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On the Dirichlet problem for a class of nonlinear degenerate elliptic equations

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Abstract

In this paper, we prove the existence and uniqueness of solutions for a class of nonlinear degenerate elliptic equations $L\,u(x) = f_0(x) - \sum_{j=1}^{n} D_j\, f_j(x)$ in the setting of the weighted Sobolev spaces, where $D_j = \partial/\partial x_j$ and L is a second order degenerate elliptic operator in divergence form in a bounded open subset of $\mathbb{R}^n$.

Keywords: nonlinear degenerate elliptic equations; weighted Sobolev spaces.

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1. Introduction

We are interested in the existence of (weak) solutions in the weighted Sobolev space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ (see Definition 2.2) for the Dirichlet problem

$$(P)\begin{cases} L\,u(x) = f_0(x) - \sum_{j=1}^{n} D_j\, f_j(x) \ \text{ in } \ \Omega, \\ u(x) = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where

$$L\,u(x) = -\operatorname{div}\left[\mathcal{A}(x,u,\nabla u)\,\omega_1 + \mathcal{B}(x,u,\nabla u)\,\nu_1\right] + \mathcal{H}(x,u,\nabla u)\,\nu_2 + |u|^{p-2}\,u\,\omega_2\,, \tag{1}$$

$D_j = \partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$; ω_1, ω_2, ν_1 and ν_2 are four weight functions (which represent the degeneration or singularity in Equation (1)), $1 < q, s < p < \infty$ and the functions $\mathcal{A}_j : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $\mathcal{B}_j : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j = 1, ..., n)$ and $\mathcal{H} : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfy the following conditions:

(H1). $x\mapsto\mathcal{A}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$; $(\eta,\xi)\mapsto\mathcal{A}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.

(H2). There exists a constant $\theta_1 > 0$ such that $\langle\mathcal{A}(x,\eta,\xi) - \mathcal{A}(x,\eta',\xi'),(\xi-\xi')\rangle \geq \theta_1\,|\xi-\xi'|^p$, whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, and $\mathcal{A}(x,\eta,\xi) = (\mathcal{A}_1(x,\eta,\xi), ..., \mathcal{A}_n(x,\eta,\xi))$, where $\langle .,.\rangle$ denotes the Euclidian scalar product in $\mathbb{R}^n$.

(H3). $\langle\mathcal{A}(x,\eta,\xi),\xi\rangle \geq \lambda_1|\xi|^p$, where λ_1 is a positive constant.

(H4). $|\mathcal{A}(x,\eta,\xi)| \leq K_1(x) + h_1(x)\,|\eta|^{p/p'} + h_2(x)|\xi|^{p/p'}$, where K_1, h_1 and h_2 are nonnegative functions, with $h_1, h_2\in L^\infty(\Omega)$ and $K_1\in L^{p'}(\Omega,\omega_1)$ provided that $1/p + 1/p' = 1$.

(H5). $x\mapsto\mathcal{B}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$; $(\eta,\xi)\mapsto\mathcal{B}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.

(H6). $\langle\mathcal{B}(x,\eta,\xi) - \mathcal{B}(x,\eta',\xi'),(\xi-\xi')\rangle > 0$, whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, where $\mathcal{B}(x,\eta,\xi) = (\mathcal{B}_1(x,\eta,\xi), ..., \mathcal{B}_n(x,\eta,\xi))$.

(H7). $\langle\mathcal{B}(x,\eta,\xi),\xi\rangle \geq \lambda_2|\xi|^q + \Lambda_2|\eta|^q$, where $\lambda_2 > 0$ and $\Lambda_2 \geq 0$ are constants.

(H8). $|\mathcal{B}(x,\eta,\xi)| \leq K_2(x) + g_1(x)|\eta|^{q/q'} + g_2(x)|\xi|^{q/q'}$, where K_2, g_1 and g_2 are nonnegative functions, with $g_1, g_2\in L^\infty(\Omega)$, and $K_2\in L^{q'}(\Omega,\nu_1)$ provided that $1/q + 1/q' = 1$.

(H9). $x\mapsto\mathcal{H}(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$; $(\eta,\xi)\mapsto\mathcal{H}(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.

(H10). $[\mathcal{H}(x,\eta,\xi) - \mathcal{H}(x,\eta',\xi')](\eta-\eta') > 0$, whenever $\eta,\eta'\in\mathbb{R}$, $\eta\neq\eta'$.

(H11). $\mathcal{H}(x,\eta,\xi)\eta \geq \lambda_3|\xi|^s + \Lambda_3|\eta|^s$, where λ_3 and Λ_3 are nonnegative constants.

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(H12). $|\mathcal{H}(x,\eta,\xi)| \le K_3(x) + h_2(x)|\eta|^{s/s'} + h_3(x)|\xi|^{s/s'}$, where K_3, h_2 and h_3 are nonnegative functions, with $K_3 \in L^{s'}(\Omega,\nu_2)$, $1/s + 1/s' = 1$ and $h_2, h_3 \in L^\infty(\Omega)$.

By the symbol $\mathcal{W}(\Omega)$ we denote the set of all those positive and finite functions $\omega = \omega(x)$, $x \in \Omega$, that are measurable almost everywhere in Ω. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,dx$, for any measurable set $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [2–4,7]). In various applications, we may meet boundary value problems for the elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g. from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [1,6]).

A well-known class of weights is the class of A_p-weights (or Muckenhoupt class) that was introduced by Muckenhoupt [15]. This class of weights has found many useful applications in harmonic analysis, see [18]. Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p, see [13]. There are, in fact, many interesting examples of weights (see [12] for p-admissible weights).

The following theorem will be proved in Section 3.

Theorem 1.1. *Let* $1 < q, s < p < \infty$ *and assume that the conditions* (H1)–(H12) *hold. If*

(i) $\omega_1, \omega_2 \in A_p$, $\nu_1, \nu_2 \in \mathcal{W}(\Omega)$, $\dfrac{\nu_1}{\omega_1} \in L^{r_1}(\Omega,\omega_1)$ *and* $\dfrac{\nu_2}{\omega_1} \in L^{r_2}(\Omega,\omega_1)$, *where* $r_1 = p/(p-q)$ *and* $r_2 = p/(p-s)$;

(ii) $f_0/\nu_1 \in L^{q'}(\Omega,\nu_1)$ *and* $f_j/\omega_1 \in L^{p'}(\Omega,\omega_1)$ *for* $j = 1, \ldots, n$;

then the problem (P) has a unique solution $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$. *Moreover, we have*

$$\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \le C \left(C_{p,q}\,\|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \right)^{1/(p-1)},$$

where $C = [(C_\Omega^p + 1)/\lambda_1]^{1/(p-1)}$, C_Ω *is the constant in Theroem 2.2 and* $C_{p,q}$ *is the constant in Remark 2.2(i).*

2. Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)dx\right)^{p-1} \le C\,,$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \le p$, then $A_q \subset A_p$ (see [10,12,18] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu(B(x;2r)) \le C\,\mu(B(x;r))$, for every ball $B = B(x;r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [12]).

Let us consider an example of A_p-weight. For $x \in \mathbb{R}^n$, the function $\omega(x) = |x|^\alpha$ is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4 of Chapter IX in [18]).

