

Weak solutions for a system of quasilinear elliptic equations

Maria Alessandra Ragusa^{1,2,*}, Abdolrahman Razani³

¹Dipartimento di Matematica e Informatica, Università di Catania, Catania 95125, Italy

²Faculty of Science, RUDN University, Moscow 117198, Russia

³Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin 34149-16818, Iran

(Received: 5 June 2020. Accepted: 1 August 2020. Published online: 4 August 2020.)

© 2020 the authors. This is an open access article under the CC BY (International 4.0) license (www.creativecommons.org/licenses/by/4.0/).

Abstract

A system of quasilinear elliptic equations on an unbounded domain is considered. The existence of a sequence of radially symmetric weak solutions is proved via variational methods.

Keywords: sequence of solutions, elliptic problem, p -Laplacian, variational methods.

2020 Mathematics Subject Classification: 34B10, 35J20, 35J50.

1. Introduction

We consider the following problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda\alpha_1(x)f_1(v) & \text{in } \mathbb{R}^N, \\ -\Delta_q v + |v|^{q-2}v = \lambda\alpha_2(x)f_2(u) & \text{in } \mathbb{R}^N, \\ u, v \in W^{1,p}(\mathbb{R}^N), \end{cases} \quad (1)$$

where $p, q > N > 1$. We assume that $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, $\alpha_1(x), \alpha_2(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ are nonnegative (not identically zero) radially symmetric maps, and λ is a real parameter. Also $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p -Laplacian operator.

Partial differential equations are used to model a wide variety of physically significant problems arising in different areas such as physics, engineering and other applied disciplines (see [7, 11, 12, 18, 25, 26, 28–31, 34–45]). Sobolev spaces play an important role in the theory of partial differential equations as well as Orlicz-Morrey space and $\dot{B}_{\infty, \infty}^{-1}$ space (see [2, 8–10, 32–34]). Laplace equation is the prototype for linear elliptic equations. This equation has a non-linear counterpart, the so-called p -Laplace equation (see [1, 6, 13, 14, 19, 21–24, 48]).

Here, by inspiration of [20], we prove the existence of a sequence of radially symmetric weak solutions for (1) in the unbounded domain \mathbb{R}^N .

The solution of (1) belongs to the product space

$$W^{1,(p,q)}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N) \times W^{1,q}(\mathbb{R}^N)$$

equipped with the norm $\|(u, v)\|_{(p,q)} = \|u\|_p + \|v\|_q$.

Definition 1.1. For fixed λ , $(u, v) : \mathbb{R}^N \rightarrow \mathbb{R}$ is said to be a weak solution of (1), if $(u, v) \in W^{1,(p,q)}(\mathbb{R}^N)$ and for every $(z, w) \in W^{1,(p,q)}(\mathbb{R}^N)$

$$\begin{aligned} & - \int_{\mathbb{R}^N} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla z(x) dx - \int_{\mathbb{R}^N} |\nabla v(x)|^{q-2} \nabla v(x) \cdot \nabla w(x) dx + \int_{\mathbb{R}^N} |u(x)|^{p-2} u(x) z(x) dx + \int_{\mathbb{R}^N} |v(x)|^{q-2} v(x) w(x) dx \\ & - \lambda \int_{\mathbb{R}^N} \alpha_1(x) f_1(v(x)) z(x) dx - \lambda \int_{\mathbb{R}^N} \alpha_2(x) f_2(u(x)) w(x) dx = 0, \end{aligned}$$

where

$$\|(u, v)\|_{W^{1,(p,q)}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx + \int_{\mathbb{R}^N} |u(x)|^p dx \right) + \left(\int_{\mathbb{R}^N} |\nabla v(x)|^q dx + \int_{\mathbb{R}^N} |v(x)|^q dx \right).$$

*Corresponding author (maragusa@dmi.unict.it)

Note that the critical points of a energy functional are exactly the weak solutions of (1).

Morrey’s theorem, implies the continuous embedding

$$W^{1,(p,q)}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N), \tag{2}$$

which says that there exists c (depends on p, q, N), such that $\|(u, v)\|_\infty \leq c\|(u, v)\|_{W^{1,(p,q)}(\mathbb{R}^N)}$, for every $(u, v) \in W^{1,p}(\mathbb{R}^N) \times W^{1,q}(\mathbb{R}^N)$, where $\|(u, v)\|_\infty := \max\{\|u\|_\infty, \|v\|_\infty\}$. Since in the low-dimensional case, every function $(u, v) \in W^{1,(p,q)}(\mathbb{R}^N)$ admits a continuous representation (see [4, p.166]). In the sequel we will replace (u, v) by this element.

