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Associated filters of quasi–ordered residuated systems

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Abstract

The concept of residuated relational systems ordered under a quasi-order relation (in short: QRS) was introduced by Bonzio and Chajda [*Asian–Eur. J. Math.* **11** (2018) Art# 1850024] as a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $(A, \cdot)$ is a commutative monoid with the identity 1 as the top element in this ordered monoid under a quasi-order R. The author has recently introduced and analyzed the concept of filters in the foregoing type of algebraic structures. In this article, as a continuation of the previous research on QRS, the concept of associated filters of QRS is proposed and analyzed.

Keywords: Quasi-ordered residuated system (QRS); filter of QRS; associated filter of QRS.

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1. Introduction

Let $(A, \cdot, 1)$ be a commutative semigroup with the identity 1. Suppose that, on the carrier A, there exists another operation $\rightarrow$ and one relation R that with the multiplication in A have a link $(x \cdot y, z) \in R \iff (x, y \rightarrow z) \in R$ for all $x, y, z \in A$. A relational system designed in this way, when R is a quasi-ordered relation on A, is in the focus of this paper.

The concept of residuated relational systems ordered under a quasi-order relation (in short: QRS) was introduced by Bonzio and Chajda [2] in 2018. Previously, this concept was discussed in [1]. Maddux suggests that the text [7] written by Tarski in 1941, is probably one of the first articles which relates to 'The calculus of relations' (see [4], page 438).

The approach outlined in [7] is worked out in more detail in [8]. In addition, according to Madduox, the first definition of relation algebras appears in [3] (cited by [4], page 441).

This paper continues the investigations of quasi-ordered residuated systems and of their filters, that were started in the articles [5, 6]. The author dealt with implicative filters in this algebraic system in the paper [6]. In particular, the concept of associated filters of a QRS is introduced here in the present paper. Also, some conditions for a filter of such a system to be an associated filter are found.

2. Preliminaries

2.1 Concept of quasi–ordered residuated systems

In the article [2], Bonzio and Chajda introduced and analyzed the concept of 'residual relational systems'.

Definition 2.1 ([2], Definition 2.1)**.** *A* residuated relational system *is a structure* $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$*, where* $\langle A, \cdot, \rightarrow, 1\rangle$ *is an algebra of type* $\langle 2, 2, 0\rangle$ *and* R *is a binary relation on* A *and satisfying the following properties:*

(1). $(A, \cdot, 1)$ *is a commutative monoid;*
(2). $(\forall x \in A)((x, 1) \in R)$*;*
(3). $(\forall x, y, z \in A)((x \cdot y, z) \in R \iff (x, y \rightarrow z) \in R)$.

We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

The basic properties of the residuated relational systems are subsumed in the following theorem, where we continue the numbering mentioned in Definition 2.1.

Theorem 2.1 ([2], Proposition 2.1)**.** *Let* $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ *be a residuated relational system. Then*

(4). $(\forall x, y \in A)(x \rightarrow y = 1 \Longrightarrow (x, y) \in R)$*,*
(5). $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$*,*

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(6). $(\forall x \in A)((1, x \to 1) \in R)$,
(7). $(\forall x, y, z \in A)(x \to y = 1 \Longrightarrow (z \cdot x, y) \in R)$,
(8). $(\forall x, y \in A)((x, y \to 1) \in R)$.

Recall that a *quasi-order relation* '$\preccurlyeq$' on a set A is a binary relation which is reflexive and transitive (some authors use the term pre-order relation).

Definition 2.2 ([2], Definition 3.1)**.** *A* quasi-ordered residuated system *is a residuated relational system* $\mathfrak{A} = \langle A, \cdot, \to, 1, \preccurlyeq\rangle$, *where* $\preccurlyeq$ *is a quasi-order relation in the monoid* $(A, \cdot)$.

The following proposition shows the basic properties of quasi-ordered residuated systems, and in this proposition we continue the numbering specified in Theorem 2.1.

Proposition 2.1 ([2], Proposition 3.1)**.** *Let A be a quasi-ordered residuated system. Then*
(9). $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y))$,
(10). $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (y \to z \preccurlyeq x \to z \wedge z \to x \preccurlyeq z \to y))$,
(11). $(\forall x, y \in A)(x \cdot y \preccurlyeq x \wedge x \cdot y \preccurlyeq y)$.

Estimating that this topic is interesting ([1,2,5,6]), it is certain that there is interest in the development of the concept of some substructures and processes in these systems.