If $\omega \in A_p$, then $\left(\dfrac{|E|}{|B|}\right)^p \le C\,\dfrac{\mu(E)}{\mu(B)}$, whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [12]). Therefore, if $\mu(E) = 0$ then $|E| = 0$. The measure μ and the Lebesgue measure $|.|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu(E) = 0$ if and only if $|E| = 0$); so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated "a.e.".

Definition 2.1. *Let* ω *be a weight, and let* $\Omega \subset \mathbb{R}^n$ *be open. For* $1 < p < \infty$, *we define* $L^p(\Omega,\omega)$ *as the set of measurable functions* f *on* Ω *satisfying* $\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f|^p \omega\,dx\right)^{1/p} < \infty$.

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [19]). Thus, it makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let ω_1 and ω_2 be A_p-weights ($1 < p < \infty$). We define the weighted Sobolev space $W^{1,p}(\Omega, \omega_1, \omega_2)$ as the set of functions $u \in L^p(\Omega, \omega_2)$ with weak derivatives $D_j u \in L^p(\Omega, \omega_1)$. The norm of u in $W^{1,p}(\Omega, \omega_1, \omega_2)$ is defined by*

$$\|u\|_{W^{1,p}(\Omega,\omega_1,\omega_2)} = \left(\int_\Omega |u|^p \,\omega_2 \,dx + \int_\Omega |\nabla u|^p \,\omega_1 \,dx \right)^{1/p}. \tag{2}$$

The space $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2). Equipped with this norm, $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a reflexive Banach space (see [14, 16] for more information about the spaces $W^{1,p}(\Omega, \omega_1, \omega_2)$). The dual of the space $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is the space defined as

$$[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^* = \{T = f_0 - \operatorname{div}(F),\ F = (f_1, ..., f_n) : \frac{f_0}{\omega_2} \in L^{p'}(\Omega,\omega_2), \frac{f_j}{\omega_1} \in L^{p'}(\Omega,\omega_1), j = 1, ..., n\}.$$

If $T \in [W_0^p(\Omega, \omega_1, \omega_2)]^*$ and $\varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, we take

$$(T|\varphi) = \int_\Omega f_0\, \varphi \,dx + \sum_{j=1}^n f_j\, D_j\varphi \,dx\,, \quad \|T\|_* = \|f_0/\omega_2\|_{L^{p'}(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\,, \quad |(T|\varphi)| \le \|T\|_* \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\,.$$

If $\omega = \omega_1 = \omega_2$, we denote $W_0^{1,p}(\Omega, \omega) = W_0^{1,p}(\Omega, \omega, \omega)$. To prove the main result of this paper, we use the following results.

Theorem 2.1. *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega, \omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega, \omega)$ such that*
(i) $u_{m_k}(x) \to u(x)$, $m_k \to \infty$ a.e. on Ω;
(ii) $|u_{m_k}(x)| \le \Phi(x)$ a.e. on Ω.

Proof. The proof of this theorem follows from the lines of Theorem 2.8.1 in [9]. □

Theorem 2.2. (The weighted Sobolev inequality) *Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ ($1 < p < \infty$). There exist positive constants C_Ω and δ such that for every $u \in W_0^{1,p}(\Omega, \omega)$ and for each k satisfying $1 \le k \le n/(n-1) + \delta$, it holds*

$$\|u\|_{L^{kp}(\Omega,\omega)} \le C_\Omega \|\, |\nabla u| \,\|_{L^p(\Omega,\omega)}, \tag{3}$$

where C_Ω depends only on n, p, the A_p-constant $C(p, \omega)$ of ω and on the diameter of Ω.

Proof. It suffices to prove (3) for every function $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [8]). In order to extend the estimates (3) to an arbitrary $u \in W_0^{1,p}(\Omega, \omega)$, let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega, \omega)$. Applying the estimates (3) to the differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{kp}(\Omega, \omega)$. Consequently, the limit function u lies in the desired spaces and satisfies (3). □

Remark 2.1. *If $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ then by Theorem 2.2 (with $k = 1$), it holds that*

$$\|u\|_{L^p(\Omega,\omega_1)} \le C_\Omega \,\|\, |\nabla u| \,\|_{L^p(\Omega,\omega_1)} \le C_\Omega \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\,.$$

Hence, $W_0^{1,p}(\Omega, \omega_1, \omega_2) \subset W_0^{1,p}(\Omega, \omega_1)$.

Proposition 2.1. *Let $1 < p < \infty$.*
(a) There exists a constant C_p such that $\big||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big| \le C_p\, |\xi - \eta|\, (|\xi| + |\eta|)^{p-2}\ \ \forall\, \xi, \eta \in \mathbb{R}^n$.
(b) There exist two positive constants β_p, γ_p such that for every $x, y \in \mathbb{R}^n$, it holds that

$$\beta_p\, (|x| + |y|)^{p-2} |x - y|^2 \le \langle |x|^{p-2}x - |y|^{p-2}y), (x - y)\rangle \le \gamma_p\, (|x| + |y|)^{p-2} |x - y|^2.$$

Proof. See Proposition 17.2 and Proposition 17.3 in [5]. □

Definition 2.3. *We say that an element $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a (weak) solution of problem (P) if*

$$\int_\Omega \langle \mathcal{A}(x,u,\nabla u), \nabla\varphi\rangle\, \omega_1 \,dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla\varphi\rangle\, \nu_1 \,dx + \int_\Omega \mathcal{H}(x,u,\nabla u)\, \varphi\, \nu_2 \,dx + \int_\Omega |u|^{p-2}\, u\, \varphi\, \omega_2 \,dx = \int_\Omega f_0\, \varphi \,dx + \sum_{j=1}^n \int_\Omega f_j\, D_j\varphi \,dx,$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$.

Remark 2.2. *(i) If* $\dfrac{\nu_1}{\omega_1} \in L^{r_1}(\Omega,\omega_1)$ *(where* $r_1 = p/(p-q)$, $1 < q < p < \infty$*) then* $\|u\|_{L^q(\Omega,\nu_1)} \leq C_{p,q}\,\|u\|_{L^p(\Omega,\omega_1)}$*, where* $C_{p,q} = \|\nu_1/\omega_1\|^{1/q}_{L^{r_1}(\Omega,\omega_1)}$*. In fact, by Hölder's inequality we obtain*

$$\|u\|^q_{L^q(\Omega,\nu_1)} = \int_\Omega |u|^q\,\nu_1\,dx = \int_\Omega |u|^q\,\frac{\nu_1}{\omega_1}\,\omega_1\,dx \leq \left(\int_\Omega |u|^{q\,p/q}\,\omega_1\,dx\right)^{q/p}\left(\int_\Omega (\nu_1/\omega_1)^{p/(p-q)}\,\omega_1\,dx\right)^{(p-q)/p}$$
$$= \|u\|^q_{L^p(\Omega,\omega_1)}\,\|\nu_1/\omega_1\|_{L^{r_1}(\Omega,\omega_1)}.$$

Hence, $\|u\|_{L^q(\Omega,\nu_1)} \leq C_{p,q}\|u\|_{L^p(\Omega,\omega_1)}$.
(ii) Analogously, if $\dfrac{\nu_2}{\omega_1} \in L^{r_2}(\Omega,\omega_1)$ *(where* $r_2 = p/(p-s)$, $1 < s < p < \infty$*) then* $\|u\|_{L^s(\Omega,\nu_2)} \leq C_{p,s}\,\|u\|_{L^p(\Omega,\omega_1)}$*, where* $C_{p,s} = \|\nu_2/\omega_1\|^{1/s}_{L^{r_2}(\Omega,\omega_1)}$.