We need the following notations (see [5] or [17] for more details):

(I) $O(N)$ stands for the orthogonal group of \mathbb{R}^N .

(II) $B(0, s)$ denotes the open N -dimensional ball of center zero, radius $s > 0$ and standard Lebesgue measure, $meas(B(0, s))$.

(III) $\|\alpha\|_{B(0, \frac{s}{2})} := \int_{B(0, \frac{s}{2})} \alpha(x)dx$.

Definition 1.2.

- A function $h : \mathbb{R}^N \rightarrow \mathbb{R}$ is radially symmetric if $h(gx) = h(x)$, for every $g \in O(N)$ and $x \in \mathbb{R}^N$.

- Let G be a topological group. A continuous map $\xi : G \times X \rightarrow X : (g, x) \rightarrow \xi(g, u) := gu$, is called the action of G on the Banach space $(X, \|\cdot\|_X)$ if

$$1u = u, \quad (gm)u = g(mu), \quad u \mapsto gu \text{ is linear.}$$

- The action is said to be isometric if $\|gu\|_X = \|u\|_X$, for every $g \in G$.

- The space of G -invariant points is defined by

$$Fix(G) := \{u \in X : gu = u, \text{ for all } g \in G\}.$$

- A map $m : X \rightarrow \mathbb{R}$ is said to be G -invariant if $mog = m$ for every $g \in G$.

The following theorem is important to study the critical point of the functional.

Theorem 1.1. [27] Assume that the action of the topological group G on the Banach space X is isometric. If $J \in C^1(X : \mathbb{R})$ is G -invariant and if u is a critical point of J restricted to $Fix(G)$, then u is a critical point of J .

The action of the group $O(N)$ on $W^{1,p}(\mathbb{R}^N)$ can be defined by $(gu)(x) := u(g^{-1}x)$, for every $g \in O(N)$ and $x \in \mathbb{R}^N$. A computation shows that this group acts linearly and isometrically, which means $\|u\| = \|gu\|$, for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$.

Definition 1.3. The subspace of radially symmetric functions of $W_r^{1,(p,q)}(\mathbb{R}^N)$ is defined by

$$\begin{aligned} X &:= W_r^{1,(p,q)}(\mathbb{R}^N) \\ &:= \{(u, v) \in W^{1,(p,q)}(\mathbb{R}^N) : (g_1u, g_2v) = (u, v), \text{ for all } (g_1, g_2) \in O(N) \times O(N)\}, \end{aligned}$$

and endowed by the norm

$$\|(u, v)\|_{W_r^{1,(p,q)}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx + \int_{\mathbb{R}^N} |u(x)|^p dx \right) + \left(\int_{\mathbb{R}^N} |\nabla v(x)|^q dx + \int_{\mathbb{R}^N} |v(x)|^q dx \right).$$

In what follows: $\|(u, v)\|_r$ denotes $\|(u, v)\|_{W_r^{1,(p,q)}(\mathbb{R}^N)}$.

The following crucial embedding result due to Kristály and principally based on a Strauss-type estimation (see [46]) (Also see [15, Theorem 3.1], and [16, 47] for related subjects).

Theorem 1.2. The embedding $W_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, is compact whenever $2 \leq N < p < +\infty$.

Here we consider the following functionals:

- $F_i(\xi) := \int_0^\xi f_i(t)dt$ for every $\xi \in \mathbb{R}$.

- $\Phi(u, v) := \frac{\|u\|_p^p}{p} + \frac{\|v\|_q^q}{q}$ for every $(u, v) \in X$.

- $\Psi(u, v) := \int_{\mathbb{R}^N} \alpha_1 F_1(v(x))dx + \int_{\mathbb{R}^N} \alpha_2 F_2(u(x))dx$, for every $(u, v) \in X$.

- $I_\lambda(u, v) := \Phi(u, v) - \lambda\Psi(u, v)$ for every $(u, v) \in X$.