Let $L(a) = \{y \in A : a \preccurlyeq y\}$ be the left class and $R(b) = \{x \in A : x \preccurlyeq b\}$ be the right class of the relation $\preccurlyeq$ generated by the elements a and b respectively. Then $R(1) = A$. Some authors use the notation $U(a)$ instead of $L(a)$ (see, for example [2]).

Example 2.1. *Let $A = \{1, 2, 3, 4\}$ and the operations '$\cdot$' and '$\to$' be defined on A as follows:*

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	4	2	4
3	3	2	2	4
4	4	4	4	4

and

$\to$	1	2	3	4
1	1	2	3	4
2	1	1	2	3
3	1	1	1	3
4	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated system, where the relation '$\preccurlyeq$' is defined as $4 \preccurlyeq 3 \preccurlyeq 2 \preccurlyeq 1$.

Example 2.2. *For a commutative monoid A, denote by $\mathfrak{P}(A)$ the powerset of A ordered by set inclusion, and let '$\cdot$' be the usual multiplication of subsets of A. Then, $\langle \mathfrak{P}(A), \cdot, \to, \{1\}, \subseteq\rangle$ is a quasi-ordered residuated system in which the residuum are given by*

$$(\forall X, Y \in \mathfrak{P}(A))(Y \to X := \{z \in A : Yz \subseteq X\}).$$

Example 2.3. *Any commutative residuated lattice, where R is a lattice quasi-order, is a quasi-ordered residuated system.*

2.2 Concepts of filters

In the article [5], in order to determine the concept of filters in quasi-ordered residuated systems, the relationships between the following conditions are analyzed:
(F0). $1 \in F$;
(F1). $(\forall u, v \in A)((u \cdot v \in F \Longrightarrow (u \in F \wedge v \in F))$;
(F2). $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \Longrightarrow v \in F)$; and
(F3). $(\forall u, v \in A)((u \in F \wedge u \to v \in F) \Longrightarrow v \in F)$.

It was shown, in Proposition 3.2 of the paper [5], that (F2) $\Longrightarrow$ (F1). Also, it was shown, in Proposition 3.4 of the article [5], that (F2) $\Longrightarrow$ (F0) is valid for every nonempty subset of F of the system $\mathfrak{A}$.

Proposition 2.2 ([5], Proposition 3.6)**.** *Let F be a submonoid of the monoid $(A, \cdot)$ in a quasi-ordered residuated system $\mathfrak{A} = \langle (A, \cdot, \to, 1, \preccurlyeq\rangle$. Then,* (F2) $\Longrightarrow$ (F3).

Based on our previous analysis of the interrelationship between the conditions (F1), F(2) and (F3) in a quasi-ordered residual system, we introduced the concept of filters.

Definition 2.3 ([5], Definition 3.1)**.** *For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$, we say that it is a* filter *of $\mathfrak{A}$ if it satisfies the conditions* (F2) *and* (F3).

Example 2.4. *Let $A = \{1, 2, 3, 4\}$ and the operations '$\cdot$' and '$\to$' be defined on A as follows:*

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	2	2	4
3	3	2	2	4
4	4	4	4	4

and

$\to$	1	2	3	4
1	1	2	3	4
2	1	1	1	1
3	1	2	1	4
4	1	2	3	1

Then, $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ *is a quasi-ordered residuated systems, where the relation '$\preccurlyeq$' is defined as follows*

$$\preccurlyeq = \{(1,1),(2,2),(3,3),(4,4),(4,1),(3,1),(2,1),(2,3),(2,4)\}.$$

Then, the subsets $\{1\}$, $\{1,3\}$ *and* $\{1,3,4\}$ *are filters of* $\mathfrak{A}$.

3. Concept of associated filters

In this section, we introduce the concept of associated filters of quasi-ordered residuared systems and analyze it.

Definition 3.1. *Let* $\mathfrak{A}$ *be a QRS and* x *be a fixed element of* A*. A non-empty subset* F *of* A *is called an* associated *filter of* $\mathfrak{A}$ with respect to x *(briefly,* x-associated filter *of* $\mathfrak{A}$*) if it satisfies the condition* (F2) *as well as the following condition:*

(FA). $(\forall y, z \in A)((x \to (y \to z) \in F \ \wedge\ x \to y \in F) \implies z \in F)$.

By an associated filter *of* $\mathfrak{A}$*, we mean an* x*-associated filter of* $\mathfrak{A}$ *for all* x *in* A.

It is immediately seen that $1 \in F$ and F satisfies the condition **(F1)** because F satisfies the condition **(F2)** and F is a non-empty subset.