3. Proof of Theorem 1.1

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem given below.

Theorem 3.1. *Let* $A : X \to X^*$ *be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space* X*. Then the following assertions hold:*
(a) For each $T \in X^*$ *the equation* $Au = T$ *has a solution* $u \in X$*;*
(b) If the operator A *is strictly monotone, then equation* $A\,u = T$ *is uniquely solvable in* X*.*

Proof. See Theorem 26.A in [21]. □

To prove Theorem 1.1, we define $\mathbf{B} : W^{1,p}_0(\Omega,\omega_1,\omega_2) \times W^{1,p}_0(\Omega,\omega_1,\omega_2) \to \mathbb{R}$ and $\mathbf{T} : W^{1,p}_0(\Omega,\omega_1,\omega_2) \to \mathbb{R}$ by

$$\mathbf{B}(u,\varphi) = \mathbf{B_1}(u,\varphi) + \mathbf{B_2}(u,\varphi) + \mathbf{B}_3(u,\varphi) + \mathbf{B}_4(u,\varphi) \quad \text{and} \quad \mathbf{T}(\varphi) = \int_\Omega f_0\,\varphi\,dx + \sum_{j=1}^n \int_\Omega f_j\,D_j\varphi\,dx\,,$$

where $\mathbf{B_i} : W^{1,p}_0(\Omega,\omega_1,\omega_2) \times W^{1,p}_0(\Omega,\omega_1,\omega_2) \to \mathbb{R}$, for $i = 1,2,3,4$, are defined as

$$\mathbf{B_1}(u,\varphi) = \int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla\varphi\rangle\,\omega_1\,dx\,, \quad \mathbf{B_2}(u,\varphi) = \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla\varphi\rangle\,\nu_1\,dx\,,$$
$$\mathbf{B}_3(u,\varphi) = \int_\Omega \mathcal{H}(x,u,\nabla u)\,\varphi\,\nu_2\,dx \ \text{ and } \ \mathbf{B_4}(u,\varphi) = \int_\Omega |u|^{p-2}\,u\,\varphi\,\omega_2\,dx\,.$$

Then $u \in W^{1,p}_0(\Omega,\omega_1,\omega_2)$ is a (weak) solution to problem (P) if $\mathbf{B}(u,\varphi) = \mathbf{B_1}(u,\varphi) + \mathbf{B_2}(u,\varphi) + \mathbf{B}_3(u,\varphi) + \mathbf{B}_4(u,\varphi) = \mathbf{T}(\varphi)$, for all $\varphi \in W^{1,p}_0(\Omega,\omega_1,\omega_2)$.
Step 1. For $j = 1,\ldots,n$ we define the operator $F_j : W^{1,p}_0(\Omega,\omega_1,\omega_2)) \to L^{p'}(\Omega,\omega_1)$ as $(F_j u)(x) = \mathcal{A}_j(x,u(x),\nabla u(x))$. We now show that the operator F_j is bounded and continuous.
(i) Using (H4), Theorem 2.2 (since $\omega_1 \in A_p$) and Remark 2.1, we obtain

$$\begin{aligned}
\|F_j u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |F_j u(x)|^{p'}\omega_1\,dx = \int_\Omega |\mathcal{A}_j(x,u,\nabla u)|^{p'}\omega_1\,dx \leq \int_\Omega \left(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\right)^{p'}\omega_1\,dx\\
&\leq C_p \int_\Omega (K_1^{p'} + h_1^{p'}|u|^p + h_2^{p'}|\nabla u|^p)\omega_1\,dx = C_p\left[\int_\Omega K_1^{p'}\,\omega_1\,dx + \int_\Omega h_1^{p'}|u|^p\,\omega_1\,dx + \int_\Omega h_2^{p'}|\nabla u|^p\omega_1\,dx\right]\\
&\leq C_p\left[\int_\Omega K_1^{p'}\,\omega_1\,dx + \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |u|^p\,\omega_1\,dx + \|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^p\,\omega_1\,dx\right]\\
&\leq C_p\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \left(C^p_\Omega\,\|h_1\|^{p'}_{L^\infty(\Omega)} + \|h_2\|^{p'}_{L^\infty(\Omega)}\right)\int_\Omega |\nabla u|^p\,\omega_1\,dx\right]\\
&\leq C_p\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \left(C^p_\Omega\,\|h_1\|^{p'}_{L^\infty(\Omega)} + \|h_2\|^{p'}_{L^\infty(\Omega)}\right)\|u\|^p_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\right].
\end{aligned} \tag{4}$$

where the constant C_p depends only on p. Therefore, in (4) we obtain

$$\|F_j u\|_{L^{p'}(\Omega,\omega_1)} \leq C_p^{1/p'}\left(\|K_1\|_{L^{p'}(\Omega,\omega_1)} + (C^{p-1}_\Omega\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\right).$$

(ii) Let $u_m \to u$ in $W^{1,p}_0(\Omega,\omega_1,\omega_2)$ as $m \to \infty$. We need to show that $F_j u_m \to F_j u$ in $L^{p'}(\Omega,\omega_1)$. We will apply the Lebesgue Dominated Convergence Theorem. If $u_m \to u$ in $W^{1,p}_0(\Omega,\omega_1,\omega_2)$, then $|\nabla u_m| \to |\nabla u|$ in $L^p(\Omega,\omega_1)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_1 \in L^p(\Omega,\omega_1)$ such that $u_{m_k}(x) \to u(x)$ a.e. in Ω, $D_j u_{m_k}(x) \to D_j u(x)$ a.e. in Ω, and $|\nabla u_{m_k}(x)| \le \Phi_1(x)$ a.e. in Ω. Next, applying (H4) and Theorem 2.2 we obtain

$$\begin{aligned}
\|F_j u_{m_k} - F_j u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{p'} \omega_1\, dx = \int_\Omega |\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k}) - \mathcal{A}_j(x,u,\nabla u)|^{p'}\, \omega_1\, dx \\
&\le C_p \int_\Omega \Big(|\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x,u,\nabla u)|^{p'} \Big)\omega_1\, dx \\
&\le C_p \left[\int_\Omega \Big(K_1 + h_1 |u_{m_k}|^{p/p'} + h_2 |\nabla u_{m_k}|^{p/p'} \Big)^{p'} \omega_1\, dx \right. \\
&\quad \left. + \int_\Omega \Big(K_1 + h_1|u|^{p/p'} + h_2 |\nabla u|^{p/p'} \Big)^{p'} \omega_1\, dx \right] \\
&\le C_p \left[\int_\Omega K_1^{p'} \omega_1\, dx + \|h_1\|^{p'}_{L^\infty(\Omega)} \int_\Omega |u_{m_k}|^p\, \omega_1\, dx + \|h_2\|^{p'}_{L^\infty(\Omega)} \int_\Omega |\nabla u_{m_k}|^p\, \omega_1\, dx \right. \\
&\quad \left. + \|h_1\|^{p'}_{L^\infty(\Omega)} \int_\Omega |u|^p\, \omega_1\, dx + \|h_2\|^{p'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^p\, \omega_1\, dx \right] \\
&\le 2C_p \left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\, C^p_\Omega \int_\Omega |\Phi_1|^p\, \omega_1\, dx + \|h_2\|^{p'}_{L^\infty(\Omega)} \int_\Omega |\Phi|^p\, \omega_1\, dx \right].
\end{aligned}$$