By standard arguments [5], we can show that Φ is Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous whose derivative at the point $(u, v) \in X$ is the functional $\Phi'(u, v) \in X^*$ given by

$$\Phi'(u, v)(z, w) = \left(\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla z \, dx + \int_{\mathbb{R}^N} |u|^{p-2} u z \, dx \right) + \left(\int_{\mathbb{R}^N} |\nabla v|^{q-2} \nabla v \cdot \nabla w \, dx + \int_{\mathbb{R}^N} |v|^{q-2} v w \, dx \right),$$

for every $(z, w) \in X$. Also standard arguments show that the functional Ψ_i are well defined, sequentially weakly upper semicontinuous and Gâteaux differentiable whose Gâteaux derivative at the point $(u, v) \in X$ and for every $(z, w) \in X$ is given by,

$$\Psi'(u, v)(z, w) = \int_{\mathbb{R}^N} \alpha_1(x) f_1(u(x)) \, dx + \int_{\mathbb{R}^N} \alpha_2(x) f_2(v(x)) \, dx.$$

2. Weak solutions

First we recall the following theorem [3, Theorem 2.1].

Theorem 2.1. *Let X be a reflexive real Banach space, $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, set*

$$\varphi(r) := \inf_{\Phi(u) < r} \frac{\sup_{\Phi(v) < r} \Psi(v) - \Psi(u)}{r - \Phi(u)}, \quad \gamma := \liminf_{r \rightarrow +\infty} \varphi(r), \quad \text{and} \quad \delta := \liminf_{r \rightarrow (\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

- (a) for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(] - \infty, r])$ admits a global minimum, which is a critical point (local minimum) of I_λ in X .
- (b) if $\gamma < +\infty$, then for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds either,
 - (b₁) I_λ possesses a global minimum, or
 - (b₂) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n \rightarrow +\infty} \Phi(u_n) = +\infty$.
- (c) if $\delta < +\infty$, then for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds either:
 - (c₁) there is a global minimum of Φ which is a local minimum of I_λ . or,
 - (c₂) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ , with $\lim_{n \rightarrow +\infty} \Phi(u_n) = \inf_{u \in X} \Phi(u)$.

For fixed $D > 0$, set

$$m(D) := \text{meas}(B(0, D)) = D^N \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})},$$

where Γ is the Gamma function defined by $\Gamma(t) := \int_0^{+\infty} z^{t-1} e^{-z} \, dz$ for all $t > 0$. Moreover,

$$\Omega := \max \left\{ \frac{m(D) \left(\frac{\sigma(N,p)}{D^p} + g(p, N) \right)}{p\lambda B_1 \|\alpha_2\|_{B(0, \frac{D}{2})}}, \frac{m(D) \left(\frac{\sigma(N,q)}{D^q} + g(q, N) \right)}{q\lambda B_2 \|\alpha_1\|_{B(0, \frac{D}{2})}} \right\} > 0, \tag{3}$$

where $\sigma(N, p) := 2^{p-N}(2^N - 1)$, $c = \frac{2p}{2-N}$, m_1, m_0 are upper and lower bounds for $M(t)$ in (1) and

$$g(p, N) := \frac{1 + 2^{N+p} N \int_{\frac{1}{2}}^1 t^{N-1} (1-t)^p \, dt}{2^N}.$$

Assume $\|\cdot\|_1$ denotes the norm of $L^1(\Omega)$ and $F(\xi) := F_1(\xi) + F_2(\xi)$.

Theorem 2.2. *Let $f_i : \mathbb{R} \rightarrow \mathbb{R}$ be two continuous and radially symmetric functions. Set*

$$A := \liminf_{(\xi_1, \xi_2) \rightarrow +\infty} \frac{\max_{|t_1| \leq \xi_1} F_2(t_1)}{|\xi_1|^p} + \frac{\max_{|t_2| \leq \xi_2} F_1(t_2)}{|\xi_2|^q}, \quad B_1 := \limsup_{\xi_2 \rightarrow +\infty} \frac{F_1(\xi_2)}{|\xi_2|^p}, \quad \text{and} \quad B_2 := \limsup_{\xi_1 \rightarrow +\infty} \frac{F_2(\xi_1)}{|\xi_1|^q}.$$

where $B := B_1 + B_2$, $\xi = (\xi_1, \xi_2)$. If $\inf_{(\xi_1, \xi_2) \geq 0} F_2(\xi_1) + F_1(\xi_2) = 0$ and $A < \Omega m_0 B$, where Ω is given by (3), for every

$$\lambda \in \Lambda := \left] \Omega, \frac{1}{(pc_1^p \|\alpha_2\|_1 + qc_2^q \|\alpha_1\|_1) A} \right[,$$

there exists an unbounded sequence of radially symmetric weak solutions for (1) in X .