Example 3.1. *Let* $A = \{1,2,3,4\}$ *and the operations '$\cdot$' and '$\to$' be defined on* A *as follows:*

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

and

$\to$	1	2	3	4
1	1	2	3	4
2	1	1	2	3
3	1	1	1	4
4	1	1	1	1

Then, $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ *is a quasi-ordered residuated systems, where the relation '$\preccurlyeq$' is defined as follows*

$$\preccurlyeq = \{(1,1),(2,2),(3,3),(4,4),(4,1),(3,1),(2,1),(3,2),(4,2),(4,3)\}.$$

Then, $\{1\}$ *is a* 1*-associated filter of* $\mathfrak{A}$*. The subset* $F := \{1,2,3\}$ *is an associated filter of* $\mathfrak{A}$ *with respect to* 1*,* 2 *and* 3*, but it is not a* 4*-associated filter of* $\mathfrak{A}$ *because* $4 \to (4 \to 4) = 4 \to 1 = 1 \in F$ *and* $4 \to 4 = 1 \in F$ *but* $4 \notin F$.

Theorem 3.1. *For any* $x \in A$*, every* x*-associated filter of a QRS* $\mathfrak{A}$ *contains* x *itself.*

Proof. For any $x \in A$, let F be an x-associated filter of $\mathfrak{A}$. Then

$$(1 \to (1 \to x) \in F \ \wedge\ 1 \to x \in F) \implies x \in F$$

by (FA). Hence $x \in F$. □

Remark 3.1. *Theorem 3.1 suggests that there are no proper associated filters in quasi-ordered residuated systems. So, the only associated filter of a QRS* $\mathfrak{A}$ *is* A.

Theorem 3.2. *Let* $\mathfrak{A}$ *be a QRS. For any* $x \in A$*, if* F *is a filter of* $\mathfrak{A}$ *which contains* x*, then it is an* x*-associated filter of* $\mathfrak{A}$.

Proof. Take $y, z \in A$ such that $x \to (y \to z) \in F$ and $x \to y \in F$. Since F is a filter of $\mathfrak{A}$, it follows that $y \to z \in F$ and $y \in F$ by the condition (F3). Then, $z \in F$ by the condition (F3) again. Therefore, F is an x-associated filter of $\mathfrak{A}$. □

In what follows, we need the following lemma, and in this lemma we continue the numbering specified in Proposition 2.1.

Lemma 3.1. *Let* F *be a subset of a quasi-ordered residuated system* $\mathfrak{A}$ *satisfying the condition* (F2)*. Then, it holds that*

(12). $(\forall u \in A)(u \in F \iff 1 \to u \in F)$.

Proof. Since $(\forall x \in A)(1 \to x \preccurlyeq x)$ and $(\forall x \in A)(x \preccurlyeq 1 \to x)$, by Proposition 2.3(d) of [2], the proof of this lemma follows from the condition (F2). □

Theorem 3.3. *Let $\mathfrak{A}$ be a QRS. Every* 1*-associated filter of* $\mathfrak{A}$ *is a filter of* $\mathfrak{A}$*, and vice versa.*

Proof. Let F be a 1-associated filter of $\mathfrak{A}$ and take $x, y \in A$ such that $x \to y \in F$ and $x \in F$. Then, $1 \to x \in F$ and $1 \to (x \to y) \in F$ by Lemma 3.1. Now, from this, it follows that $y \in F$ by the condition (FA). So, F is a filter of $\mathfrak{A}$.

Conversely, let F be a filter of $\mathfrak{A}$. Assume that $1 \to (x \to y) \in F$ and $1 \to x \in F$ for all $x, y \in A$. Then, $x \to y \in F$ and $x \in F$ by Lemma 3.1, which imply $y \in F$ by the condition (F3). Thus, F is a 1-associated filter of $\mathfrak{A}$. □

Theorem 3.4. *Let F be a filter of a QRS $\mathfrak{A}$ and take $x, x' \in A$ such that $x \preccurlyeq x'$. If F is an x-associated filter of $\mathfrak{A}$, then it is a x'-associated filter of $\mathfrak{A}$.*

Proof. Take $x, x' \in A$ such that $x \preccurlyeq x'$ and let F be an x-associated filter of $\mathfrak{A}$. Take $y, z \in A$ such that $x' \to (y \to z) \in F$ and $x' \to y \in F$. Since $a \preccurlyeq x'$, it follows that $x' \to (y \to z) \preccurlyeq x \to (y \to z)$ and $x' \to y \preccurlyeq x \to y$ by (10). Using the condition (F2), we have $x \to (y \to z) \in F$ and $x \to y \in F$. From there, it follows that $z \in F$ by the condition (FA). Therefore, F is a x'-associated filter of $\mathfrak{A}$. □

In the following two theorems, we show some conditions for the filter F of a QRS $\mathfrak{A}$ to be a x-associated filter of $\mathfrak{A}$ for a given $x \in A$. In the remaining results of this section, we continue the numbering specified in Lemma 3.1, in a sequence.