By condition (H1), we have $F_j u_{m_k}(x) = \mathcal{A}_j(x,u_{m_k}(x),\nabla u_{m_k}(x)) \to \mathcal{A}_j(x,u(x),\nabla u(x)) = F_j u(x)$, as $m_k \to +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain $\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega_1)} \to 0$, that is, $F_j u_{m_k} \to F_j u$ in $L^{p'}(\Omega,\omega_1)$. We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [20]) that

$$F_j u_m \to F_j u \;\text{ in }\; L^{p'}(\Omega,\omega_1). \tag{5}$$

Step 2. We define the operator $G_j : W^{1,p}_0(\Omega,\omega_1,\omega_2) \to L^{q'}(\Omega,\nu_1)$ by $(G_j u)(x) = \mathcal{B}_j(x,u(x),\nabla u(x))$.
This operator is continuous and bounded. In fact,
(i) Using (H8), Remark 2.2(i) and Theorem 2.2 (since $\omega_1 \in A_p$) we obtain

$$\begin{aligned}
\|G_j u\|^{q'}_{L^{q'}(\Omega,\nu_1)} &= \int_\Omega |G_j u(x)|^{q'} \nu_1\, dx = \int_\Omega |\mathcal{B}_j(x,u,\nabla u)|^{q'} \nu_1\, dx \le \int_\Omega \Big(K_2 + g_1 |u|^{q/q'} + g_2 |\nabla u|^{q/q'} \Big)^{q'} \nu_1\, dx \\
&\le C_q \int_\Omega \Big[(K_2^{q'} + g_1^{q'} |u|^q + g_2^{q'} |\nabla u|^q)\nu_1 \Big] dx = C_q \left[\int_\Omega K_2^{q'}\, \nu_1\, dx + \int_\Omega g_1^{q'} |u|^q\, \nu_1\, dx + \int_\Omega g_2^{q'} |\nabla u|^q \nu_1\, dx \right] \\
&\le C_q \Big(\|K_2\|^{q'}_{L^{q'}(\Omega,\nu_1)} + \|g_1\|^{q'}_{L^\infty(\Omega)} \|u\|^q_{L^q(\Omega,\nu_1)} + \|g_2\|^{q'}_{L^\infty(\Omega)} \|\, |\nabla u|\, \|^q_{L^q(\Omega,\nu_1)} \Big) \\
&\le C_q \Big(\|K_2\|^{q'}_{L^{q'}(\Omega,\nu_1)} + \|g_1\|^{q'}_{L^\infty(\Omega)} C^q_{p,q} \|u\|^q_{L^p(\Omega,\omega_1)} + C^q_{p,q} \|g_2\|^{q'}_{L^\infty(\Omega)} \|\, |\nabla u|\, \|^q_{L^p(\Omega,\omega_1)} \Big) \\
&\le C_q \Big(\|K_2\|^{q'}_{L^{q'}(\Omega,\nu_1)} + C^q_{p,q} \big(C^q_\Omega\, \|g_1\|^{q'}_{L^\infty(\Omega)} + \|g_2\|^{q'}_{L^\infty(\Omega)} \big) \|u\|^q_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} \Big),
\end{aligned} \tag{6}$$

where C_q depends only on q. Therefore, from (6) we obtain

$$\|G_j u\|_{L^{q'}(\Omega,\nu_1)} \le C_q^{1/q'} \left(\|K_2\|_{L^{q'}(\Omega,\nu_1)} + C^{q-1}_{p,q} \big(C^{q-1}_\Omega \|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)} \big) \|u\|^{q-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} \right).$$

(ii) Let $u_m \to u$ in $W^{1,p}_0(\Omega,\omega_1,\omega_2)$ as $m \to \infty$. We need to show that $G_j u_m \to G_j u$ in $L^{q'}(\Omega,\nu_1)$. We will apply the Lebesgue Dominated Theorem. If $u_m \to u$ in $W^{1,p}_0(\Omega,\omega_1,\omega_2)$, then $|\nabla u_m| \to |\nabla u|$ in $L^p(\Omega,\omega_1)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_1 \in L^p(\Omega,\omega_1)$ such that $D_j u_{m_k}(x) \to D_j u(x)$ a.e. in Ω and $|\nabla u_{m_k}(x)| \le \Phi_1(x)$ a.e. in Ω. Then, by Remark 2.2(i) and Theorem 2.2 (with $k = 1$) we obtain

$$\|u_{m_k}\|_{L^q(\Omega,\nu_1)} \le C_{p,q}\, \|u_{m_k}\|_{L^p(\Omega,\omega_1)} \le C_{p,q}\, C_\Omega \|\, |\nabla u_{m_k}|\, \|_{L^p(\Omega,\omega_1)} \le C_{p,q}\, C_\Omega\, \|\Phi_1\|_{L^p(\Omega,\omega_1)}.$$

Next, applying (H8) and Remark 2.2(i) we obtain

$$\begin{aligned}
\|G_j u_{m_k} - G_j u\|^{q'}_{L^{q'}(\Omega,\nu_1)} &= \int_\Omega |G_j u_{m_k}(x) - G_j u(x)|^{q'} \nu_1\, dx = \int_\Omega |\mathcal{B}_j(x,u_{m_k},\nabla u_{m_k}) - \mathcal{B}_j(x,u,\nabla u)|^{q'}\, \nu_1\, dx \\
&\le C_q \int_\Omega \Big(|\mathcal{B}_j(x,u_{m_k},\nabla u_{m_k})|^{q'} + |\mathcal{B}_j(x,u,\nabla u)|^{q'} \Big) \nu_1\, dx \\
&\le C_q \left[\int_\Omega \Big(K_2 + g_1 |u_{m_k}|^{q/q'} + g_2 |\nabla u_{m_k}|^{q/q'} \Big)^{q'} \nu_1\, dx + \int_\Omega \Big(K_2 + g_1 |u|^{q/q'} + g_2 |\nabla u|^{q/q'} \Big)^{q'} \nu_1\, dx \right]
\end{aligned}$$

$$\le C_q\Big[\int_\Omega K_2^{q'}\,\nu_1\,dx + \|g_1\|^{q'}_{L^\infty(\Omega)}\int_\Omega |u_{m_k}|^q\nu_1\,dx + \|g_2\|^{q'}_{L^\infty(\Omega)}\int_\Omega |\nabla u_{m_k}|^q\,\nu_1\,dx$$
$$+\int_\Omega K_2^{q'}\,\nu_1\,dx + \|g_1\|^{q'}_{L^\infty(\Omega)}\int_\Omega |u|^q\nu_1\,dx + \|g_2\|^{q'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^q\,\nu_1\,dx\Big]$$
$$\le 2\,C_q\Big[\|K_2\|_{L^{q'}(\Omega,\nu_1)} + \|g_1\|^{q'}_{L^\infty(\Omega)}\,C^q_{p,q}\,C^q_\Omega\,\|\Phi\|^{p/q}_{L^p(\Omega,\omega_1)} + \|g_2\|^{q'}_{L^\infty(\Omega)}\|\Phi\|^{p/q}_{L^p(\Omega,\omega_1)}\Big].$$