Proof. For fixed $\lambda \in \Lambda$, we consider Φ, Ψ and I_λ as in the last section. Knowing that Φ and Ψ satisfy the regularity assumptions in Theorem 2.1. In order to study the critical points of I_λ in X , we show that $\lambda < \frac{1}{\gamma} < +\infty$, where $\gamma = \liminf_{r \rightarrow +\infty} \phi(r)$. Let $\{t_n\}$ be a sequence of positive numbers such that $\lim_{n \rightarrow \infty} t_n = +\infty$, $r_{1n} := \frac{t_n^p}{pc_1^p}$ and $r_{2n} := \frac{t_n^q}{qc_2^q}$, for all $n \in \mathbb{N}$. Set $r_n = \min\{r_{n1}, r_{n2}\}$. Considering Theorem 1.2 (by relation (2)), a computation shows that

$$\begin{aligned} \Phi^{-1}([-\infty, r_n]) &= \{(z, w) \in X : \Phi(z, w) < r_n\} \\ &= \{(z, w) \in X : \frac{\|z\|_r^p}{p} + \frac{\|w\|_r^q}{q} < r_n\} \\ &\subset \{(z, w) \in X; \|(z, w)\|_\infty < t_n\}, \end{aligned} \tag{4}$$

where $t_n = \min\{t_{n1}, t_{n2}\}$.

Since $\Phi(0, 0) = \Psi(0, 0) = 0$, by a computation one can show

$$\varphi(r_n) = \inf_{\Phi(u,v) < r_n} \frac{(\sup_{\Phi(z,w) < r_n} \Psi(z, w)) - \Psi(u, v)}{r_n - \Phi(u, v)} \leq (pc_1^p \|\alpha_2\|_1 + qc_2^q \|\alpha_1\|_1) A.$$

Hence

$$\gamma \leq \liminf_{n \rightarrow +\infty} \varphi(r_n) \leq (pc_1^p \|\alpha_2\|_1 + qc_2^q \|\alpha_1\|_1) A < +\infty.$$

Now, we show that I_λ is unbounded from below. Let $\{d_{1n}\}$ and $\{d_{2n}\}$ be two sequences of positive numbers such that $\lim_{n \rightarrow +\infty} d_{1n} = \lim_{n \rightarrow +\infty} d_{2n} = +\infty$ and

$$B_1 = \lim_{n \rightarrow +\infty} \frac{F_1(d_{2n})}{d_{2n}^q}, \quad B_2 = \lim_{n \rightarrow +\infty} \frac{F_2(d_{1n})}{d_{1n}^p} \tag{5}$$

Define $\{(H_{1n}, H_{2n})\} \in X$ by

$$H_{in}(x) := \begin{cases} 0 & \mathbb{R}^N \setminus B(0, D) \\ d_{in} & B(0, \frac{D}{2}) \\ \frac{2d_{in}}{D}(D - |x|) & B(0, D) \setminus B(0, \frac{D}{2}), \end{cases}$$

for every $n \in \mathbb{N}$ and $i = 1, 2$. By a similar argument and computations in [5, P.1017] one can show that

$$\|H_{2n}\|_r^q = d_{2n}^q m(D) \left(\frac{\sigma(N, p)}{D^q} + g(q, N) \right) \quad \text{and} \quad \|H_{1n}\|_r^p = d_{1n}^p m(D) \left(\frac{\sigma(N, p)}{D^p} + g(p, N) \right).$$

