3×1 parameter vector. For determining the maximum likelihood estimators (MLEs) of $\mathbf{\Psi}$, we have the log-likelihood function

$$\begin{aligned} \ell \;&=\; \ell(\mathbf{\Psi}) = n\log 2 + n\log\theta + n\log\alpha - n\log \mathrm{c}_{[1]} + (\beta-1)\sum\nolimits_{i=1}^{n} + (2\alpha-1)\sum\nolimits_{i=1}^{n}\log\varrho_\beta(w_i) + n\log\beta \\ &-3\sum\nolimits_{i=1}^{n}\log\left[1-\varrho_\beta^\alpha(w_i)\right] - \sum\nolimits_{i=1}^{n}\left(w_i^\beta + p_i\right) + (\theta-1)\sum\nolimits_{i=1}^{n}\log\left[1-\exp(-p_i)\right] - \sum\nolimits_{i=1}^{n}\left[1-\exp(-p_i)\right]^\theta, \end{aligned}$$

where $p_i = \left[\frac{\varrho_\beta^\alpha(w_i)}{1-\varrho_\beta^\alpha(w_i)}\right]^2$. The components of the score vector can be derived easily. The above equation can be maximized directly by using R (the optim function).

Table 2: MLEs and SEs for failure times data.

Distribution	Estimates				
QP-BXEW(θ,α,β)	**0.7256**	**4.496**	**0.529**		
	(0.322)	**(1.528)**	**(0.082)**		
MOE-W(γ,β,α)	488.90	0.28	1261.97		
	(189.34)	(0.01)	(351.07)		
Ga-W(α,β,γ)	2.38	0.85	3.53		
	(0.38)	(0.0005)	(0.67)		
Kw-W(α,β,a,b)	14.43	0.20	34.66	81.85	
	(27.10)	(0.04)	(17.53)	(52.01)	
W-Fr(α,β,a,b)	630.94	0.30	416.10	1.17	
	(697.94)	(0.032)	(232.36)	(0.36)	
B-W(α,β,a,b)	1.36	0.30	34.18	11.50	
	(1.002)	(0.06)	(14.838)	(6.73)	
TM-W$(\alpha,\beta,\gamma,\lambda)$	0.27	1	4.6×10^{-6}	0.47	
	(0.014)	(5.2×10^{-5})	(1.9×10^{-4})	(0.17)	
TExG-W(α,β,λ,b)	4.26	0.15	0.098	1173.33	
	(33.40)	(0.017)	(0.61)	(9.79)	
KwT-W$(\alpha,\beta,\lambda,a,b)$	27.7912	0.178	0.445	29.53	168.1
	(33.40)	(0.02)	(0.61)	(9.79)	(129.2)
MB-W(α,β,a,b,c)	10.15	0.16	57.42	19.39	2.004
	(18.70)	(0.02)	(14.06)	(10.02)	(0.66)
Mc-W(α,β,a,b,c)	1.94	0.31	17.69	33.64	16.72
	(1.01)	(0.05)	(6.2)	(19.99)	(9.72)

6. Applications

In this section, we will analyze three real data sets to show the flexibility of the QP-BXEW model. However many other useful real data sets can be found in [1, 18–20, 27–31]. Other real data sets for regression modeling analysis can be found in [5, 6, 24, 33, 39–41, 46]. In order to compare the fits of the QP-BXEW model with other Weibull extensions, we consider the the Anderson-Darling ($\mathtt{C}_1$) and Cramér-Von Mises ($\mathtt{C}_2$) statistics which are used to determine how closely a specific CDF (or PDF) fits the empirical distribution of a given data set, where $z_i = F(y_{k_1})$ and the y_{k_1}'s values are the ordered observations. The MLEs and the standard errors (SEs) are given in Tables 2, 4 and 6. The statistics $\mathtt{C}_1$ and $\mathtt{C}_2$ are listed in Tables 3, 5 and 7.