By condition (H5), we have $G_j u_{m_k}(x) = \mathcal{B}_j(x, u_{m_k}(x), \nabla u_{m_k}(x)) \to \mathcal{B}_j(x, u(x), \nabla u(x)) = G_j u(x)$, as $m_k \to +\infty$. Hence, by the Lebesgue Dominated Convergence Theorem, we obtain $\|G_j u_{m_k} - G_j u\|_{L^{q'}(\Omega,\nu_1)} \to 0$, i.e., $G_j u_{m_k} \to G_j u$ in $L^{q'}(\Omega,\nu_1)$. We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [20]) that

$$G_j u_m \to G_j u \ \text{ in } \ L^{q'}(\Omega,\nu_1). \tag{7}$$

Step 3. We define the operator $H : W_0^{1,p}(\Omega,\omega_1,\omega_2) \to L^{s'}(\Omega,\nu_2)$ by $(Hu)(x) = \mathcal{H}(x,u(x),\nabla u(x))$. We also have that the operator H is continuous and bounded. In fact,
(i) Using $(H12)$, Remark 2.2(ii) and Theorem 2.2 we obtain

$$\|Hu\|^{s'}_{L^{s'}(\Omega,\nu_2)} = \int_\Omega |Hu|^{s'}\,\nu_2\,dx = \int_\Omega |\mathcal{H}(x,u,\nabla u)|^{s'}\,\nu_2\,dx \le \int_\Omega \Big(K_3 + h_2|u|^{s/s'} + h_3|\nabla u|^{s/s'}\Big)^{s'}\,\nu_2\,dx$$
$$\le C_s\int_\Omega (K_3^{s'} + h_2^{s'}|u|^s + h_3^{s'}|\nabla u|^s)\nu_2\,dx$$
$$\le C_s\int_\Omega K_3^{s'}\,\nu_2\,dx + \|h_2\|^{s'}_{L^\infty(\Omega)}\int_\Omega |u|^s\,\nu_2\,dx + \|h_2\|^{s'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^s\nu_2\,dx\Big]$$
$$\le C_s\Big(\|K_3\|^{s'}_{L^{s'}(\Omega,\nu_2)} + \|h_2\|^{s'}_{L^\infty(\Omega)}C^s_{p,s}\|u\|^s_{L^p(\Omega,\omega_1)} + \|h_3\|^{s'}_{L^\infty(\Omega)}C^s_{p,s}\|\,|\nabla u|\,\|^s_{L^p(\Omega,\omega_1)}\Big)$$
$$\le C_s\Big(\|K_3\|^{s'}_{L^{s'}(\Omega,\nu_2)} + \|h_2\|^{s'}_{L^\infty(\Omega)}C^s_{p,s}C^s_\Omega\|\,\nabla u\,|\|^s_{L^p(\Omega,\omega_1)} + \|h_3\|^{s'}_{L^\infty(\Omega)}C^s_{p,s}\|\,|\nabla u|\,\|^s_{L^p(\Omega,\omega_1)}\Big)$$
$$\le C_s\Big(\|K_3\|^{s'}_{L^{s'}(\Omega,\nu_2)} + C^s_{p,s}(C^s_\Omega\|h_2\|^{s'}_{L^\infty(\Omega)} + \|h_3\|^{s'}_{L^\infty(\Omega)})\,\|u\|^s_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\Big), \tag{8}$$

where the constant C_s depends only on s. Hence, from (8), we obtain

$$\|Hu\|_{L^{s'}(\Omega,\nu_2)} \le C_s\Big[\|K_3\|_{L^{s'}(\Omega,\nu_2)} + C^{s-1}_{p,s}(C^{s-1}_\Omega\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)})\|u\|^{s-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\Big].$$

(ii) Applying (H12) and Remark 2.2(ii), by the same argument used in Step 1(ii), we obtain analogously, if $u_m \to u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ then

$$Hu_m \to Hu, \ \text{ in } \ L^{s'}(\Omega,\nu_2). \tag{9}$$

Step 4. We define the operator $J : W_0^{1,p}(\Omega,\omega_1,\omega_2) \to L^{p'}(\Omega,\omega_2)$ by $(Ju)(x) = |u(x)|^{p-2}u(x)$. We also have that the operator J is continuous and bounded. In fact:
(i) For all $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, we have

$$\|Ju\|^{p'}_{L^{p'}(\Omega,\omega_2)} = \int_\Omega |Ju|^{p'}\,\omega_2\,dx = \int_\Omega |u|^{(p-1)p'}\omega_2\,dx = \int_\Omega |u|^p\,\omega_2\,dx \le \|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

(ii) Let $u_m \to u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$. Then $u_m \to u$ in $L^p(\Omega,\omega_2)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_2 \in L^p(\Omega,\omega_2)$ such that $u_{m_k}(x) \to u(x)$ a.e. in Ω and $|u_{m_k}(x)| \le \Phi_2(x)$ a.e. in Ω. Next, applying Proposition 2.1(a), we have

$$\|Ju_{m_k} - Ju\|^{p'}_{L^{p'}(\Omega,\omega_2)} = \int_\Omega |Ju_{m_k} - Ju|^{p'}\omega_2\,dx = \int_\Omega \big||u_{m_k}|^{p-2}u_{m_k} - |u|^{p-2}u\big|^{p'}\,\omega_2\,dx$$
$$\le \int_\Omega \Big[C_p|u_{m_k} - u|(|u_{m_k}| + |u|)^{p-2}\Big]^{p'}\,\omega_2\,dx \le C_p^{p'}\int_\Omega |u_{m_k} - u|^{p'}(|u_{m_k}| + |u|)^{(p-2)\,p'}\,\omega_2\,dx$$
$$\le 2^{(p-2)\,p'}\,C_p^{p'}\int_\Omega |u_{m_k} - u|^{p'}\,\Phi_2^{(p-2)\,p'}\,\omega_2\,dx$$
$$\le 2^{(p-2)\,p'}C_p^{p'}\Big(\int_\Omega |u_{m_k} - u|^{p'(p/p')}\,\omega_2\,dx\Big)^{p'/p}\Big(\int_\Omega \Phi_2^{(p-2)\,p'\,p/(p-p')}\,\omega_2\,dx\Big)^{(p-p')/p}$$
$$= 2^{(p-2)\,p'}C_p^{p'}\Big(\int_\Omega |u_{m_k} - u|^p\,\omega_2\,dx\Big)^{p'/p}\Big(\int_\Omega \Phi_2^p\,\omega_2\,dx\Big)^{(p-p')/p}$$

$$= 2^{(p-2)\,p'} C_p^{p'} \|u_{m_k} - u\|^{p'}_{L^p(\Omega,\omega_2)} \|\Phi_2\|^{p-p'}_{L^p(\Omega,\omega_2)}.$$