Condition (i), implies

$$\begin{aligned} \int_{\mathbb{R}^N} \alpha_1(x) F_1(H_{2n}(x)) dx &\geq \int_{B(0, \frac{D}{2})} \alpha_1(x) F_1(d_{2n}) dx = F_1(d_{2n}) \|\alpha_1\|_{B(0, \frac{D}{2})}, \quad \text{and} \\ \int_{\mathbb{R}^N} \alpha_2(x) F_2(H_{1n}(x)) dx &\geq \int_{B(0, \frac{D}{2})} \alpha_2(x) F_2(d_{1n}) dx = F_2(d_{1n}) \|\alpha_2\|_{B(0, \frac{D}{2})}, \end{aligned}$$

for every $n \in \mathbb{N}$. Then

$$\begin{aligned} I_\lambda(H_{1n}, H_{2n}) &= \Phi(H_{1n}, H_{2n}) - \lambda \Psi(H_{1n}, H_{2n}) \\ &= \frac{\|H_{1n}\|_r^p}{p} + \frac{\|H_{2n}\|_r^q}{q} - \lambda \int_{\mathbb{R}^N} \alpha_1(x) F_1(H_{2n}(x)) dx - \lambda \int_{\mathbb{R}^N} \alpha_2(x) F_2(H_{1n}(x)) dx \\ &\leq \frac{d_{1n}^p m(D) \left(\frac{\sigma(N, p)}{D^p} + g(p, N) \right)}{p} + \frac{d_{2n}^q m(D) \left(\frac{\sigma(N, q)}{D^q} + g(q, N) \right)}{q} \\ &\quad - \lambda \left(F_1(d_{2n}) \|\alpha_1\|_{B(0, \frac{D}{2})} + F_2(d_{1n}) \|\alpha_2\|_{B(0, \frac{D}{2})} \right). \end{aligned}$$

If $B < +\infty$ ($B_1, B_2 < +\infty$), the conditions (5) implies that

- there exists N_1 such that for all $n \geq N_1$ we have $F_1(d_{2n}) > \varepsilon B_1 d_{2n}^p$, and
- there exists N_2 such that for all $n \geq N_2$ we have $F_2(d_{1n}) > \varepsilon B_2 d_{1n}^q$.

Then for every $n \geq N_\varepsilon := \max\{N_1, N_2\}$,

$$I_\lambda(H_{1n}, H_{2n}) \leq \frac{d_{1n}^p m(D) \left(\frac{\sigma(N, p)}{D^p} + g(p, N) \right)}{p} + \frac{d_{2n}^q m(D) \left(\frac{\sigma(N, q)}{D^q} + g(q, N) \right)}{q} - \lambda \varepsilon \left(d_{1n}^p B_2 \|\alpha_1\|_{B(0, \frac{D}{2})} + d_{2n}^q B_1 \|\alpha_2\|_{B(0, \frac{D}{2})} \right)$$

$$= d_{1n}^p \left(\frac{m(D) \left(\frac{\sigma(N,p)}{D^p} + g(p, N) \right)}{p} - \lambda \varepsilon B_1 \|\alpha_2\|_{B(0, \frac{D}{2})} \right) + d_{2n}^q \left(\frac{m(D) \left(\frac{\sigma(N,q)}{D^q} + g(q, N) \right)}{q} - \lambda \varepsilon B_2 \|\alpha_1\|_{B(0, \frac{D}{2})} \right).$$

If we set

$$\Omega := \max \left\{ \frac{m(D) \left(\frac{\sigma(N,p)}{D^p} + g(p, N) \right)}{p \lambda B_1 \|\alpha_2\|_{B(0, \frac{D}{2})}}, \frac{m(D) \left(\frac{\sigma(N,q)}{D^q} + g(q, N) \right)}{q \lambda B_2 \|\alpha_1\|_{B(0, \frac{D}{2})}} \right\},$$

then for $\varepsilon \in (\Omega, 1)$ one can get $\lim_{n \rightarrow +\infty} I_\lambda(H_{1n}, H_{2n}) = -\infty$.

