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Combinations of some spectral invariants and Hamiltonian properties of graphs*

Rao Li†

Department of Mathematical Sciences, University of South Carolina Aiken, Aiken, SC 29801, USA

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Abstract

In this note, spectral conditions involving the eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues are derived for Hamiltonian properties of graphs.

Keywords: eigenvalue; Laplacian eigenvalue; signless Laplacian eigenvalue; Hamiltonian properties.

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1. Introduction

The graphs considered in this note are finite undirected graphs containing neither multiple edges nor loops. Terminology and notation, not defined here, can be found in [1]. For a graph $G = (V, E)$, the number of vertices in G is denoted by n. The complete graph of order n is denoted by K_n. Denote by G^c the complement of a graph G. For the two graphs H and K, denote by $H \vee K$ the join of H and K. Define $G_1(n, k) := K_k \vee K^c_{k+1}$ for $k \geq 2$ and $G_2(n, k) := K_k \vee K^c_{k+2}$ for $k \geq 1$. A cycle in a graph G that contains all the vertices of G is called a Hamiltonian cycle of G. A graph containing a Hamiltonian cycle is known as a Hamiltonian graph. A path in a graph G that consists of all the vertices of G is referred to as a Hamiltonian path of G. A graph containing a Hamiltonian path is called a traceable graph. Obviously, $G_1(n, k)$ is not a Hamiltonian graph and $G_2(n, k)$ is not a traceable graph.

The eigenvalues of the adjacency matrix $A(G)$ of a graph G are called the the eigenvalues of G. Let $\lambda_1(G), \lambda_2(G), ..., \lambda_n(G)$ satisfying $\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_n(G)$, be the eigenvalues of the graph G. Let $D(G)$ be the diagonal matrix $diag(d_1, d_2, ..., d_n)$ of G, where $d_1, d_2, ..., d_n$ are the degrees of vertices in the graph G. For a graph G, the eigenvalues of the matrix

$$L(G) := D(G) - A(G)$$

are denoted by $\mu_1(G), \mu_2(G), ..., \mu_n(G)$, where $\mu_1(G) \geq \mu_2(G) \geq ... \geq \mu_n(G) = 0$, and these eigenvalues are called the Laplacian eigenvalues of G. For a graph G, the eigenvalues of the matrix

$$Q(G) := D(G) + A(G)$$

are called the signless Laplacian eigenvalues of G and these eigenvalues are denoted by $q_1(G), q_2(G), ..., q_n(G)$, where $q_1(G) \geq q_2(G) \geq ... \geq q_n(G) \geq 0$.

Notice that we can use Theorem 2.8 given on Page 57 of [2] to find $a_i := \lambda_i(G_1(n, k))$ and $a'_i := \lambda_i(G_2(n, k))$, where i is any integer satisfying the inequality $1 \leq i \leq n$. Also, we can use Theorem 2.1 given on Page 225 of [5] to find $b_i := \mu_i(G_1(n, k))$ and $b'_i := \mu_i(G_2(n, k))$, where i is any integer satisfying the inequality $1 \leq i \leq n$. Moreover, we can use (1) mentioned on Page 992 of [4] to find $c_i := q_i(G_1(n, k))$ and $c'_i := q_i(G_2(n, k))$, where i is any integer with $1 \leq i \leq n$.

In this note, spectral conditions involving the eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues are presented for Hamiltonian properties of graphs. Next, we give the statements of the main results of this note.

Theorem 1.1. *For $k \geq 2$, let G be a k-connected graph of order n. For each i with $1 \leq i \leq n$, let α_1, β_i and γ_i be non-negative real numbers such that α_1 and γ_1 cannot be equal to zero simultaneously. If*

$$\alpha_1\lambda_1 + \sum_{i=1}^{n} \beta_i\mu_i + \sum_{i=1}^{n} \gamma_i q_i \geq \alpha_1 a_1 + \sum_{i=1}^{n} \beta_i b_i + \sum_{i=1}^{n} \gamma_i c_i,$$

then G is Hamiltonian or $K_k \vee K^c_{k+1}$.

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†Email address: raol@usca.edu

Theorem 1.2. *Let G be a k-connected graph of order n, where $k \geq 1$. For each i with $1 \leq i \leq n$, let α_1, β_i and γ_i be non-negative real numbers such that α_1 and γ_1 cannot be equal to zero simultaneously. If*

$$\alpha_1\lambda_1 + \sum_{i=1}^{n}\beta_i\mu_i + \sum_{i=1}^{n}\gamma_i q_i \geq \alpha_1 a_1' + \sum_{i=1}^{n}\beta_i b_i' + \sum_{i=1}^{n}\gamma_i c_i',$$

then G is traceable or $K_k \vee K_{k+2}^c$.

2. Some lemmas

We need the following results as lemmas to prove our theorems. The following lemma is a part of Corollary 1.5 given on Page 11 of [6].

Lemma 2.1. *For any edge e of a connected graph G, $\lambda_1(G-e) < \lambda_1(G)$.*

The next result follows from Theorem 1.14 given on Page 13 of [6] and from the fact that $\mu_n(H) = 0$ for any graph H.

Lemma 2.2. *For any edge e of a connected graph G, $\mu_i(G-e) \leq \mu_i(G)$, for each i with $1 \leq i \leq n$.*

the following lemma is a part of Corollary 1.16 given on Page 14 of [6].