Hence $\|Ju_{m_k} - Ju\|_{L^{p'}(\Omega,\omega_2)} \to 0$ as $m_k \to \infty$. We conclude from the Convergence Principle in Banach spaces that

$$Ju_m \to Ju \ \text{ in } L^{p'}(\Omega,\omega_2). \tag{10}$$

Step 5. Since $\dfrac{f_0}{\nu_1} \in L^{q'}(\Omega,\nu_1)$ and $\dfrac{f_j}{\omega_1} \in L^{p'}(\Omega,\omega_1)$ $(j = 1, ..., n)$ then $\mathbf{T} \in [W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$. Moreover, by Remark 2.2(i), we have

$$\begin{aligned}
|\mathbf{T}(\varphi)| &\le \int_\Omega |f_0||\varphi|\,dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\,dx = \int_\Omega \frac{|f_0|}{\nu_1}|\varphi|\nu_1\,dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega_1}|D_j\varphi|\,\omega_1\,dx \\
&\le \|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)} \|\varphi\|_{L^q(\Omega,\nu_1)} + \left(\sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\right) \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \\
&\le \left(C_{p,q}\,\|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
|\mathbf{B}(u,\varphi)| &\le |\mathbf{B_1}(u,\varphi)| + |\mathbf{B_2}(u,\varphi)| + |\mathbf{B}_3(u,\varphi)| + |\mathbf{B}_4(u,\varphi)| \\
&\le \int_\Omega |\mathcal{A}(x,u,\nabla u)||\nabla\varphi|\,\omega_1\,dx + \int_\Omega |\mathcal{B}(x,u,\nabla u)|\,|\nabla\varphi|\,\nu_1\,dx + \int_\Omega |\mathcal{H}(x,u,\nabla u)|\,\nu_2 + \int_\Omega |u|^{p-1}|\varphi|\,\omega_2\,dx.
\end{aligned} \tag{11}$$

From (11), we get by using (H4), Theorem 2.2 (with $k = 1$) and Remark 2.1,

$$\begin{aligned}
\int_\Omega |\mathcal{A}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega_1\,dx &\le \int_\Omega \left(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\right)|\nabla\varphi|\,\omega_1\,dx \\
&\le \|K_1\|_{L^{p'}(\Omega,\omega_1)} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)} \|u\|^{p/p'}_{L^p(\Omega,\omega_1)} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \\
&\quad + \|h_2\|_{L^\infty(\Omega)} \|\,|\nabla u|\,\|^{p/p'}_{L^p(\Omega,\omega_1)} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \\
&\le \left(\|K_1\|_{L^{p'}(\Omega,\omega_1)} + (C_\Omega^{1/p'}\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},
\end{aligned}$$

and by (H8) and Remark 2.2(i), we have

$$\begin{aligned}
\int_\Omega |\mathcal{B}(x,u,\nabla u)|\,|\nabla\varphi|\,\nu_1\,dx &\le \int_\Omega \left(K_2 + g_1|u|^{q/q'} + g_2|\nabla u|^{q/q'}\right)|\nabla\varphi|\,\nu_1\,dx \\
&\le \|K_2\|_{L^{q'}(\Omega,\nu_1)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu_1)} + \|g_1\|_{L^\infty(\Omega)} \|u\|^{q/q'}_{L^q(\Omega,\nu_1)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu_1)} \\
&\quad + \|g_2\|_{L^\infty(\Omega)} \|\,|\nabla u|\,\|^{q/q'}_{L^q(\Omega,\nu_1)} \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu_1)} \\
&\le C_{p,q}\|K_2\|_{L^{q'}(\Omega,\nu_1)} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} + C_{p,q}^{q-1}\|g_1\|_{L^\infty(\Omega)}\,\|u\|^{q-1}_{L^p(\Omega,\omega_1)}\,C_{p,q}\,\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \\
&\quad + \|g_2\|_{L^\infty(\Omega)} C_{p,q}^{q-1} \|\,|\nabla u|\,\|^{q-1}_{L^p(\Omega,\omega_1)} C_{p,q} \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \\
&\le \left[C_{p,q}\|K_2\|_{L^{q'}(\Omega,\nu_1)} + \left(C_{p,q}^q\|g_1\|_{L^\infty(\Omega)} + C_{p,q}^q\|g_2\|_{L^\infty(\Omega)}\right)\|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\right]\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},
\end{aligned}$$

and by (H12) and Remark 2.2(ii), we obtain

$$\begin{aligned}
\int_\Omega |\mathcal{H}(x,u,\nabla u)|\,|\varphi|\,\nu_2\,dx &\le \int_\Omega \left(K_3 + h_2|u|^{s/s'} + h_3|\nabla u|^{s/s'}\right)|\varphi|\,\nu_2\,dx \\
&\le \int_\Omega K_3\,|\varphi|\,\nu_2\,dx + \|h_2\|_{L^\infty(\Omega)} \int_\Omega |u|^{s/s'}|\varphi|\,\nu_2\,dx + \|h_3\|_{L^\infty(\Omega)} \int_\Omega |\nabla u|^{s/s'}|\varphi|\,\nu_2\,dx \\
&\le \|K_3\|_{L^{s'}(\Omega,\nu_2)}\|\varphi\|_{L^s(\Omega,\nu_2)} + \|h_2\|_{L^\infty(\Omega)}\|u\|^{s/s'}_{L^s(\Omega,\nu_2)}\|\varphi\|_{L^s(\Omega,\nu_2)} \\
&\le C_{p,s}\|K_3\|_{L^{s'}(\Omega)}\|\varphi\|_{L^p(\Omega,\omega_1)} + \|h_2\|_{L^\infty(\Omega)} C_{p,s}^{s-1}\|u\|^{s-1}_{L^p(\Omega,\omega_1)} C_{p,s}\|\varphi\|_{L^p(\Omega,\omega_1)} \\
&\quad + \|h_3\|_{L^\infty(\Omega)} C_{p,s}^{s-1}\|\,|\nabla u|\,\|^{s-1}_{L^p(\Omega,\omega_1)} C_{p,s}\|\varphi\|_{L^p(\Omega,\omega_1)} \\
&\le \left[C_{p,s}\,\|K_3\|_{L^{s'}(\Omega,\nu_2)} + C_{p,s}^s(\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)})\|u\|^{s-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\right]\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}, \ \text{ and}
\end{aligned}$$

$$\int_\Omega |u|^{p-1}|\varphi|\,\omega_2\,dx \le \left(\int_\Omega |u|^p\,\omega_2\,dx\right)^{1/p'}\left(\int_\Omega |\varphi|^p\,\omega_2\,dx\right)^{1/p} \le C_\Omega\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Hence, by (11), for all $u, \varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ it holds that

$$|\mathbf{B}(u,\varphi)| \leq \Big[\|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)} \|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} + C_{p,q} \|K_2\|_{L^{q'}(\Omega,\nu_1)} + C^q_{p,q}\big(\|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}\big) \|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} + C_{p,s}\|K_3\|_{L^{s'}(\Omega,\nu_2)} + C^s_{p,s}(\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)}) \|u\|^{s-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} + C_\Omega \|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \Big] \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Since $\mathbf{B}(u,.)$ is linear, for each $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, there exists a linear and continuous functional on $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ denoted by Au such that $(Au|\varphi) = \mathbf{B}(u, \varphi)$ for all $u, \varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ (here $(f|x)$ denotes the value of the linear functional f at the point x). Moreover, we have

$$\|Au\|_* \leq \|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)} \|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} + C_{p,q} \|K_2\|_{L^{q'}(\Omega,\nu_1)} + C^q_{p,q}\big(\|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}\big) \|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} + C_{p,s}\|K_3\|_{L^{s'}(\Omega,\nu_2)} + C^s_{p,s}(\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)}) \|u\|^{s-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} + C_\Omega \|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

where $\|Au\|_* = \sup\{|(Au|\varphi)| = |B(u,\varphi)| : \varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2), \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} = 1\}$ is the norm of the operator Au. Hence, we obtain the operator

$$A : W_0^{1,p}(\Omega, \omega_1, \omega_2) \to [W_0^{1,p}(\Omega, \omega_1, \omega_2)]^*$$
$$u \mapsto Au.$$

Consequently, problem (P) is equivalent to the operator equation $Au = \mathbf{T}$, $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$.