If at least one of the B_1 or B_2 are $+\infty$. Let $B_1 = +\infty$, and consider $M_1 > \Omega$, then by (5) there exists N_{M_1} such that for every $n > N_{M_1}$, we have $F_1(d_{1n}) > M_1 d_{1n}^p$. Moreover, for every $n > N_{M_1}$

$$\begin{aligned} I_\lambda(H_{1n}, H_{2n}) &\leq \frac{d_{1n}^p m(D) \left(\frac{\sigma(N,p)}{D^p} + g(p, N) \right)}{p} + \frac{d_{2n}^q m(D) \left(\frac{\sigma(N,q)}{D^q} + g(q, N) \right)}{q} - \lambda \left(d_{1n}^p M_1 \|\alpha_2\|_{B(0, \frac{D}{2})} + d_{2n}^q M_1 \|\alpha_1\|_{B(0, \frac{D}{2})} \right) \\ &= d_{1n}^p \left(\frac{m(D) \left(\frac{\sigma(N,p)}{D^p} + g(p, N) \right)}{p} - \lambda M_1 \|\alpha_2\|_{B(0, \frac{D}{2})} \right) + d_{2n}^q \left(\frac{m(D) \left(\frac{\sigma(N,q)}{D^q} + g(q, N) \right)}{q} - \lambda M_1 \|\alpha_1\|_{B(0, \frac{D}{2})} \right). \end{aligned}$$

This implies that $\lim_{n \rightarrow +\infty} I_\lambda(H_{1n}, H_{2n}) = -\infty$.

Now, Theorem 2.1(b) implies, the functional I_λ admits an unbounded sequence $\{u_n\} \subset X$ of critical points. Considering Theorem 1.1, these critical points are also critical points for the smooth and $O(N)$ -invariant functional $I_\lambda : W^{1,p}(\mathbb{R}^N) \rightarrow \mathbb{R}$. Therefore, there is a sequence of radially symmetric weak solutions for the problem (1), which are unbounded in $W^{1,p}(\mathbb{R}^N)$. \square

Here we prove our second result which says that under different conditions the problem (1) has a sequence of weak solutions, which converges weakly to zero.

Theorem 2.3. *Let $f_i : \mathbb{R} \rightarrow \mathbb{R}$ be two continuous and radially symmetric functions. Set*

$$A' := \liminf_{(\xi_1, \xi_2) \rightarrow 0^+} \frac{\max_{|t_1| \leq \xi_1} F_2(t_1)}{|\xi_1|^p} + \frac{\max_{|t_2| \leq \xi_2} F_1(t_2)}{|\xi_2|^q}, \quad B'_1 := \limsup_{\xi_2 \rightarrow 0^+} \frac{F_1(\xi_2)}{|\xi_2|^p}, \quad \text{and} \quad B'_2 := \limsup_{\xi_1 \rightarrow 0^+} \frac{F_2(\xi_1)}{|\xi_1|^q},$$

where $B' := B'_1 + B'_2$, $\xi = (\xi_1, \xi_2)$. If $\inf_{(\xi_1, \xi_2) \geq 0} F_2(\xi_1) + F_1(\xi_2) = 0$ and $A' < \Omega m_0 B'$, where Ω is given by (3), for every

$$\lambda \in \Lambda' := \left] \Omega, \frac{1}{(p c_1^p \|\alpha_2\|_1 + q c_2^q \|\alpha_1\|_1) A'} \right[,$$

there exists an unbounded sequence of radially symmetric weak solutions for (1) in X .

Proof. For fixed $\lambda \in \Lambda'$, we consider Φ , Ψ and I_λ as in Section 2. Knowing that Φ and Ψ satisfy the regularity assumptions in Theorem 2.1, we show that $\lambda < \frac{1}{\delta}$. We know that $\inf_X \Phi = 0$. Set $\delta := \liminf_{r \rightarrow 0^+} \varphi(r)$. A computation similar to the one in the Theorem 2.2 implies $\delta < \infty$ and if $\lambda \in \Lambda'$ then $\lambda < \frac{1}{\delta}$. A compaction (similar in the Theorem 2.2) shows that $I_\lambda(H_{1n}, H_{2n}) < 0$ for n large enough and thus zero is not a local minimum of I_λ . Therefore, there exists a sequence $\{u_n\} \subset X$ of critical points of I_λ which converges weakly to zero in X as $\lim_{n \rightarrow +\infty} \Phi(u_n) = 0$.

Again, considering Theorem 1.1, these critical points are also critical points for the smooth and $O(N)$ -invariant functional $I_\lambda : W^{1,p}(\mathbb{R}^N) \rightarrow \mathbb{R}$. Therefore, there is a sequence of radially symmetric weak solutions for the problem (1), which converges weakly to zero in $W^{1,p}(\mathbb{R}^N)$. \square

Acknowledgment

This work is supported by I.N.D.A.M - G.N.A.M.P.A. 2019 and the ‘‘RUDN University Program 5-100’’.