The first real data set that we consider is the failure times of 84 aircraft windshield. This data is available and recently analyzed in [23, 29]. Based on Table 3, we conclude that the QP-BXEW model provides adequate fits as compared to other Weibull models with the smallest values for C_1 and C_2. The QP-BXEW proposed model is much better than the Gamma-Weibull (GaW), transmuted modified-Weibull (TMW), Beta-Weibull (BW), transmuted exponentiated generalized Weibull (TExGW), Weibull-Fréchet (WFr), Kumaraswamy-Weibull (KwW), Marshall Olkin extended-Weibull (MOEW), Kumaraswamy transmuted-Weibull (KwTW), Modified beta-Weibull (MBW) and Mcdonald-Weibull (McW) distributions. The PDFs of the competitive models are given in [23, 29].

Table 3: C_1 and C_2 for the failure times data set.

Distribution	C_1	C_2
QP-BXEW	**0.723**	**0.076**
MOE-W	4.448	0.399
Ga-W	1.949	0.255
Kw-W	1.506	0.185
W-Fr	1.957	0.254
B-W	3.220	0.465
TM-W	11.205	0.806
KwT-W	1.363	0.164
MB-W	3.266	0.472
Mc-W	1.591	0.199
TExG-W	6.233	1.008

The second data set that we consider is the remission times of bladder cancer patients. This data is available and recently analyzed in [23, 29]. From Table 5, we conclude that the QP-BXEW model is much better than the transmuted additive Weibull distribution (TAW) and the exponentiated transmuted generalized Rayleigh (ETGR) distributions. The PDFs of the competitive models are given in [23, 29].

Table 4: MLEs and SEs for the remission times data.

Distribution	Estimates				
W(α,β)	9.56	1.05			
	(0.85)	(0.07)			
QP-BXEW(θ,α,β)	**4.499**	**1.421**	**0.151**		
	(0.211)	**(0.043)**	**(0.111)**		
ET-GR$(\alpha,\beta,\delta,\lambda)$	7.38	0.05	0.05	0.118	
	(5.39)	(3.9×10^{-3})	(0.04)	(0.26)	
TM-W$(\alpha,\beta,\gamma,\lambda)$	0.121	0.90	0.0002	0.25	
	(0.024)	(0.63)	(0.011)	(0.41)	
MB-W(α,β,a,b,c)	0.150	0.16	57.42	19.39	2.004
	(22.44)	(0.044)	(37.32)	(13.49)	(0.79)
T-AW$(\alpha,\beta,\gamma,\theta,\lambda)$	0.114	0.97	3.09×10^{-5}	1.007	-0.16
	(0.03)	(0.13)	(6.1×10^{-3})	(0.035)	(0.28)

Table 5: C_1 and C_2 for the remission times data set.

Distribution	C_1	C_2
QP-BXEW	**0.126**	**0.021**
W	0.663	0.106
TM-W	0.760	0.125
MB-W	0.720	0.107
T-AW	0.703	0.113
ET-GR	2.361	0.398

The third real data set that we consider is the survival times of guinea pigs. This data is available and recently analyzed in [23, 29]. Based on Table 7, we conclude that the QP-BXEW model is much better than the Burr X Exponentiated Weibull (BXEW), Weibull-Weibull (WW), gamma exponentiated-exponential (GaEE) and exponential exponential-geometric (EEGc) models. The PDFs of the competitive models are given in [23, 29]. The total time test (TTT) plots are given in Figure 2. From Figure 2, we note that the HRF for the three data sets are "increasing", "upside down" and "increasing". The histograms are given in Figure 3 and the estimated CDFs are given in Figure 4. The P-P plots are given in Figure 5. From Figures 3, 4 and 5, we conclude that the QP-BXEW has a suitable fit to the used data sets.

Table 6: MLEs and SEs for survival times data.