Step 6. Using (H2), (H6), (H10) and Proposition 2.1(b), we obtain (for $u_1, u_2 \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, $u_1 \neq u_2$)

$$\begin{aligned}(Au_1 - Au_2|u_1 - u_2) &= \mathbf{B}(u_1, u_1 - u_2) - \mathbf{B}(u_2, u_1 - u_2)\\
&= \int_\Omega \langle \mathcal{A}(x, u_1 \nabla u_1), \nabla(u_1 - u_2)\rangle\, \omega_1\, dx + \int_\Omega \langle \mathcal{B}(x, u_1, \nabla u_1), \nabla(u_1 - u_2)\rangle\, \nu_1\, dx\\
&+ \int_\Omega \mathcal{H}(x, u_1, \nabla u_1)(u_1 - u_2)\, \nu_2\, dx + \int_\Omega |u_1|^{p-2}\, u_1\, (u_1 - u_2)\, \omega_2\, dx\\
&- \int_\Omega \langle \mathcal{A}(x, u_2, \nabla u_2), \nabla(u_1 - u_2)\rangle\, \omega_1\, dx - \int_\Omega \langle \mathcal{B}(x, u_2, \nabla u_2, \nabla(u_1 - u_2)\rangle\, \nu_1\, dx\\
&- \int_\Omega \mathcal{H}(x, u_2, \nabla u_2)(u_1 - u_2)\, \nu_2\, dx - \int_\Omega |u_2|^{p-2}\, u_2\, (u_1 - u_2)\, \omega_2\, dx\\
&= \int_\Omega \langle \mathcal{A}(x, u_1 \nabla u_1) - \mathcal{A}(x, u_2, \nabla u_2), \nabla(u_1 - u_2)\rangle\, \omega_1\, dx\\
&+ \int_\Omega \langle \mathcal{B}(x, u_1, \nabla u_1) - \mathcal{B}(x, u_2, \nabla u_2), \nabla(u_1 - u_2)\rangle\, \nu_1\, dx\\
&+ \int_\Omega \Big(\mathcal{H}(x, u_1, \nabla u_1) - \mathcal{H}(x, u_2, \nabla u_2)\Big)(u_1 - u_2)\, \nu_2\, dx + \int_\Omega (|u_1|^{p-2}\, u_1 - |u_2|^{p-2}\, u_2)\, (u_1 - u_2)\, \omega_2\, dx\\
&\geq \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^p\, \omega_1\, dx + \beta_p \int_\Omega \big(|u_1| + |u_2|\big)^{p-2} |u_1 - u_2|^2\, \omega_2\, dx\\
&\geq \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^p\, \omega_1\, dx + \beta_p \int_\Omega |u_1 - u_2|^{p-2}\, |u_1 - u_2|^2\, \omega_2\, dx\\
&\geq \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^p\, \omega_1\, dx + \beta_p \int_\Omega |u_1 - u_2|^p\, \omega_2\, dx \geq \gamma_1\, \|u_1 - u_2\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},\end{aligned}$$

where $\gamma_1 = \min\{\theta_1, \beta_p\}$. Therefore, the operator A is strictly monotone. Moreover, from (H3),(H7), (H11), we obtain

$$\begin{aligned}(Au|u) &= \mathbf{B}(u,u) = \mathbf{B_1}(u,u) + \mathbf{B_2}(u,u) + \mathbf{B}_3(u,u) + \mathbf{B}_4(u,u)\\
&= \int_\Omega \langle \mathcal{A}(x,u,\nabla u), \nabla u\rangle\, \omega_1\, dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla u\rangle\, \nu_1\, dx + \int_\Omega \mathcal{H}(x,u,\nabla u)\, u\, \nu_2\, dx + \int_\Omega |u|^p\, \omega_2\, dx\\
&\geq \lambda_1 \int_\Omega |\nabla u|^p\, \omega_1\, dx + \lambda_2 \int_\Omega |\nabla u|^q\, \nu_1\, dx + \Lambda_2 \int_\Omega |u|^q\, \nu_1\, dx + \lambda_3 \int_\Omega |\nabla u|^s\, \nu_2\, dx + \Lambda_3 \int_\Omega |u|^s\, \nu_2\, dx + \int_\Omega |u|^p\, \omega_2\, dx\\
&\geq \lambda_1 \int_\Omega |\nabla u|^p\, \omega_1\, dx + \int_\Omega |u|^p\, \omega_2\, dx \geq \gamma_2\, \|u\|^p_{W_0^{1,p}(\Omega,\omega_1),\omega_2},\end{aligned}$$

where $\gamma_2 = \min\{\lambda_1, 1\}$. Hence, since $1 < q, s < p < \infty$, we have

$$\frac{(Au|u)}{\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}} \to +\infty, \text{ as } \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \to +\infty,$$

that is, A is coercive.

Step 7. We need to show that the operator A is continuous. Let $u_m \to u$ in X as $m \to \infty$. We have,

$$\begin{aligned}|\mathbf{B_1}(u_m,\varphi)-\mathbf{B_1}(u,\varphi)| &\le \sum_{j=1}^{n}\int_\Omega |\mathcal{A}_j(x,u_m,\nabla u_m)-\mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\,\omega_1\,dx = \sum_{j=1}^{n}\int_\Omega |F_ju_m-F_ju||D_j\varphi|\,\omega_1\,dx\\ &\le \left(\sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_1)}\right)\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \le \left(\sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_1)}\right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},\end{aligned}$$

and by Remark 2.2(i), we get

$$\begin{aligned}|\mathbf{B_2}(u_m,\varphi)-\mathbf{B_2}(u,\varphi)| &\le \sum_{j=1}^{n}\int_\Omega |\mathcal{B}_j(x,u_m,\nabla u_m)-\mathcal{B}_j(x,u,\nabla u)||D_j\varphi|\,\nu_1\,dx = \sum_{j=1}^{n}\int_\Omega |G_ju_m-G_ju||D_j\varphi|\,\nu_1\,dx\\ &\le \left(\sum_{j=1}^{n}\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\nu_1)}\right)\|\,|\nabla\varphi|\,\|_{L^q(\Omega,\nu_1)}\\ &\le C_{p,q}\left(\sum_{j=1}^{n}\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\nu_1)}\right)\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)}\\ &\le C_{p,q}\left(\sum_{j=1}^{n}\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\nu_1)}\right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},\end{aligned}$$

and by Remark 2.2(ii), we have

$$\begin{aligned}|\mathbf{B}_3(u_m,\varphi)-\mathbf{B}_3(u,\varphi)| &\le \int_\Omega |\mathcal{H}(x,u_m,\nabla u_m)-\mathcal{H}(x,u,\nabla u)|\,|\varphi|\,\nu_2\,dx = \int_\Omega |Hu_m-Hu|\,|\varphi|\,\nu_2\,dx\\ &\le \|Hu_m-Hu\|_{L^{s'}(\Omega,\nu_2)}\,\|\varphi\|_{L^s(\Omega,\nu_2)} \le C_{p,s}\,\|Hu_m-Hu\|_{L^{s'}(\Omega,\nu_2)}\,\|\varphi\|_{L^p(\Omega,\omega_1)}\\ &\le C_{p,s}\,\|Hu_m-Hu\|_{L^{s'}(\Omega,\nu_2)}\,\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},\end{aligned}$$