References

- [1] F. Behboudi, A. Razani, Two weak solutions for a singular (p, q) -Laplacian problem, *Filomat* **33** (2019) 3399–3407.
- [2] S. Benbernou, S. Gala, M. A. Ragusa, On the regularity criteria for the 3D magnetohydrodynamic equations via two components in terms of BMO space, *Math. Methods Appl. Sci.* **37** (2014) 2320–2325.
- [3] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.* **2009** (2009) Art# 670675.
- [4] H. Brezis, *Analyse fonctionnelle: Théorie et applications*, Masson, Paris, 1963.
- [5] P. Candito, G. Molica Bisci, Radially symmetric weak solutions for elliptic problems in \mathbb{R}^N , *Differential Integral Equations* **26** (2013) 1009–1026.

- [6] Q. Chen, C. Chen, Infinitely many solutions for a class of p -biharmonic equation in \mathbb{R}^N , *Bull. Iranian Math. Soc.* **43** (2017) 205–215.
- [7] D. Ehsani, M. Mokhtarzadeh, A. Razani, Expansions of Dirichlet to Neumann operators under perturbations, *Appl. Anal.* **89** (2010) 789–799.
- [8] S. Gala, Q. Liu, M. A. Ragusa, A new regularity criterion for the nematic liquid crystal flows, *Appl. Anal.* **91** (2012) 1741–1747.
- [9] S. Gala, M. A. Ragusa, A logarithmically improved regularity criterion for the 3D MHD equations in Morrey–Campanato space, *AIMS Math.* **2** (2017) 16–23.
- [10] S. Gala, M. A. Ragusa, Logarithmically improved regularity criterion for the Boussinesq equations in Besov spaces with negative indices, *Appl. Anal.* **95** (2016) 1271–1279.
- [11] M. Hesaaraki, A. Razani, Detonative travelling waves for combustion, *Appl. Anal.* **77** (2001) 405–418.
- [12] M. Hesaaraki, A. Razani, On the existence of Chapman–Jouguet detonation Waves, *Bull. Aust. Math. Soc.* **63** (2001) 485–496.
- [13] S. M. Khalkhali, S. Heidarkhani, A. Razani, Infinitely many solutions for a fourth-order boundary-value problem, *Electron. J. Differential Equations* **2012** (2012) Art# 164.
- [14] S. M. Khalkhali, A. Razani, Multiple solutions for a quasilinear (p, q) -elliptic system, *Electron. J. Differential Equations* **2013** (2013) Art# 144.
- [15] A. Kristály, Infinitely many solutions for a differential inclusion problem in \mathbb{R}^N , *J. Differential Equations* **220** (2006) 511–530.
- [16] A. Kristály, V. D. Rădulescu, C. G. Varga, *Variational Principles in Mathematical Physics, Geometry, and Conomics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems (Encyclopedia of Mathematics and its Applications, Vol. 136)*, Cambridge University Press, Cambridge, 2010.
- [17] Y. Li, Multiple solutions for perturbed p -Laplacian problems on \mathbb{R}^N , *Electron. J. Differential Equations* **2015** (2015) Art# 23.
- [18] E. Lindgren, A. Razani, The N -membranes problem, *Bull. Iranian Math. Soc.* **35** (2009) 31–40.
- [19] R. Mahdavi Khanghahi, A. Razani, Solutions for a singular elliptic problem involving the $p(x)$ -Laplacian, *Filomat* **32** (2018) 4841–4850.
- [20] M. Makvand Chaharlang, M. A. Ragusa, A. Razani, A sequence of radially symmetric weak solutions for some nonlocal elliptic problem in \mathbb{R}^N , *Mediterr. J. Math.* **17** (2020) Art# 53.
- [21] M. Makvand Chaharlang, A. Razani, A fourth order singular elliptic problem involving p -biharmonic operator, *Taiwanese J. Math.* **23** (2019) 589–599.
- [22] M. Makvand Chaharlang, A. Razani, Existence of infinitely many solutions for a class of nonlocal problems with Dirichlet boundary condition, *Commun. Korean Math. Soc.* **34** (2019) 155–167.