Lemma 2.3. *For any edge e of a connected graph G, $q_1(G-e) < q_1(G)$.*

Next lemma follows directly from Theorem 2.1 given on Page 13 of [3].

Lemma 2.4. *For any edge e of a connected graph G, $q_i(G-e) \leq q_i(G)$, for each i with $1 \leq i \leq n$.*

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Suppose that G is a graph satisfying the constraints mentioned in Theorem 1.1. Suppose that G is not a Hamiltonian graph. Then G is not a complete graph. We further have that $n \geq 2k+1$ otherwise $2\delta \geq 2k \geq n$ and G is Hamiltonian. Since $k \geq 2$, G contains a cycle. Let C be a longest cycle in the graph G and take an orientation on the cycle C. As G is not a Hamiltonian graph, there is at least one vertex $x_0 \in V(G) \setminus V(C)$. Because of Menger's theorem, there are s pairwise disjoint (except for the vertex x_0) paths P_1, P_2, ..., P_s between x_0 and $V(C)$, where $s \geq k$. For $1 \leq i \leq s$, assume that u_i is an end vertex of P_i lying on C. For $1 \leq i \leq s$, denote by u_i^+ the successor of the vertex u_i along the orientation of the cycle C. Then, $\{x_0, u_1^+, u_2^+, ..., u_s^+\}$ is an independent set, otherwise G has at least one cycle of length greater than that of the cycle C. Therefore, $S := \{x_0, u_1^+, u_2^+, ..., u_k^+\}$ is an independent set and $|S| = k+1$. Take

$$T := V(G) - S = \{\, v_1, v_2, ..., v_r \,\}.$$

Thus,

$$|T| = r = n - |S| = n - (k+1) \geq k.$$

It is clear that $xy \in E$ for every $x \in S$ and for every $y \in T$, and $v_iv_j \in E$ where $1 \leq i \neq j \leq r$. Otherwise, from Lemmas 2.1, 2.2, 2.3 and 2.4, we have

$$\alpha_1 a_1 + \sum_{i=1}^{n}\beta_i b_i + \sum_{i=1}^{n}\gamma_i c_i \leq \alpha_1\lambda_1 + \sum_{i=1}^{n}\beta_i\mu_i + \sum_{i=1}^{n}\gamma_i q_i < \alpha_1 a_1 + \sum_{i=1}^{n}\beta_i b_i + \sum_{i=1}^{n}\gamma_i c_i,$$

a contradiction. If $r \geq (k+1)$, it is obvious that G is Hamiltonian. Thus $r \leq k$. Namely, $r = k$ and G is $G_1(n,k)$.

□

Proof of Theorem 1.2. Suppose that G is a graph satisfying the conditions of Theorem 1.2. assume that G is not a traceable graph. Then, G is not complete. We further have that $n \geq 2k+2$ otherwise $2\delta \geq 2k \geq n-1$ and G is a traceable graph. Let P be a longest path in the graph G and take an orientation on the path P. Let $y, z \in V(G)$ be the end vertices of the path P. As G is not a traceable graph, there is at least one vertex $x_0 \in V(G) \setminus V(P)$. Because of Menger's theorem, there are s pairwise disjoint (except for x_0) paths P_1, P_2, ..., P_s between x_0 and $V(P)$, where $s \geq k$. For $1 \leq i \leq s$, assume that u_i is the end vertex of the path P_i lying on P. As P is a longest path in the graph G, $y \neq u_i$ and $z \neq u_i$, for each i with $1 \leq i \leq s$, otherwise G has at least one path of length greater than that of the path P. For $1 \leq i \leq s$, denote by u_i^+ the successor of the vertex u_i along the orientation of the path P. Then, $\{x_0, y, u_1^+, u_2^+, ..., u_s^+\}$ is an independent set otherwise

G has at least one path of length greater than that of the path P. Therefore, $S := \{x_0, y, u_1^+, u_2^+, ..., u_k^+\}$ is independent and $|S| = k+2$. Now, take $T := V(G) - S = \{ v_1, v_2, ..., v_r \}$. Thus, $|T| = r = n - |S| = n - (k+2) \geq k$.

It is clear that $xy \in E$ for every $x \in S$ and for every $y \in T$, and $v_i v_j \in E$ where $1 \leq i \neq j \leq r$. Otherwise, from Lemmas 2.1, 2.2, 2.3 and 2.4, we have that

$$\alpha_1 a_1' + \sum_{i=1}^{n} \beta_i b_i' + \sum_{i=1}^{n} \gamma_i c_i' \leq \alpha_1 \lambda_1 + \sum_{i=1}^{n} \beta_i \mu_i + \sum_{i=1}^{n} \gamma_i q_i < \alpha_1 a_1' + \sum_{i=1}^{n} \beta_i b_i' + \sum_{i=1}^{n} \gamma_i c_i',$$

a contradiction. If $r \geq (k+1)$, it is obvious that G is traceable. Thus $r \leq k$. Namely, $r = k$ and G is $G_2(n,k)$.

□

Since we have infinitely many choices for the values of α_1, $\beta_i \geq 0$ $(1 \leq i \leq n)$, and $\gamma_i \geq 0$ $(1 \leq i \leq n)$ in Theorems 1.1 and 1.2, we can obtain infinitely many sufficient conditions for Hamiltonian and traceable graphs.

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