Theorem 3.5. *Let F be a filter of a QRS $\mathfrak{A}$. If F satisfies the following additional condition:*

(13). $(\forall y, z \in A)(x \to (y \to z) \in F \implies (x \to y) \to z \in F)$,

then F is an x-associated filter of $\mathfrak{A}$.

Proof. If F is a filter of $\mathfrak{A}$ that satisfies the condition (13), then

$$(x \to y) \to z \in F \wedge x \to y \in F) \implies z \in F$$

by the condition (F3). Thus, F is a x-associated filter of $\mathfrak{A}$. □

In what follows, we need the following two lemmas.

Lemma 3.2 ([5], Proposition 3.3)**.** *Let F be a filter of a QRS $\mathfrak{A}$. Then,*

(14). $(\forall u, v \in A)(u \preccurlyeq v \implies u \to v \in F)$.

Lemma 3.3 ([6], Lemma 3.4)**.** *Let F be a filter of a QRS $\mathfrak{A}$. Then, it holds that*

(15). $(\forall u, v, z \in A)(u \to (v \to z) \in F \iff v \to (u \to z) \in F)$.

Theorem 3.6. *Let F be a filter of a QRS $\mathfrak{A}$. If F satisfies the additional condition*

(16). $(\forall z \in A)(x \to (x \to z) \in F \implies z \in F)$,

then F is an x-associated filter of $\mathfrak{A}$.

Proof. Let $x, y, z \in A$ be arbitrary elements. Let us prove the condition (FA). Suppose that $(x \to (y \to z) \in F$ and $x \to y \in F$. From

$$x \to (y \to z) \preccurlyeq x \to (y \to z),$$

it follows that

$$((y \to z) \to ((x \to y) \to (x \to z))) \preccurlyeq (x \to ((x \to y) \to (x \to z)))$$

by (10). Thus,

(a). $((y \to z) \to ((x \to y) \to (x \to z))) \to (x \to ((x \to y) \to (x \to z))) \in F$

by (14). On the other side, since from

$$y \to z \preccurlyeq (x \to y) \to (x \to z),$$

it follows that

$$(y \to z) \to ((x \to y) \to (x \to z)) \in F$$

by (14), we from (a) conclude that

$$(x \to ((x \to y) \to (x \to z))) \in F$$

by the condition (F3). The latter is equivalent to

$$(x \to y) \to (x \to (x \to z)) \in F$$

according to (15). Hence, from $x \to y \in F$ it follows that $(x \to (x \to z)) \in F$ according to (F3). Taking into account hypothesis (16), we get $z \in F$ from here. Thus, F is an x-associated filter in $\mathfrak{A}$. □

Remark 3.2. *The converse of Theorem 3.6 is obvious because the implication (16) can be obtained from (FA) by choosing* $y = z$ *and* $z = y$*.*

4. Further work

It seems that designing some new types of filters in quasi-ordered residuated systems and investigating their properties could be of interest to a wider academic audience. For example, it would be interesting to explore a substructure that satisfies the following condition

$$(\forall x, y, z \in A)((x \to (y \to z) \in F \wedge x \in F) \implies ((z \to y) \to y) \to z \in F)$$

instead of the condition (F3).

References

[1] S. Bonzio, *Algebraic structures from quantum and fuzzy logics*, Ph.D. Thesis, Università degli Studi di Cagliari, Cagliari, 2015.
[2] S. Bonzio, I. Chajda, Residuated relational systems, *Asian–Eur. J. Math.* **11** (2018) Art# 1850024.
[3] B. Jónsson, A. Tarski, Representation problems for relation algebras (Abstract 89T), *Bull. Amer. Math. Soc.* **54** (1948) pp. 80 and pp. 1192.
[4] R. D. Maddux, The origin of relation algebras in the development and axiomatization of the calculus of relations, *Studia Logica* **50** (1991) 421–455.
[5] D. A. Romano, Filters in residuated relational system ordered under quasi-order, *Bull. Int. Math. Virtual Inst.* **10** (2020) 529–534.
[6] D. A. Romano, Implicative filters in quasi-ordered residuated system, *Proyecciones*, To appear.
[7] A. Tarski, On the calculus of relations, *J. Symb. Log.* **6** (1941) 73–89.
[8] A. Tarski, S. Givant, *A Formalization of Set Theory Without Variables*, Amer. Math. Soc., Providence, 1987.