- [23] M. Makvand Chaharlang, A. Razani, Infinitely many solutions for a fourth order singular elliptic problem, *Filomat* **32** (2018) 5003–5010.
- [24] M. Makvand Chaharlang, A. Razani, Two weak solutions for some Kirchhoff–type problem with Neumann boundary condition, *Georgian Math. J.*, DOI: 10.1515/gmj-2019-2077, In press.
- [25] M. R. Mokhtarzadeh, M. R. Pournaki, A. Razani, A note on periodic solutions of Riccati equations, *Nonlinear Dyn.* **62** (2010) 119–125.
- [26] M. R. Mokhtarzadeh, M. R. Pournaki, A. Razani, An existence–uniqueness theorem for a class of boundary value problems, *Fixed Point Theory* **13** (2012) 583–592.
- [27] R. S. Palais, The principle of symmetric criticality, *Commun. Math. Phys.* **69** (1979) 19–30.
- [28] S. Polidoro, M. A. Ragusa, Harnack inequality for hypoelliptic ultraparabolic equations with a singular lower order term, *Rev. Mat. Iberoam* **24** (2008) 1011–1046.
- [29] M. R. Pournaki, A. Razani, Erratum to “On the existence of periodic solutions for a class of generalized forced Liénard equations” [Appl. Math. Lett. 20(3) (2007) 248–254], *Appl. Math. Lett.* **21** (2008) 880.
- [30] M. R. Pournaki, A. Razani, On the existence of periodic solutions for a class of generalized forced Liénard equations, *Appl. Math. Lett.* **20** (2007) 248–254.
- [31] M. A. Ragusa, Local Hölder regularity for solutions of elliptic systems, *Duke Math. J.* **113** (2002) 385–397.
- [32] M. A. Ragusa, On weak solutions of ultraparabolic equations, *Nonlinear Anal.* **47** (2001) 503–511.
- [33] M. A. Ragusa, Parabolic Herz spaces and their applications, *Appl. Math. Lett.* **25** (2012) 1270–1273.
- [34] M. A. Ragusa, A Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, *Adv. Nonlinear Anal.* **9** (2020) 710–728.
- [35] A. Razani, An existence theorem for ordinary differential equation in Menger probabilistic metric space, *Miskolc Math. Notes* **15** (2014) 711–716.
- [36] A. Razani, Chapman–Jouguet detonation profile for a qualitative model, *Bull. Aust. Math. Soc.* **66** (2002) 393–403.
- [37] A. Razani, Chapman–Jouguet travelling wave for a two–steps reaction scheme, *Ital. J. Pure Appl. Math.* **39** (2018) 544–553.
- [38] A. Razani, Existence of Chapman–Jouguet detonation for a viscous combustion model, *J. Math. Anal. Appl.* **293** (2004) 551–563.
- [39] A. Razani, On the existence of premixed laminar flames, *Bull. Aust. Math. Soc.* **69** (2004) 415–427.
- [40] A. Razani, Shock waves in gas dynamics, *Surv. Math. Appl.* **2** (2007) 59–89.
- [41] A. Razani, Structure of shock wave for a viscous combustion model, *Filomat* **26** (2012) 389–395.
- [42] A. Razani, Subsonic detonation waves in porous media, *Phys. Scr.* **94** (2019) Art# 085209.
- [43] A. Razani, Weak and strong detonation profiles for a qualitative model, *J. Math. Anal. Appl.* **276** (2002) 868–881.
- [44] A. Razani, Z. Goodarzi, A solution of Volterra–Hamerstain integral equation in partially ordered sets, *Int. J. Ind. Math.* **3** (2011) 277–281.
- [45] A. Scapellato, Regularity of solutions to elliptic equations on Herz spaces with variable exponents, *Bound. Value Probl.* **2019** (2019) Art# 2.
- [46] W. A. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.* **55** (1977) 149–162.
- [47] M. Willem, *Minimax Theorems (Progress in Nonlinear Differential Equations and Their Applications, Vol. 24)*, Birkhäuser, Boston, 1996.
- [48] R. Yang, W. Zhang, X. Liu, Sign–changing solutions for p -biharmonic equations with Hardy potential in \mathbb{R}^N , *Acta Math. Sci. Ser. B* **37** (2017) 593–606.