Distribution	Estimates		
QP-BXEW(θ, α, β)	**0.456**	**4.075**	**0.153**
	(0.391)	**(1.502)**	**(1.115)**
BX-EW(θ, α, β)	3.18	5.54	0.17
	(2.12)	(2.44)	(0.02)
W-W (β, γ, λ)	2.660	0.690	0.030
	(0.713)	(0.171)	(0.02)
OW-W(β, γ, λ)	11.16	0.09	0.46
	(4.545)	(0.04)	(0.08)
Ga-EE$(\lambda, \alpha, \theta)$	2.114	2.6	0.008
	(1.33)	(0.56)	(0.005)
EE-Gc(α, θ, p)	2.59	0.0004	0.99
	(0.48)	(0.004)	(0.104)

Table 7: C_1 and C_2 for the survival times data set.

Distribution	C_1	C_2
QP-BXEW	**0.434**	**0.081**
BX-EW	0.567	0.091
W-W	0.781	0.143
OW-W	2.476	0.449
Ga-EE	1.721	0.315
EE-Gc	0.579	0.105

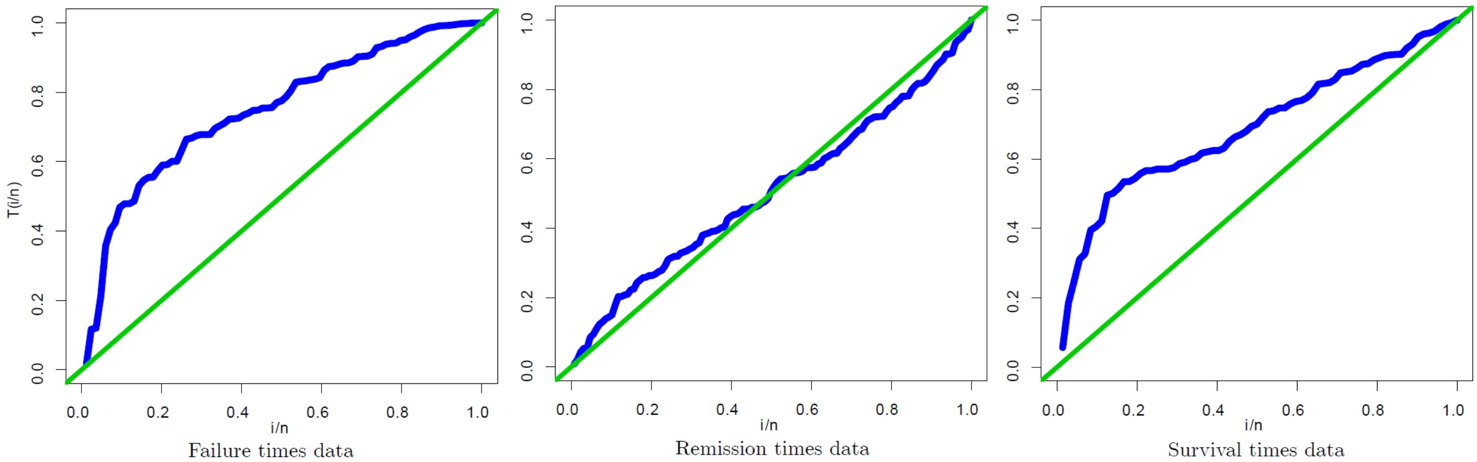

Figure 2: TTT plots.