and by Step 4, we get

$$\begin{aligned}|\mathbf{B}_4(u_m,\varphi)-\mathbf{B}_4(u,\varphi)| &\le \int_\Omega |\,|u_m|^{p-2}u_m-|u|^{p-2}u\,|\,|\,\varphi|\,\omega_2\,dx = \int_\Omega |Ju_m-Ju|\,|\varphi|\,\omega_2\,dx\\ &\le \|Ju_m-Ju\|_{L^{p'}(\Omega,\omega_2)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},\end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$. Hence,

$$\begin{aligned}|\mathbf{B}(u_m,\varphi)-\mathbf{B}(u,\varphi)| &\le |\mathbf{B_1}(u_m,\varphi)-\mathbf{B_1}(u,\varphi)|+|\mathbf{B_2}(u_m,\varphi)-\mathbf{B_2}(u,\varphi)|\\ &\quad+|\mathbf{B_3}(u_m,\varphi)-\mathbf{B_3}(u,\varphi)|+|\mathbf{B_4}(u_m,\varphi)-\mathbf{B_4}(u,\varphi)|\\ &\le \Bigg[\sum_{j=1}^{n}\left(\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_1)}+C_{p,q}\,\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\nu_1)}\right)\\ &\quad+C_{p,s}\|Hu_m-Hu\|_{L^{s'}(\Omega,\nu_2)}+\|Ju_m-Ju\|_{L^{p'}(\Omega,\omega_2)}\Bigg]\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\|Au_m-Au\|_* &\le \sum_{j=1}^{n}\left(\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_1)}+C_{p,q}\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\nu_1)}\right)\\ &\quad+C_{p,s}\|Hu_m-Hu\|_{L^{s'}(\Omega,\nu_2)}+\|Ju_m-Ju\|_{L^{p'}(\Omega,\omega_2)}.\end{aligned}$$

Hence, using (5), (7), (9) and (10) we have $\|Au_m-Au\|_* \to 0$ as $m \to +\infty$, that is, A is continuous and this implies that A is hemicontinuous. Therefore, by Theorem 3.1, the operator equation $Au = \mathbf{T}$ has a unique solution $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ and it is the unique solution for problem (P).

Step 8. Estimates for $\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}$. In particular, by setting $\varphi = u$ in Definition 2.3, we have

$$\mathbf{B}(u,u)=\mathbf{B_1}(u,u)+\mathbf{B_2}(u,u)+\mathbf{B}_3(u,u)+\mathbf{B}_4(u,u)=\mathbf{T}(u). \tag{12}$$

Hence, using (H3),(H7), (H11) and Remark 2.2(ii) we obtain

$$\mathbf{B_1}(u,u)+\mathbf{B_2}(u,u)+\mathbf{B}_3(u,u)+\mathbf{B_4}(u,u)=\int_\Omega \langle \mathcal{A}(x,u,\nabla u),\nabla u\rangle\,\omega_1\,dx+\int_\Omega \langle \mathbf{B}(x,u,\nabla u),\nabla u\rangle\,\nu_1\,dx$$

$$
\begin{aligned}
&+\int_\Omega H(x,u,\nabla u)\,u\,\nu_2\,dx+\int_\Omega |u|^{p-2}\,u^2\,\omega_2\\
&\geq \lambda_1\int_\Omega |\nabla u|^p\,\omega_1\,dx+\lambda_2\int_\Omega |\nabla u|^q\,\nu_1\,dx+\Lambda_2\int_\Omega |u|^q\,\nu_1\,dx\\
&+\lambda_3\int_\Omega |\nabla u|^s\,\nu_2\,dx+\Lambda_3\int_\Omega |u|^s\,\nu_2\,dx+\int_\Omega |u|^p\,\omega_2\,dx\\
&\geq \lambda_1\int_\Omega |\nabla u|^p\,\omega_1\,dx+\int_\Omega |u|^p\,\omega_2\,dx\geq \gamma_2\,\|u\|^p_{W^{1,p}_0(\Omega,\omega_1,\omega_2)},
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
\mathbf{T}(u)&=\int_\Omega f_0\,u\,dx+\sum_{j=1}^n\int_\Omega f_j\,D_ju\,dx\leq \|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)}\|u\|_{L^q(\Omega,\nu_1)}+\left(\sum_{j=1}^n\|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\right)\|\,|\nabla u|\,\|_{L^p(\Omega,\omega_1)}\\
&\leq\left(C_{p,q}\,\|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)}+\sum_{j=1}^n\|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\right)\|u\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}=M\,\|u\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)},
\end{aligned}
\tag{14}
$$

where $M=C_{p,q}\,\|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)}+\sum_{j=1}^n\|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}$. Hence, using (13) and (14) in (12), we obtain

$$
\gamma_2\,\|u\|^p_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\leq M\,\|u\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}.
$$

Therefore,

$$
\|u\|_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\leq\left(\frac{M}{\gamma_2}\right)^{1/(p-1)}=C\left(C_{p,q}\,\|f_0/\nu_1\|_{L^{q'}(\Omega,\nu_1)}+\sum_{j=1}^n\|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\right)^{1/(p-1)},
$$

where $C=(1/\gamma_2)^{1/(p-1)}$.

Example 3.1. *Take $\Omega=\{(x,y)\in\mathbb{R}^2: x^2+y^2<1\}$ and consider the weight functions $\omega_1(x,y)=(x^2+y^2)^{-1/2}$, $\omega_2(x,y)=(x^2+y^2)^{-3/2}$, $\nu_1(x,y)=(x^2+y^2)^{-1/3}$ and $\nu_2(x,y)=(x^2+y^2)^{-1}$ ($\omega_1,\omega_2\in A_4$, $p=4$, $q=3$ and $s=2$) and the function $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$ defined by $\mathcal{A}((x,y),\eta,\xi)=h_1(x,y)\,|\xi|^2\,\xi$, where $h_1(x,y)=2\,\mathrm{e}^{(x^2+y^2)}$; and $\mathcal{B}:\Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$ defined by $\mathcal{B}((x,y),\eta,\xi)=g_2(x,y)\,|\xi|\,\xi$, where $g_2(x,y)=2+\cos(x^2+y^2)$; and $\mathcal{H}:\Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ defined by $\mathcal{H}((x,y),\eta,\xi)=\eta\,h_2(x,y)$, where $h_2(x,y)=1+\cos^2(xy)$. Let us consider the partial differential operator*

$$
Lu(x,y)=-\mathrm{div}\,\big(\mathcal{A}((x,y),\nabla u)\,\omega_1(x,y)+\mathcal{B}((x,y),u,\nabla u)\,\nu_1(x,y)\big)+\mathcal{H}((x,y),u,\nabla u)\,\nu_2(x,y)+|u|^2\,u\,\omega_2(x,y).
$$

Therefore, by Theorem 1.1, the problem

$$
(P)\begin{cases} Lu(x) &=\dfrac{\cos(xy)}{(x^2+y^2)}-\dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{(x^2+y^2)}\right)-\dfrac{\partial}{\partial y}\left(\dfrac{\sin(xy)}{(x^2+y^2)}\right)\ \text{in}\ \Omega,\\ u(x) &=0\ \text{on}\ \partial\Omega,\end{cases}
$$

has a unique solution $u\in W^{1,4}_0(\Omega,\omega_1,\omega_2)$.

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