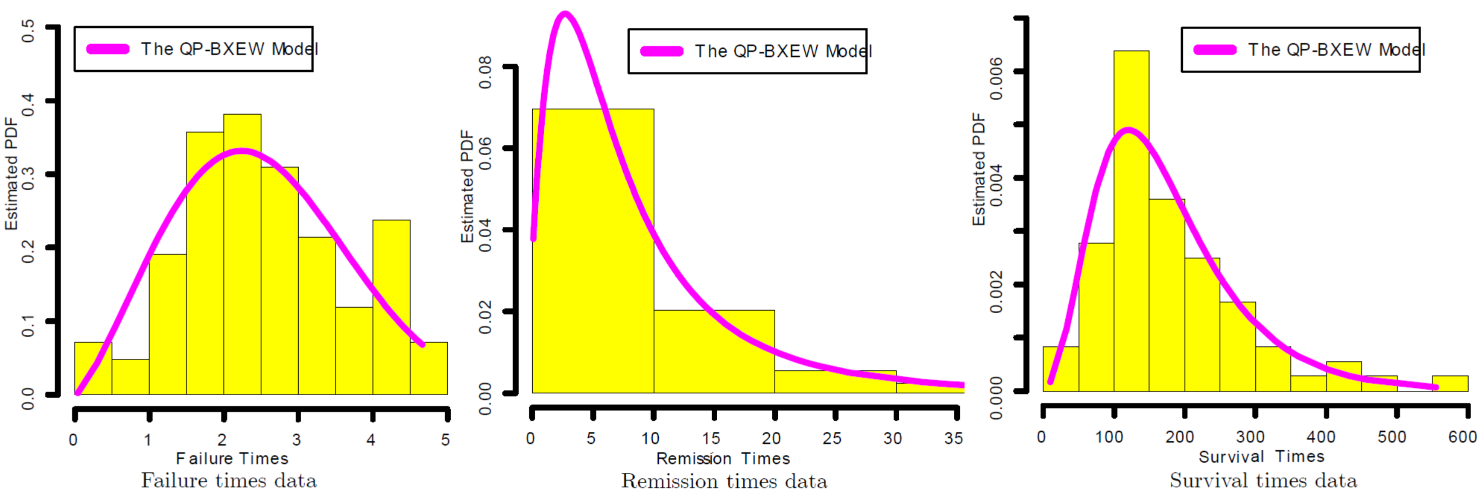

Figure 3: Estimated PDFs.

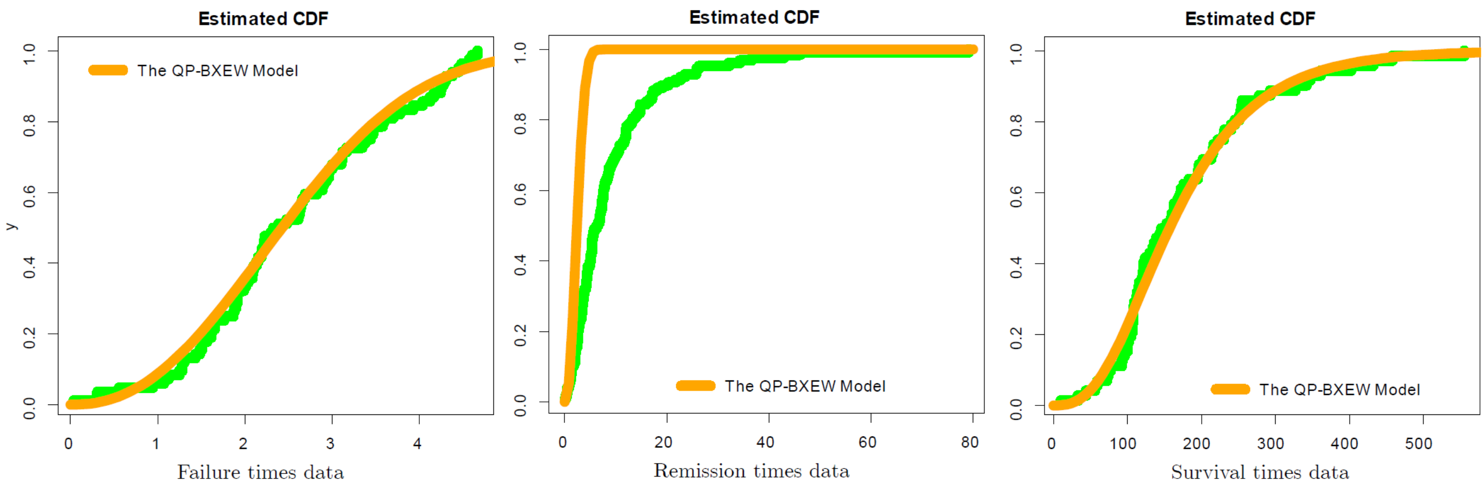

Figure 4: Estimated CDFs.

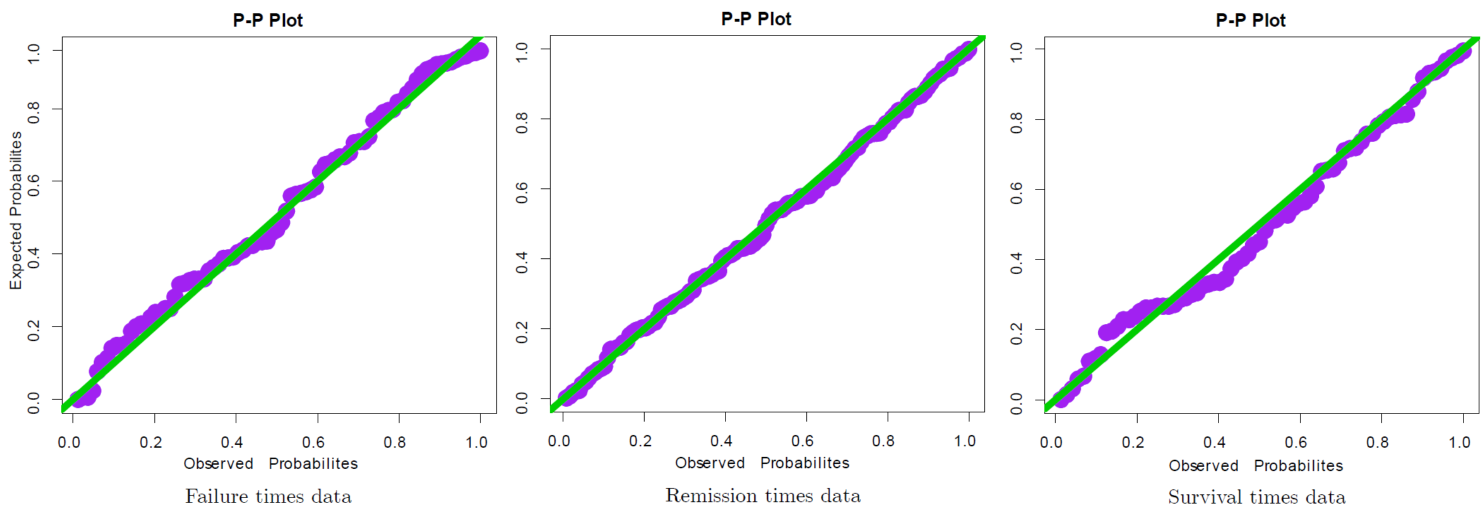

Figure 5: P-P plots.

7. Concluding remarks

We introduced a new continuous distribution, namely the quasi Poisson Burr X exponentiated Weibull distribution, based on the zero truncated Poisson model which accommodates many important failure rates. Some of its mathematical properties are derived. The density of the new model is expressed as a combination of exponentiated Weibull densities. The quasi Poisson Burr X exponentiated Weibull distribution model may be suitable for modeling “asymmetric left skewed”, “symmetric”, “asymmetric right skewed” and “asymmetric unimodal” data sets. The hazard rate function of the quasi Poisson Burr X exponentiated Weibull distribution can be “monotonically increasing” or “bathtub” or “monotonically decreasing” or “upside down” or “J shaped”. We derived some new bivariate type distributions using Farlie Gumbel Morgenstern copula, modified FGM copula, Clayton copula and Renyi’s entropy copula; the Multivariate type is also presented. The method of the maximum likelihood is used to estimate the proposed distribution’s parameters. We also demonstrated the importance and flexibility of the new distribution by modeling three data sets.

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